Recall that an autonomous Hamiltonian system on a $2n$-dimensional symplectic manifold $(Z, \Omega)$ is called completely integrable (henceforth CIS) if it admits $n$ independent integrals of motion $\{H_1, \ldots, H_n\}$ in involution. Let $M$ be its regular connected invariant submanifold. The classical Liouville–Arnold theorem [1–3] and its generalization [4,5] for noncompact invariant submanifolds state that an open neighborhood $U_M$ of $M$ can be provided with the action-angle coordinates $(J_a, y^a)$ such that a symplectic form on $U_M$ reads $\Omega = dJ_a \wedge dy^a$, and the integrals of motion $H_a$ together with a Hamiltonian $H$ are expressed only in the action coordinates $(J_a)$.

However, integrals of motion of a Hamiltonian system need not commute. A Hamiltonian system on a symplectic manifold $(Z, \Omega)$ is called a noncommutative CIS if it admits $n \leq k < 2n$ integrals of motion $\{H_1, \ldots, H_k\}$ which obey the following conditions.

(i) The smooth real functions $H_i$ are independent on $Z$, i.e., the $k$-form $\wedge dH_i$ nowhere vanishes. Their common level surfaces are regular invariant submanifolds which make $Z$ into a fibred manifold

$$H : Z \to N \subset \mathbb{R}^k.$$  

(ii) There exist smooth real functions $s_{ij} : N \to \mathbb{R}$ such that the Poisson bracket of integrals of motion reads

$$\{H_i, H_j\} = s_{ij} \circ H, \quad i, j = 1, \ldots, k,$$

where the matrix function $(s_{ij})$ is of constant corank $m = 2n - k$ at all points of $N$.

If $k = n$, we are in the case of an Abelian CIS. A noncommutative CIS is exemplified by a spherical top possessing the Lie algebra $\mathfrak{so}(3)$ of three independent integrals of motion on a certain four-dimensional reduced subspace of the momentum phase space.

Let us additionally assume that the Hamiltonian vector fields $\theta_i$ of integrals of motion $H_i$ are complete and their invariant manifolds are connected and mutually diffeomorphic. Then the classical Mishchenko–Fomenko theorem [6–8] and its generalization [9] for noncompact invariant submanifolds state that every invariant submanifold $M$ is diffeomorphic to a toroidal cylinder $\mathbb{R}^{m-r} \times T^r$, $m = 2n - k$, coordinated by $(y^a)$, and it admits an open fibered neighborhood $H : U_M \to N_M$ endowed with action-angle coordinates $(J_a, p_A, q^A, y^a)$ such that a symplectic form on $U_M$ reads

$$\Omega = dJ_a \wedge dy^a + dp_A \wedge dq^A,$$

and a Hamiltonian $\mathcal{H}$ depends only on the action coordinates $J_a$. 

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One can say something more. The base $N(1)$ is provided with a unique coinduced Poisson structure, whose $N$ of rank $2(k - n)$ such that $H$ is a Poisson morphism. Furthermore, every invariant submanifold $M$ is a maximal integral manifold of the involutive distribution spanned by the Hamiltonian vector fields $\mathfrak{u}_a$ of the pull-back $H^*C_a(\xi) = C_a(H_\xi(\xi))$ onto $U_M$ of $m$ independent Casimir functions $C_1, \ldots, C_m$ on an open neighborhood $N_M$ of the point $H(M) \subset N$. The original integrals of motion are smooth functions of coordinates $(J_a, q^A, p_A)$, but the Casimir functions

$$C_a(H(J_b, q^A, p_A)) = C_a(J_b)$$

depend only on the action coordinates $J_a$. Moreover, a Hamiltonian $\hat{\mathcal{H}}(J_b) = \mathcal{H}(C_a(J_b))$ is expressed in action variables $J_a$ through the Casimir functions (4).

We aim to quantize a noncommutative CIS written in the action-angle variables around its invariant submanifold. Since $(J_a, p_A, q^A)$ are coordinates on $N_M$, they are integrals of motion which constitute a noncommutative CIS

$$\{J_a, p_A\} = \{J_a, q^A\} = 0, \quad \{p_A, q^B\} = \delta^B_A,$$ (5)

on $U_M$ equivalent to the original one (2). Furthermore, this CIS can be treated as a particular Abelian CIS possessing $n$ integrals of motion $(J_a, p_A)$ and action-angle coordinates $(J_a, p_A, q^A, y^a)$ on $U_M$, where $(q^A, y^a)$ are angle coordinates on its invariant submanifold

$$\mathcal{M} = V_M \times \mathbb{R}^{m-r} \times T^r \subset \mathbb{R}^{n-r} \times T^r,$$ (6)

where $V_M$ is a base of the fibration $U_M \ni (J_a, p_A, q^A) \to (q^A) \in V_M$. Therefore, the noncommutative CIS (5) can be quantized as the Abelian one. Strictly speaking, this quantization fails to be a quantization of the original CIS (2) because $H(J_a, q^A, p_A)$ are not linear functions and, consequently, the algebras (2) and (5) are not isomorphic in general. As a result, one however can obtain the Hamilton operator $\hat{\mathcal{H}}$ and the Casimir operators $\hat{C}_a$ of an original CIS and their spectra.

There are different approaches to quantization of Abelian CISs [10–14]. It should be emphasized that action-angle coordinates need not be globally defined on the momentum phase space of a CIS, but form an algebra of Poisson canonical coordinates. Moreover, the action-angle coordinates $(J_a, p_A, q^A, y^a)$ are canonical for the symplectic form $Ω(3)$.

\begin{align*}
\hat{f} = & -i\nabla_{\theta_f} f + f,
\end{align*} (7)

on sections of $C \to T^*\mathcal{M}$, where $\theta_f$ is the Hamiltonian vector field of $f$ and $\nabla$ is the covariant differential with respect to a suitable $U(1)$-principal connection $A$ on $C$. This connection preserves the Hermitian metric $g(c, c') = \bar{c}c'$ on $C$, and its curvature obeys the prequantization condition $R = i\Omega$. It reads

$$A = A_0 + ic(p_A d\lambda^A + J_j d\gamma^j + J_{\mu} d\alpha^\mu) \otimes \partial_c,$$ (8)

where $A_0$ is a flat $U(1)$-principal connection on $C \to T^*\mathcal{M}$. The classes of gauge nonconjugated flat principal connections on $C$ are indexed by the set $\mathbb{R}/\mathbb{Z}$ of homomorphisms of the de Rham cohomology group

$$H^1(T^*\mathcal{M}) = H^1(M) = H^1(T^*R) \cong \mathbb{R}$$

of $T^*\mathcal{M}$ to $U(1)$. We choose their representatives of the form

$$A_0[O(\mu)] = dp_A \otimes d\lambda^A + dJ_j \otimes d\gamma^j + dJ_{\mu} \otimes d\alpha^\mu + dq^A \otimes d\lambda_A + dy^j \otimes d\gamma_j + d\alpha^\mu \otimes (\partial_c + i\lambda_{\mu}c\partial_c),$$

\begin{align*}
\lambda_{\mu} & \in [0, 1).
\end{align*}

Accordingly, the relevant connection (8) on $C$ reads

\begin{align*}
A[O(\mu)] = & dp_A \otimes d\lambda^A + dJ_j \otimes d\gamma^j + dJ_{\mu} \otimes d\alpha^\mu \\
& + dq^A \otimes (\partial_c + i\lambda_{\mu}c\partial_c) + dy^j \otimes (\partial_j + iJ_{j}c\partial_c) \\
& + d\alpha^\mu \otimes (\partial_c + i(J_{\mu} + \lambda_{\mu})c\partial_c).\end{align*} (9)
For the sake of simplicity, we further assume that the numbers \( \lambda_{\mu} \) in the expression (9) belong to \( \mathbb{R} \), but bear in mind that connections \( A(\lambda_{\mu}) \) and \( A(\lambda'_{\mu}) \) with \( \lambda_{\mu} - \lambda'_{\mu} \in \mathbb{Z} \) are gauge conjugated.

Let us choose the above mentioned angle polarization \( VT^*M \). Then the corresponding quantum algebra \( \mathcal{A} \) of \( T^*M \) consists of affine functions

\[
f = a^A(\varphi, y^j, \alpha^\nu) p_A + a^I(\varphi, y^j, \alpha^\nu) J_I + \alpha^\mu(\varphi, y^j, \alpha^\nu) J_\mu + b(\varphi, y^j, \alpha^\nu)
\]

in action coordinates \((p_A, J_I, J_\mu)\). Given a connection (9), the corresponding operators (7) read

\[
\hat{f} = \left( -ia^A \partial_A - \frac{i}{2} \partial_A a^A \right) + \left( -ia^I \partial_I - \frac{i}{2} \partial_I a^I \right) + \left( -ia^\mu \partial_\mu - \frac{i}{2} \partial_\mu a^\mu - a^\mu \lambda_\mu \right) + b.
\]

They are self-adjoint operators in the pre-Hilbert space \( \mathbb{C}^\infty(M) \) of smooth complex functions of compact support on \( M \) endowed with the Hermitian form

\[
\langle \psi | \psi' \rangle = \left( \frac{1}{2\pi} \right)^r \int_{\mathcal{M}} \psi^* \psi' d^{m-r}q d^r \alpha,
\]

\( \psi, \psi' \in \mathbb{C}^\infty(M) \).

Note that any function \( \psi \in \mathbb{C}^\infty(M) \) is expanded into the series

\[
\psi = \sum_{(n_\mu)} \phi(q^R, y^j) |(n_\mu)\rangle \exp\{in_\mu \alpha^\mu\},
\]

\( (n_\mu) = (n_1, \ldots, n_r) \in \mathbb{Z}^r \),

(11)

where \( \phi(q^R, y^j) |(n_\mu)\rangle \) are functions of compact support on \( \mathbb{R}^{r-m} \). In particular, the action operators (10) read

\[
\hat{p}_A = -i \partial_A, \quad \hat{J}_I = -i \partial_I, \quad \hat{J}_\mu = -i \partial_\mu - \lambda_\mu.
\]

(12)

It should be emphasized that

\[
\hat{a} \hat{p}_A \neq \hat{a} \hat{p}_A, \quad \hat{a} \hat{J}_I \neq \hat{a} \hat{J}_I, \quad \hat{a} \hat{J}_\mu \neq \hat{a} \hat{J}_\mu, \quad a \in \mathbb{C}^\infty(M).
\]

(13)

The operators (10) provide the desired quantization of a noncommutative CIS with respect to the action-angle coordinates. They satisfy the Dirac condition

\[
[\hat{f}, \hat{f}'] = -i \{ f, f' \}, \quad f, f' \in \mathcal{A}.
\]

(14)

However, both a Hamiltonian \( \mathcal{H} \) and original integrals of motion \( H_I \) do not belong to the quantum algebra \( \mathcal{A} \), unless they are affine functions in the action coordinates \((p_A, J_I, J_\mu)\). It is a well-known problem of the Schrödinger representation. In some particular cases, integrals of motion \( H_I \) can be represented by differential operators, but this representation fails to be unique because of inequalities (13), and the Dirac condition (14) need not be satisfied. At the same time, both a Hamiltonian \( \mathcal{H} \) and the Casimir functions \( C_I \) depend only on action variables \( J_I, J_\mu \). If they are polynomial in \( J_I \), one can associate to them the operators \( \hat{\mathcal{H}} = \mathcal{H}(\hat{J}_I, \hat{J}_\mu) \), \( \hat{C}_I = C_I(\hat{J}_I, \hat{J}_\mu) \) acting in the space \( \mathbb{C}^\infty(M) \) by the law

\[
\hat{H}\psi = \sum_{(n_\mu)} \mathcal{H}(\hat{J}_I, n_\mu - \lambda_\mu) \phi(q^A, y^j) |(n_\mu)\rangle \exp\{in_\mu \alpha^\mu\},
\]

\[
\hat{C}_I\psi = \sum_{(n_\mu)} C_I(\hat{J}_I, n_\mu - \lambda_\mu) \phi(q^A, y^j) |(n_\mu)\rangle \exp\{in_\mu \alpha^\mu\}.
\]

Let us mention a particular class of CISs whose integrals of motion \( \{H_1, \ldots, H_3\} \) form a \( k \)-dimensional real Lie algebra \( \mathfrak{g} \) of rank \( m \) with the commutation relations

\[
\{H_I, H_J\} = c_{ij}^h H_h, \quad c_{ij}^h = \text{const}.
\]

(15)

In this case, nonvanishing complete Hamiltonian vector fields \( \partial_i \) of \( H_I \) define a free Hamiltonian action on \( Z \) of some connected Lie group \( G \) whose Lie algebra \( \mathfrak{g} \) is isomorphic to \( \mathfrak{g}^* \). Orbits of \( G \) coincide with \( k \)-dimensional maximal integral manifolds of the regular distribution on \( Z \) spanned by Hamiltonian vector fields \( \partial_i \). Furthermore, one can treat \( H(1) \) as an equivariant moment map of \( Z \) to the lie coalgebra \( \mathfrak{g}^* \), provided with the coordinates \( g(z) = H_I(z), z \in Z \). In this case, the coinduced Poisson structure \( \{\cdot,\cdot\}_N \) on the base \( N \) coincides with the canonical Lie–Poisson structure on \( \mathfrak{g}^* \) given by the Poisson bivector field

\[
w = \frac{1}{2} c_{ij}^h x_h \partial_i \wedge \partial_j.
\]

Recall that the coadjoint action of \( G \) on \( \mathfrak{g}^* \) reads

\[
\varepsilon(x_j) = c_{ij}^h x_h.
\]

(15)

Casimir functions of the Lie–Poisson structure are exactly the coadjoint invariant functions on \( \mathfrak{g}^* \). They are constant on orbits of the coadjoint action of \( G \) on \( \mathfrak{g}^* \). Given a point \( z \in Z \) and the orbit \( G_z \) of \( G \) in \( Z \) through \( z \), the fibration \( H(1) \) projects this orbit onto the orbit \( G_H(z) \) of the coadjoint action of \( G \) in \( \mathfrak{g}^* \) through \( H(z) \). Moreover, the inverse image \( H^{-1}(G_H(z)) \) of \( G_H(z) \) coincides with the orbit \( G_z \). It follows that any orbit of \( G \) in \( Z \) is fibered in invariant submanifolds.

The Mishchenko–Fomenko theorem has been mainly applied to CISs whose integrals of motion form a compact Lie algebra. The group \( G \) generated by flows of their Hamiltonian vector fields is compact, and every orbit of \( G \) in \( Z \) is compact. Since a fibration of a compact manifold possesses compact fibers, any invariant submanifold of such a noncommutative CIS is compact.

For instance, let us consider the above mentioned noncommutative CIS with the Lie algebra \( \mathfrak{g} = so(3) \) of integrals of motion \( \{H_1, H_2, H_3\} \) on a four-dimensional symplectic manifold \( (Z, \Omega) \), namely,

\[
\{H_1, H_2\} = H_3, \quad \{H_2, H_3\} = H_1, \quad \{H_3, H_1\} = H_2.
\]

(16)

The rank of this Lie algebra equals one. Since it is compact, an invariant submanifold of a CIS in question is a circle \( M = S^1 \). We have a fibered manifold \( \hat{H}: Z \to N \) onto an open subset \( N \subset \mathfrak{g}^* \) of the Lie coalgebra \( \mathfrak{g}^* \). This fibered manifold is a fiber bundle since its fibers are compact [21]. The base \( N \) is endowed with the coordinates \((x_1, x_2, x_3)\) such that integrals of motion
\{H_1, H_2, H_3\} on \(Z\) read
\[
H_1 = x_1, \quad H_2 = x_2, \quad H_3 = x_3.
\]
As was mentioned above, the coinduced Poisson structure on \(N\) is the Lie–Poisson structure
\[
w = x_2 \partial^3 \wedge \partial^1 + x_3 \partial^1 \wedge \partial^2 + x_1 \partial^2 \wedge \partial^3.
\]
(17)
The coadjoint action (15) of \(so(3)\) reads
\[
e_1 = x_3 \partial^2 - x_2 \partial^3, \quad e_2 = x_1 \partial^3 - x_3 \partial^1,
\]
\[
e_3 = x_2 \partial^1 - x_1 \partial^2.
\]
An orbit of the coadjoint action of dimension 2 is given by the equation
\[
(x_1^2 + x_2^2 + x_3^2) = \text{const}.
\]
Let \(M\) be an invariant submanifold such that the point \(H(M) \in g^*\) belongs to an orbit of the coadjoint action of maximal dimension 2. Let us consider an open fibered neighborhood \(U_M = N_M \times S^1\) of \(M\) which is a trivial bundle over an open contractible neighborhood \(N_M\) of \(H(M)\) endowed with the coordinates \((r, x, \gamma)\) defined by the equalities
\[
r = \left(x_1^2 + x_2^2 + x_3^2\right)^{1/2},
\]
\[
x_2 = (r^2 - x_1^2)^{1/2} \sin \gamma, \quad x_3 = (r^2 - x_1^2)^{1/2} \cos \gamma.
\]
(18)
Here, \(r\) is a Casimir function on \(g^*\). It is readily observed that the coordinates (18) are the Darboux coordinates of the Lie–Poisson structure (17) on \(N_M\), namely,
\[
w = \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}.
\]
(19)
Let \(\partial_r\) be the Hamiltonian vector field of the Casimir function \(r\) (18). It is a combination
\[
\partial_r = \frac{1}{r}(x_1 \partial_1 + x_2 \partial_2 + x_3 \partial_3)
\]
of the Hamiltonian vector fields \(\partial_i\) of integrals of motion \(H_i\).
Its flows are invariant submanifolds. Let \(\alpha\) be a parameter along the flows of this vector field, i.e.,
\[
\partial_r = \frac{\partial}{\partial \alpha}.
\]
Then \(U_M\) is provided with the action-angle coordinates \((r, x_1, \gamma, \alpha)\) such that the Poisson bivector associated to the symplectic form \(\Omega\) on \(U_M\) reads
\[
W = \frac{\partial}{\partial r} \wedge \frac{\partial}{\partial \alpha} + \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial \gamma}.
\]
(20)
Accordingly, Hamiltonian vector fields of integrals of motion take the form
\[
\partial_1 = \frac{\partial}{\partial \gamma},
\]
\[
\partial_2 = r(r^2 - x_1^2)^{-1/2} \sin \gamma \frac{\partial}{\partial \alpha} - x_1 (r^2 - x_1^2)^{-1/2}
\]
\[
\times \sin \gamma \frac{\partial}{\partial \gamma} - (r^2 - x_1^2)^{1/2} \cos \gamma \frac{\partial}{\partial x_1},
\]
\[
\partial_3 = r(r^2 - x_1^2)^{-1/2} \cos \gamma \frac{\partial}{\partial \alpha} - x_1 (r^2 - x_1^2)^{-1/2}
\]
\[
\times \cos \gamma \frac{\partial}{\partial \gamma} + (r^2 - x_1^2)^{1/2} \sin \gamma \frac{\partial}{\partial x_1}.
\]
(21)
The action-angle variables \((r, H_1 = x_1, \gamma)\) constitute a noncommutative CIS \(\{r, H_1\} = 0, \quad \{r, \gamma\} = 0, \quad \{H_1, \gamma\} = 1\)
on \(U_M\). This noncommutative CIS is related to the original one by the transformations
\[
r = (H_1^2 + H_2^2 + H_3^2)^{1/2},
\]
\[
H_2 = (r^2 - H_1^2)^{1/2} \sin \gamma, \quad H_3 = (r^2 - H_1^2)^{1/2} \cos \gamma.
\]
Its Hamiltonian is expressed only in the action variable \(r\).
Let us quantize the noncommutative CIS (21). We obtain the algebra of operators
\[
\hat{r} = \hat{a}(-i\frac{\partial}{\partial \alpha} - \lambda) - i\frac{\partial}{\partial \gamma} - \frac{\partial}{2}(\frac{\partial}{\partial a} + \frac{\partial}{\partial b}) + c,
\]
where \(a, b, c\) are smooth functions of angle coordinates \((\gamma, \alpha)\) on the cylinder \(\mathbb{R} \times S^1\). In particular, the action operators read
\[
\hat{r} = -i\frac{\partial}{\partial \alpha} - \lambda, \quad \hat{H}_1 = -i\frac{\partial}{\partial \gamma}.
\]
(22)
These operators act in the space of smooth complex functions
\[
\psi(\gamma, \alpha) = \sum_k \phi(\gamma) k \exp[i k \alpha]
\]
of compact support on \(\mathbb{R} \times S^1\). A Hamiltonian \(\hat{H}(r)\) of a classical CIS can also be represented by the operator
\[
\hat{H}(r) \psi = \sum_k \hat{H}(k - \lambda) \phi(\gamma) k \exp[i k \alpha]
\]
on this space.
For instance, let us consider a spherical top whose integrals of motion \(\{H_1, H_2, H_3\}\) are angular momenta, and a Hamiltonian reads
\[
\mathcal{H} = \frac{1}{2} \left(H_1^2 + H_2^2 + H_3^2\right) = \frac{1}{2} I r^2,
\]
where \(I\) is a rotational constant. The momentum phase space of a spherical top is the cotangent bundle \(Z' = T^* \mathbb{RP}^3\) of the group space \(\mathbb{RP}^3\) of \(SO(3)\). It is a trivial bundle \(Z' = \mathbb{RP}^3 \times g^*\) provided with the symplectic structure given by the non-degenerate Poisson bracket
\[
\{x_i, x_j\} = \epsilon^i_{jk} x_k, \quad \{\alpha^i, \alpha^j\} = 0, \quad \{x_i, \alpha^j\} = \delta^j_i,
\]
where \(\alpha^i\) are group parameters. Note that it is not the canonical symplectic structure on the cotangent bundle. Let us consider a four-dimensional submanifold \(Z \subset Z'\) of points which belong to the one-dimensional trajectories of a spherical top passing through the unit of \(SO(3)\). These trajectories are exactly the invariant submanifolds of the noncommutative CIS (16), and \(Z\) is the corresponding fibered manifold \(H : Z \to N = g^* \setminus \{0\}\). This
fibered manifold is not trivial. In particular, the restriction of \( Z \) to a coadjoint orbit \( r = \text{const of } N \) is a nontrivial fiber bundle \( SO(3) = \mathbb{RP}^3 \to SO(3)/SO(2) = S^2 \). Its restriction to a cycle \( S^1, r = \text{const, } x_1 = \text{const} \), is isomorphic to the trivial bundle \( T^2 \to S^1 \). However, the parameter \( \alpha \) along the flows of the Hamiltonian vector field \( \vartheta_r \) need not perform such a trivialization. Therefore, the action-angle coordinate chart \( (r, x_1, \gamma, \alpha) \) is defined on an open neighborhood \( U_M = N_M \times S^1 \) of an invariant submanifold \( M \) where \( N_M \) is an open contractible neighborhood of \( H(M) \) diffeomorphic to \( \mathbb{R}^3 \).

A familiar quantization of a spherical top in fact reduces to a linear representation of the Lie algebra \( so(3) \) by differential operators \( \{ \hat{H}_1, \hat{H}_2, \hat{H}_3 \} \) in the space of smooth complex functions on a sphere \( S^2 \). In comparison with this quantization, the operators (22) provide a representation of the algebra of canonical commutation relations (21) (but not the Lie algebra \( so(3) \)) in the space of smooth complex functions of compact support on \( \mathbb{R} \times S^1 \).

References