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Global action-angle coordinates for completely integrable systems with noncompact invariant submanifolds

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The obstruction to the existence of global action-angle coordinates of Abelian and noncommutative (non-Abelian) completely integrable systems with compact invariant submanifolds has been studied. We extend this analysis to the case of noncompact invariant submanifolds. © *2007 American Institute of Physics.*
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I. INTRODUCTION

We consider Abelian and noncommutative (non-Abelian) completely integrable Hamiltonian systems (henceforth CISs) on symplectic manifolds. The Liouville-Arnold (or Liouville-Mineur-Arnold) theorem for Abelian CISs (Refs. 1–4) and the Mishchenko-Fomenko theorem for noncommutative ones[5–8] state the existence of action-angle coordinates around a compact invariant submanifold of a CIS. These theorems have been extended to the case of noncompact invariant submanifolds.[9–12] In particular, this is the case of time-dependent CISs.[13,14] Any time-dependent CIS of m degrees of freedom can be represented as the autonomous one of $m+1$ degrees of freedom on a homogeneous momentum phase space, where time is a generalized angle coordinate. Therefore, we further consider only autonomous CISs.

If invariant submanifolds of a CIS are compact, a topological obstruction to the existence of global action-angle coordinates has been analyzed.[8,15,16] Here, we aim to extend this analysis to the case of noncompact invariant submanifolds (Theorems 3–5).

Throughout the paper, all functions and maps are smooth, and symplectic manifolds are real smooth and paracompact. We are not concerned with the real-analytic case because a paracompact real-analytic manifold admits the partition of unity by smooth, not analytic, functions. As a consequence, sheaves of modules over real-analytic functions need not be acyclic, which is essential for our consideration.

Definition 1: Let (Z,Ω) be a $2n$-dimensional connected symplectic manifold, and let $(C^\infty(Z),\{,\})$ be the Poisson algebra of smooth real functions on Z. A subset $H=(H_1,\ldots,H_k)$, $n \leq k < 2n$, of $C^\infty(Z)$ is called a (noncommutative) CIS if the following conditions hold:

(i) All the functions H_i are independent, i.e., the k-form $\overset{k}{\wedge} dH_i$ nowhere vanishes on Z. It follows that the map $H:Z\to\mathbb{R}^k$ is a submersion, i.e.,

$$H:Z \to N = H(Z) \tag{1}$$

is a fibered manifold over a connected open subset $N\subset\mathbb{R}^k$.

(ii) There exist smooth real functions s_{ij} on N such that

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$$\{H_i, H_j\} = s_{ij} \circ H, \quad i,j = 1, \ldots, k. \tag{2}$$

(iii) The matrix function $\mathbf{s}$ with the entries s_{ij} (2) is of constant corank $m=2n-k$ at all points of N.

In Hamiltonian mechanics, one can think of the functions H_i as being integrals of motion of a CIS which are in involution with its Hamiltonian. Their level surfaces (fibers of H) are invariant submanifolds of a CIS.

If $k=n$, then $\mathbf{s}=0$, and we are in the case of an Abelian CIS. If $k>n$, the matrix $\mathbf{s}$ is necessarily nonzero, and a CIS is said to be noncommutative.

Note that, in many physical models, the condition (i) of Definition 1 fails to hold. In a general setting, one supposes that the subset $Z_R \subset Z$ of regular points, where $\overset{k}{\wedge} dH_i \neq 0$, is open and dense. Then one considers a CIS on this subset. However, a CIS on Z_R fails to be equivalent to the original one because there is no morphism of Poisson algebras $C^\infty(Z_R) \to C^\infty(Z)$. In particular, canonical quantization of the Poisson algebra $C^\infty(Z_R)$, e.g., with respect to action-angle variables, essentially differs from that of $C^\infty(Z)$.[17–19] For instance, let M be a connected compact invariant manifold of an Abelian CIS through a regular point $z \in Z_R \subset Z$. There exists its open saturated neighborhood $U_M \subset Z_R$ (i.e., a fiber of H through a point of U_M belongs to U_M), which is a trivial fiber bundle in tori. By virtue of the above mentioned Liouville-Arnold theorem, U_M is provided with the Darboux action-angle coordinates. Then one treats quantization of the Poisson algebra $C^\infty(U_M)$ with respect to these coordinates as quantization "around" an invariant submanifold M.

Given a CIS in accordance with Definition 1, the above mentioned generalization of the Mishchenko-Fomenko theorem to noncompact invariant submanifolds states the following:[12]

Theorem 1: Let the Hamiltonian vector fields ϑ_i of the functions H_i be complete, and let the fibers of the fibered manifold H (1) be connected and mutually diffeomorphic. Then the following hold:

(I) The fibers of H (1) are diffeomorphic to a toroidal cylinder

$$\mathbb{R}^{m-r} \times T^r. \tag{3}$$

(II) Given a fiber M of H (1), there exists an open saturated neighborhood U_M of it which is a trivial principal bundle with the structure group (3).

(III) The neighborhood U_M is provided with the bundle (generalized action-angle) coordinates $(I_\lambda, p_A, q^A, y'^\lambda)$, $\lambda=1, \ldots, m$, $A=1, \ldots, n-m$, where (y'^λ) are coordinates on a toroidal cylinder, such that the symplectic form Ω on U_M reads

$$\Omega = \mathrm{d}I_\lambda \wedge \mathrm{d}y'^\lambda + \mathrm{d}p_A \wedge \mathrm{d}q^A,$$

and a Hamiltonian of a CIS is a smooth function only of the action coordinates I_λ.

Theorem 1 restarts the Mishchenko-Fomenko one if its condition is replaced with one in which the fibers of the fibered manifold H (1) are compact and connected.

The proof of Theorem 1 is based on the following facts.[8,12] Any function constant on fibers of the fibration H (1) is the pullback of some function on its base N. Due to item (ii) of Definition 1, the Poisson bracket $\{f, f'\}$ of any two functions $f, f' \in C^\infty(Z)$ constant on fibers of H is also of this type. Consequently, the base N of H is provided with a unique coinduced Poisson structure $\{,\}_N$ such that H is a Poisson morphism.[20] By virtue of condition (iii) of Definition 1, the rank of this coinduced Poisson structure equals $2(n-m) = 2\dim N - \dim Z$. Furthermore, one can show the following.[8,21]

Lemma 2: The fibers of the fibration H (1) are maximal integral manifolds of the involutive distribution spanned by the Hamiltonian vector fields of the pullback H^*C of Casimir functions C of the coinduced Poisson structure on N.

In particular, a Hamiltonian of a CIS is the pullback onto Z of some Casimir function of the coinduced Poisson structure on N.

It follows from Lemma 2 that invariant submanifolds of a noncommutative CIS are maximal integral manifolds of a certain Abelian partially integrable system (henceforth PIS).

Definition 2: A collection $\{S_1, \dots, S_m\}$ of $m \leqslant n$ independent smooth real functions in involution on a $2n$-dimensional symplectic manifold (Z, Ω) is called a PIS.

Let us consider the map

$$S:Z \to W \subset \mathbb{R}^m. \tag{4}$$

Since functions S_λ are everywhere independent, this map is a submersion onto an open subset $W \subset R^m$, i.e., S [Eq. (4)] is a fibered manifold of fiber dimension $2n-m$. Hamiltonian vector fields υ_λ of functions S_λ are mutually commutative and independent. Consequently, they span an m-dimensional involutive distribution on Z whose maximal integral manifolds constitute a foliation $\mathcal{F}$ of Z. Because functions S_λ are constant on leaves of this foliation, each fiber of a fibered manifold $Z \to W$ [Eq. (4)] is foliated by the leaves of the foliation $\mathcal{F}$. If $m=n$, we are in the case of an Abelian CIS, and the leaves of $\mathcal{F}$ are connected components of fibers of the fibered manifold (4). The Poincaré-Lyapounov-Nekhoroshev theorem[22–24] generalizes the Liouville-Arnold one to a PIS if leaves of the foliation $\mathcal{F}$ are compact. It imposes a sufficient condition which Hamiltonian vector fields υ_λ must satisfy in order that the foliation $\mathcal{F}$ is a fibered manifold.[24,25] Extending the Poincaré-Lyapounov-Nekhoroshev theorem to the case of noncompact integral submanifolds, we in fact assumed from the beginning that these submanifolds formed a fibration.[11,14,18] Here, we aim to prove the following global variant of Theorem 6 in Ref. 11.

Theorem 3: Let a PIS $\{S_1, \dots, S_m\}$ on a symplectic manifold (Z, Ω) satisfy the following conditions:

(i) The Hamiltonian vector fields υ_λ of S_λ are complete.
(ii) The foliation $\mathcal{F}$ is a fiber bundle $\mathcal{F}:Z \to N$.
(iii) Its base N is simply connected.
(iv) The cohomology $H^2(N, \mathbb{Z})$ of N with coefficients in the constant sheaf $\mathbb{Z}$ is trivial.

Then the following hold:

(I) The fiber bundle $\mathcal{F}$ is a trivial principal bundle with the structure group (3), and we have a composite fibered manifold

$$S = \zeta \circ \mathcal{F}:Z \to N \to W, \tag{5}$$

where $N \to W$, however, need not be a fiber bundle.

(II) The fibered manifold (5) is provided with the adapted fibered (generalized action-angle) coordinates

$$(I_\lambda, x^A, y'^\lambda) \to (I_\lambda, x^A) \to (I_\lambda), \quad \lambda = 1, \dots, m, \quad A = 1, \dots, 2(n-m),$$

such that the coordinates (I_λ) possess identity transition functions, and the symplectic form Ω reads

$$\Omega = \mathrm{d}I_\lambda \wedge \mathrm{d}y'^\lambda + \Omega_A^\lambda \mathrm{d}I_\lambda \wedge \mathrm{d}x^A + \Omega_{AB}\mathrm{d}x^A \wedge \mathrm{d}x^B. \tag{6}$$

If one supposes from the beginning that leaves of the foliation $\mathcal{F}$ are compact, condition (i) of Theorem 3 always holds, and assumption (ii) can be replaced with the requirement that $\mathcal{F}$ is a fibered manifold with mutually diffeomorphic connected fibers. Recall that any fibered manifold whose fibers are diffeomorphic either to $\mathbb{R}^r$ or a compact connected manifold K is a fiber bundle.[26] However, a fibered manifold whose fibers are diffeomorphic to a product $\mathbb{R}^r \times K$ (e.g., a toroidal cylinder) need not be a fiber bundle (see Ref. 27, Example 1.2.2).

Theorem 3 is proved in Sec. II. Since m-dimensional fibers of the fiber bundle $\mathcal{F}$ admit m complete independent vector fields, they are locally affine manifolds diffeomorphic to a toroidal cylinder (3). Then condition (iii) of Theorem 3 guarantees that the fiber bundle $\mathcal{F}$ is a principal bundle with the structure group (3). Furthermore, this principal bundle is trivial due to condition (iv), and it is provided with the bundle action-angle coordinates. Note that conditions (ii) and (iii) of Theorem 3 are sufficient, but not necessary.

If $m=n$, the following corollary of Theorem 3 states the existence of global action-angle coordinates of an Abelian CIS.

Theorem 4: Let an Abelian CIS $\{H_1,\ldots,H_n\}$ on a symplectic manifold (Z,Ω) satisfy the following conditions:

(i) The Hamiltonian vector fields ϑ_i of H_i are complete.
(ii) The fibered manifold H (1) is a fiber bundle with connected fibers over a simply connected base N whose cohomology $H^2(N,\mathbb{Z})$ is trivial.

Then the following hold:

(I) The fiber bundle H (1) is a trivial principal bundle with the structure group (3).
(II) The symplectic manifold Z is provided with the global Darboux coordinates (I_λ,y'^λ) such that $\Omega=\mathrm{d}I_\lambda\wedge\mathrm{d}y'^\lambda$.

Due to Lemma 2, a manifested global generalization of Theorem 1 is a corollary of Theorem 3 (see Sec. III).

Theorem 5: Given a noncommutative CIS in accordance with Definition 1, let us assume the following:

(i) Hamiltonian vector fields ϑ_i of integrals of motion H_i are complete.
(ii) The fibration H (1) is a fiber bundle with connected fibers.
(iii) Let V be an open subset of the base N of this fiber bundle which admits m independent Casimir functions of the coinduced Poisson structure on N.
(iv) Let V be simply connected, and let the cohomology $H^2(V,\mathbb{Z})$ be trivial.

Then the following hold:

(I) The fibers of H (1) are diffeomorphic to a toroidal cylinder (3).
(II) The restriction Z_V of the fiber bundle H (1) to V is a trivial principal bundle with the structure group (3).
(III) The fiber bundle Z_V is provided with the bundle (generalized action-angle) coordinates $(I_\lambda,x^A,y'^\lambda)$ such that the action-angle coordinates (I_λ,y'^λ) possess identity transition functions and the symplectic form Ω on Z_V reads

$$\Omega=\mathrm{d}I_\lambda\wedge\mathrm{d}y'^\lambda+\Omega_{AB}\mathrm{d}x^A\wedge\mathrm{d}x^B. \tag{7}$$

Note that if invariant submanifolds of a CIS are assumed to be connected and compact, condition (i) of Theorem 5 is unnecessary since vector fields v_λ on compact fibers of H are complete. In this case, condition (ii) of Theorem 5 also holds because, as was mentioned above, a fibered manifold with compact mutually diffeomorphic fibers is a fiber bundle.

In the case of an Abelian CIS, the coinduced Poisson structure on N equals zero, the integrals of motion H_λ are the pullback of n independent functions on N, and Theorem 5 reduces to Theorem 4.

Following the original Mishchenko-Fomenko theorem, let us mention noncommutative CISs whose integrals of motion $\{H_1,\ldots,H_k\}$ form a k-dimensional real Lie algebra $\mathcal{G}$ of rank m with the commutation relations

$$\{H_i,H_j\}=c_{ij}^hH_h,\qquad c_{ij}^h=\mathrm{const}.$$

In this case, complete Hamiltonian vector fields ϑ_i of H_i define a locally free Hamiltonian action on Z of some simply connected Lie group G whose Lie algebra is isomorphic to $\mathcal{G}$.[28,29] Orbits of G coincide with k-dimensional maximal integral manifolds of the regular distribution on Z spanned by Hamiltonian vector fields ϑ_i.[30] Furthermore, one can treat H (1) as an equivariant momentum mapping of Z to the Lie coalgebra $\mathcal{G}^*$, provided with the coordinates $x_i(H(z))=H_i(z)$, $z\in Z$.[18,31] In this case, the coinduced Poisson structure $\{,\}_N$ coincides with the canonical Lie-Poisson structure on $\mathcal{G}^*$ given by the Poisson bivector field

$$w=\frac{1}{2}c_{ij}^h x_h \partial^i \wedge \partial^j.$$

Casimir functions of the Lie-Poisson structure are exactly the coadjoint invariant functions on $\mathcal{G}^*$. They are constant on orbits of the coadjoint action of G on $\mathcal{G}^*$, which coincide with leaves of the symplectic foliation of $\mathcal{G}^*$. Let V be an open subset of $\mathcal{G}^*$, which obeys conditions (iii) and (iv) of Theorem 5. Then the open subset $H^{-1}(V)\subset Z$ is provided with the action-angle coordinates.

II. PROOF OF THEOREM 3

In accordance with the well-known theorem,[28,29] complete Hamiltonian vector fields υ_λ define an action of a simply connected Lie group on Z. Because vector fields υ_λ are mutually commutative, it is the additive group $\mathbb{R}^m$ whose group space is coordinated by parameters s^λ with respect to the basis $\{e_\lambda=\upsilon_\lambda\}$ for its Lie algebra. The orbits of the group $\mathbb{R}^m$ in Z coincide with the fibers of the fiber bundle

$$\mathcal{F}:Z\to N. \tag{8}$$

Since vector fields υ_λ are independent of Z, the action of $\mathbb{R}^m$ on Z is locally free, i.e., isotropy groups of points of Z are discrete subgroups of the group $\mathbb{R}^m$. Given a point $x\in N$, the action of $\mathbb{R}^m$ on a fiber $M_x=\mathcal{F}^{-1}(x)$ factorizes as

$$\mathbb{R}^m\times M_x\to G_x\times M_x\to M_x \tag{9}$$

through the free transitive action of the factor group $G_x=\mathbb{R}^m/K_x$, where K_x is the isotropy group of an arbitrary point of M_x. It is the same group for all points of M_x because $\mathbb{R}^m$ is a commutative group. Since the fibers M_x are mutually diffeomorphic, all isotropy groups K_x are isomorphic to the group $\mathbb{Z}_r$ for some fixed $0\leqslant r\leqslant m$. Accordingly, the groups G_x are isomorphic to the Abelian group

$$G=\mathbb{R}^{m-r}\times T^r, \tag{10}$$

and fibers of the fiber bundle (8) are diffeomorphic to the toroidal cylinder (10).

Let us bring the fiber bundle (8) into a principal bundle with the structure group (10). Generators of each isotropy subgroup K_x of $\mathbb{R}^m$ are given by r linearly independent vectors $u_i(x)$ of the group space $\mathbb{R}^m$. These vectors are assembled into an r-fold covering $K\to N$. This is a subbundle of the trivial bundle

$$N\times R^m\to N, \tag{11}$$

whose local sections are local smooth sections of the fiber bundle (11). Such a section over an open neighborhood of a point $x\in N$ is given by a unique local solution $s^\lambda(x')e_\lambda$ of the equation

$$g(s^\lambda)\sigma(x')=\exp(s^\lambda\upsilon_\lambda)\sigma(x')=\sigma(x'),\quad s^\lambda(x)e_\lambda=u_i(x),$$

where σ is an arbitrary local section of the fiber bundle $Z\to N$ over an open neighborhood of x. Since N is simply connected, the covering $K\to N$ admits r everywhere different global sections u_i, which are global smooth sections $u_i(x)=u_i^\lambda(x)e_\lambda$ of the fiber bundle (11). Let us fix a point of N further denoted by $\{0\}$. One can determine linear combinations of the functions S_λ, say again S_λ, such that $u_i(0)=e_i$, $i=m-r,\ldots,m$, and the group G_0 is identified to the group G (10). Let E_x denote the r-dimensional subspace of $\mathbb{R}^m$ passing through the points $u_1(x),\ldots,u_r(x)$. The spaces E_x, $x\in N$, constitute an r-dimensional subbundle $E\to N$ of the trivial bundle (11). Moreover, the latter is split into the Whitney sum of vector bundles $E\oplus E'$, where $E'_x=\mathbb{R}^m/E_x$.[32] Then there is a global smooth section γ of the trivial principal bundle $N\times GL(m,\mathbb{R})\to N$ such that $\gamma(x)$ is a morphism of E_0 onto E_x, where $u_i(x)=\gamma(x)(e_i)=\gamma_i^\lambda e_\lambda$. This morphism is also an automorphism of the group $\mathbb{R}^m$ sending K_0 onto K_x. Therefore, it provides a group isomorphism $\rho_x: G_0\to G_x$. With these isomorphisms, one can define the fiberwise action of the group G_0 on Z given by the law

$$G_0\times M_x \to \rho_x(G_0)\times M_x\to M_x. \tag{12}$$

Namely, let an element of the group G_0 be the coset $g(s^\lambda)/K_0$ of an element $g(s^\lambda)$ of the group $\mathbb{R}^m$. Then it acts on M_x by the rule (12) just as the coset $g((\gamma(x)^{-1})^\lambda_\beta s^\beta)/K_x$ of an element $g((\gamma(x)^{-1})^\lambda_\beta s^\beta)$ of $\mathbb{R}^m$ does. Since entries of the matrix γ are smooth functions on N, the action (12) of the group G_0 on Z is smooth. It is free, and $Z/G_0=N$. Thus, $Z\to N$ (8) is a principal bundle with the structure group $G_0=G$ (10).

Furthermore, this principal bundle over a paracompact smooth manifold N is trivial as follows. In accordance with the well-known theorem,[32] its structure group G (10) is reducible to the maximal compact subgroup T^r, which is also the maximal compact subgroup of the group product $\overset{r}{\times} GL(1,\mathbb{C})$. Therefore, the equivalence classes of T^r-principal bundles ξ are defined as

$$c(\xi)=c(\xi_1\oplus\cdots\oplus\xi_r)=(1+c_1(\xi_1))\cdots(1+c_1(\xi_r))$$

by the Chern classes $c_1(\xi_i)\in H^2(N,\mathbb{Z})$ of $U(1)$-principal bundles ξ_i over N.[32] Since the cohomology group $H^2(N,\mathbb{Z})$ of N is trivial, all Chern classes c_1 are trivial and the principal bundle $Z\to N$ is also trivial. This principal bundle can be provided with the following coordinate atlas.

Let us consider the fibered manifold $S: Z\to W$ (4). Because functions S_λ are constant on fibers of the fiber bundle $Z\to N$ (8), the fibered manifold (4) factorizes through the fiber bundle (8), and we have the composite fibered manifold (5). Let us provide the principal bundle $Z\to N$ with a trivialization

$$Z=N\times\mathbb{R}^{m-r}\times T^r\to N, \tag{13}$$

whose fibers are endowed with the standard coordinates $(y^\lambda)=(t^a,\varphi^i)$ on the toroidal cylinder (10). Then the composite fibered manifold (5) is provided with the fibered coordinates

$$(J_\lambda,x^A,t^a,\varphi^i),\quad \lambda=1,\ldots,m,\quad A=1,\ldots,2(n-m),\quad a=1,\ldots,m-r,\quad i=1,\ldots,r, \tag{14}$$

where $J_\lambda=S_\lambda(z)$ are coordinates on the base W induced by Cartesian coordinates on $\mathbb{R}^m$, and (J_λ,x^A) are fibered coordinates on the fibered manifold $\zeta: N\to W$. The coordinates J_λ on $W\subset\mathbb{R}^m$ and the coordinates (t^a,φ^i) on the trivial bundle (13) possess the identity transition functions, while the transition function of coordinates (x^A) depends on the coordinates (J_λ) in general.

The Hamiltonian vector fields υ_λ on Z relative to the coordinates (14) take the form

$$\upsilon_\lambda=\upsilon^a_\lambda(x)\partial_a+\upsilon^i_\lambda(x)\partial_i. \tag{15}$$

Since these vector fields commute (i.e., fibers of $Z\to N$ are isotropic), the symplectic form Ω on Z reads

$$\Omega=\Omega^\alpha_\beta dJ_\alpha\wedge dy^\beta+\Omega_{\alpha A}dy^\alpha\wedge dx^A+\Omega^{\alpha\beta}dJ_\alpha\wedge dJ_\beta+\Omega^\alpha_A dJ_\alpha\wedge dx^A+\Omega_{AB}dx^A\wedge dx^B. \tag{16}$$

Lemma 6: The symplectic form Ω (16) is exact.

Proof: In accordance with the well-known Künneth formula, the de Rham cohomology group of the product (13) reads

$$H^2(Z)=H^2(N)\oplus H^1(N)\otimes H^1(T^r)\oplus H^2(T^r).$$

By the de Rham theorem,[32] the de Rham cohomology $H^2(N)$ is isomorphic to the cohomology $H^2(N,\mathbb{R})$ of N with coefficients in the constant sheaf $\mathbb{R}$. It is trivial since $H^2(N,\mathbb{R})=H^2(N,\mathbb{Z})\otimes\mathbb{R}$, where $H^2(N,\mathbb{Z})$ is trivial. The first cohomology group $H^1(N)$ of N is trivial because N is simply connected. Consequently, $H^2(Z)=H^2(T^r)$. Then the closed form Ω (16) is exact since it does not contain the term $\Omega_{ij}\mathrm{d}\varphi^i\wedge\mathrm{d}\varphi^j$.

Thus, we can write

$$\Omega = \mathrm{d}\Xi, \quad \Xi = \Xi^\lambda(J_\alpha,x^B,y^\alpha)\mathrm{d}J_\lambda + \Xi_\lambda(J_\alpha,x^B)\mathrm{d}y^\lambda + \Xi_A(J_\alpha,x^B,y^\alpha)\mathrm{d}x^A. \tag{17}$$

Up to an exact summand, the Liouville form Ξ (17) is brought into the form

$$\Xi = \Xi^\lambda(J_\alpha,x^B,y^\alpha)\mathrm{d}J_\lambda + \Xi_i(J_\alpha,x^B)\mathrm{d}\varphi^i + \Xi_A(J_\alpha,x^B,y^\alpha)\mathrm{d}x^A, \tag{18}$$

i.e., it does not contain the term $\Xi_a\mathrm{d}t^a$.

The Hamiltonian vector fields v_λ (15) obey the relations $v_\lambda\rfloor\Omega=-\mathrm{d}J_\lambda$, which result in the coordinate conditions

$$\Omega^\alpha_\beta\vartheta^\beta_\lambda = \delta^\alpha_\lambda, \quad \Omega_{A\beta}\vartheta^\beta_\lambda = 0. \tag{19}$$

The first of them shows that Ω^α_β is a nondegenerate matrix independent of coordinates y^λ. Then the second one implies $\Omega_{A\beta}=0$.

Since $\Xi_a=0$ and Ξ_i are independent of φ^i, it follows from the relations

$$\Omega_{A\beta} = \partial_A\Xi_\beta - \partial_\beta\Xi_A = 0$$

that Ξ_A are independent of coordinates t^a and at most affine in φ^i. Since φ^i are cyclic coordinates, Ξ_A are independent of φ^i. Hence, Ξ_i are independent of coordinates x^A, and the Liouville form Ξ (18) reads

$$\Xi = \Xi^\lambda(J_\alpha,x^B,y^\alpha)\mathrm{d}J_\lambda + \Xi_i(J_\alpha)\mathrm{d}\varphi^i + \Xi_A(J_\alpha,x^B)\mathrm{d}x^A. \tag{20}$$

Because entries Ω^α_β of $\mathrm{d}\Xi=\Omega$ are independent of y^λ, we obtain the following:

(i) $\Omega^\lambda_i=\partial^\lambda\Xi_i-\partial_i\Xi^\lambda$. Consequently, $\partial_i\Xi^\lambda$ are independent of φ^i, and so are Ξ^λ since φ^i are cyclic coordinates. Hence, $\Omega^\lambda_i=\partial^\lambda\Xi_i$ and $\partial_i\rfloor\Omega=-\mathrm{d}\Xi_i$. A glance at the last equality shows that ∂_i are Hamiltonian vector fields. It follows that, from the beginning, one can separate r integrals of motion, say H_i again, whose Hamiltonian vector fields are tangent to invariant tori. In this case, the Hamiltonian vector fields v_λ (15) read

$$v_a = \partial_a, \quad v_i = v^k_i(x)\partial_k, \tag{21}$$

where the matrix function $v^k_i(x)$ is nondegenerate. Moreover, the coordinates t^a are exactly the flow parameters s^a. Substituting expressions (21) into the first condition (19), we obtain

$$\Omega = \mathrm{d}J_a\wedge\mathrm{d}s^a + (v^{-1})^i_k\mathrm{d}J_i\wedge\mathrm{d}\varphi^k + \Omega^{\alpha\beta}\mathrm{d}J_\alpha\wedge\mathrm{d}J_\beta + \Omega^\lambda_A\mathrm{d}J_\lambda\wedge\mathrm{d}x^A + \Omega_{AB}\mathrm{d}x^A\wedge\mathrm{d}x^B.$$

It follows that Ξ_i are independent of J_a, and so are $(v^{-1})^k_i=\partial^k\Xi_i$.

(ii) $\Omega^\lambda_a=-\partial_a\Xi^\lambda=\delta^\lambda_a$. Hence, $\Xi^a=-t^a+E^a(J_\lambda,x^B)$ and Ξ^i are independent of t^a.

In view of items (i) and (ii), the Liouville form Ξ (20) reads

$$\Xi = (-t^a + E^a(J_\lambda,x^B))\mathrm{d}J_a + E^i(J_\lambda,x^B)\mathrm{d}J_i + \Xi_i(J_j)\mathrm{d}\varphi^i + \Xi_A(J_\lambda,x^B)\mathrm{d}x^A.$$

Since the matrix $\partial^k\Xi_i$ is nondegenerate, we can perform the coordinate transformations $I_a=J_a$, $I_i=\Xi_i(J_j)$ together with the coordinate transformations

$$t'^a = -t^a + E^a(J_\lambda,x^B), \quad \varphi'^i = \varphi^i - E^j(J_\lambda,x^B)\frac{\partial J_j}{\partial I_i}. \tag{22}$$

These transformations bring Ω into the form (6).

III. PROOF OF THEOREM 5

Given a fibration H (1), let V be an open subset of its base N which satisfies condition (iii) of Theorem 5, i.e., there is a set $\{C_1, \ldots, C_m\}$ of m independent Casimir functions of the coinduced Poisson structure $\{,\}_N$ on V. Note that such functions always exist around any point of N. Let Z_V be the restriction of the fiber bundle $Z \to N$ onto $V \subset Z$. By virtue of Lemma 2, $Z_V \to V$ is a fibration in invariant submanifolds of a PIS $\{H^* C_\lambda\}$, where $H^* C_\lambda$ are the pullback of the Casimir functions C_λ onto Z_V.

Let v_λ be Hamiltonian vector fields of functions $H^* C_\lambda$. Since

$$H^* C_\lambda(z) = (C_\lambda \circ H)(z) = C_\lambda(H_i(z)), \quad z \in Z_V, \tag{23}$$

the Hamiltonian vector fields v_λ restricted to any fiber M of Z_V are linear combinations of the Hamiltonian vector fields ϑ_i of integrals of motion H_i. It follows that v_λ are elements of a finite-dimensional real Lie algebra of vector fields on M generated by the vector fields ϑ_i. Since vector fields ϑ_i are complete, the vector fields v_λ on M are also complete.[28] Consequently, the Hamiltonian vector fields v_λ are complete on Z_V. Then the conditions of Theorem 3 for a PIS $\{H^* C_\lambda\}$ on the symplectic manifold (Z_V, Ω) hold.

In accordance with Theorem 3, we have a composite fibered manifold

$$Z_V \xrightarrow{H} V \xrightarrow{C} W, \tag{24}$$

where $C: V \to W$ is a fibered manifold of level surfaces of the Casimir functions C_λ. The fibered manifold (24) is provided with the adapted fibered coordinates $(J_\lambda, x^A, y^\lambda)$ (14), where J_λ are values of the Casimir functions and $(y^\lambda) = (t^a, \varphi^i)$ are coordinates on a toroidal cylinder. Since $C_\lambda = J_\lambda$ are Casimir functions on V, the symplectic form Ω (16) on Z_V reads

$$\Omega = \Omega^\alpha_\beta \mathrm{d}J_\alpha \wedge \mathrm{d}y^\beta + \Omega_{\alpha A} \mathrm{d}y^\alpha \wedge \mathrm{d}x^A + \Omega_{AB} \mathrm{d}x^A \wedge \mathrm{d}x^B. \tag{25}$$

In particular, it follows that transition functions of coordinates x^A on V are independent of coordinates J_λ, i.e., $C: V \to W$ is a trivial bundle.

By virtue of Lemma 6, the symplectic form (25) is exact, i.e., $\Omega = \mathrm{d}\Xi$, where the Liouville form Ξ (20) is

$$\Xi = \Xi^\lambda(J_\alpha, y^\lambda)\mathrm{d}J_\lambda + \Xi_i(J_\alpha)\mathrm{d}\varphi^i + \Xi_A(x^B)\mathrm{d}x^A.$$

It is brought into the form

$$\Xi = (-t^a + E^a(J_\lambda))\mathrm{d}J_a + E^i(J_\lambda)\mathrm{d}J_i + \Xi_i(J_j)\mathrm{d}\varphi^i + \Xi_A(x^B)\mathrm{d}x^A.$$

Then the coordinate transformations (22)

$$I_a = J_a, \quad I_i = \Xi_i(J_j),$$

$$t'^a = -t^a + E^a(J_\lambda), \quad \varphi'^i = \varphi^i - E^j(J_\lambda)\frac{\partial J_j}{\partial I_i}, \tag{26}$$

bring Ω (25) into form (7). In comparison with the general case (22), the coordinate transformations (26) are independent of coordinates x^A. Therefore, the angle coordinates φ'^i possess identity transition functions on V.

Theorem 5 restarts Theorem 1 if one considers an open subset U of V admitting the Darboux coordinates x^A on the symplectic leaves of U.

The proof of Theorem 5 gives something more. Let $\mathcal{H}$ be a Hamiltonian of a CIS. It is the pullback onto Z_V of some Casimir function on V. Since (I_λ, x^A) are coordinates on V, they are also integrals of motion of $\mathcal{H}$. Though the original integrals of motion H_i are smooth functions of coordinates (I_λ, x^A), the Casimir functions (23)

$$C_\lambda(H_i(I_\mu, x^A)) = C_\lambda(I_\mu)$$

and, in particular, a Hamiltonian $\mathcal{H}$ depend only on the action coordinates I_λ. Hence, the equations of motion of a CIS take the form

$$\dot{y}'^\lambda = \frac{\partial \mathcal{H}}{\partial I_\lambda}, \quad I_\lambda = \text{const.}, \quad x^A = \text{const.}$$

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