SPACES OF POLYTOPES AND COBORDISM OF QUASITORIC MANIFOLDS

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*To Askold Khovanskii, a brilliant mathematician and pioneer of toric geometry,
on the occasion of his 60th birthday*

ABSTRACT. Our aim is to bring the theory of analogous polytopes to
bear on the study of quasitoric manifolds, in the context of stably com-
plex manifolds with compatible torus action. By way of application,
we give an explicit construction of a quasitoric representative for every
complex cobordism class as the quotient of a free torus action on a real
quadratic complete intersection. We suggest a systematic description
for omnioriented quasitoric manifolds in terms of combinatorial data,
and explain the relationship with non-singular projective toric varieties
(otherwise known as toric manifolds). By expressing the first and third
authors' approach to the representability of cobordism classes in these
terms, we simplify and correct two of their original proofs concerning
quotient polytopes; the first relates to framed embeddings in the posi-
tive cone, and the second involves modifying the operation of connected
sum to take account of orientations. Analogous polytopes provide an
informative setting for several of the details.

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ted sum, framing, omniorientation, quasitoric manifold, stable tangent bundle.

1. INTRODUCTION

The theory of analogous polytopes was initiated by Alexandrov [1] in the 1930s,
and extended more recently by Khovanskii and Pukhlikov [13]. Our aim is to apply
this theory to the algebraic topology of torus actions.

Davis and Januszkiewicz [7] explain how to construct a $2n$-dimensional man-
ifold M from a characteristic pair (P, λ), where P is a simple convex polytope
of dimension n, and λ is a function with certain special properties which assigns

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a subcircle of the torus T^n to each facet of P. By construction M admits a locally standard T^n action, whose quotient space is homeomorphic to P. Davis and Januszkiewicz describe such manifolds as *toric*; more recently, the term *quasitoric* has been adopted, to avoid confusion with the non-singular compact toric varieties of algebraic geometry. We follow this convention below, and refer to such M as *quasitoric manifolds*.

Every simple polytope P is equivalent to an arrangement $\mathcal{H}$ of m closed half-spaces in an n-dimensional vector space V, whose bounding hyperplanes meet only in general position. The intersection of the half-spaces is assumed to be bounded, and defines P. The $(n-1)$-dimensional faces F_j are the facets of P, where $1 \leqslant j \leqslant m$, and general position ensures that any face of codimension k is the intersection of precisely k facets. In particular, every vertex is the intersection of n facets, and lies in an open neighbourhood isomorphic to the positive cone $\mathbb{R}_{\geqslant}^n$. For any characteristic pair (P, λ), it is possible to vary P within its combinatorial equivalence class without affecting the equivariant diffeomorphism type of the quasitoric manifold M.

For a fixed arrangement, we consider the vector $d_{\mathcal{H}}$ of signed distances from the origin O to the bounding hyperplanes in V; a coordinate is positive when O lies in the interior of the corresponding half-space, and negative in the complement. We then identify the m-dimensional vector space $\mathbb{R}^m$ with the space of arrangements *analogous to* $\mathcal{H}$. Under this identification, $d_{\mathcal{H}}$ corresponds to $\mathcal{H}$ itself, and every other vector corresponds to the arrangement obtained by the appropriate parallel displacement of half-spaces. For small displacements, the intersections of the half-spaces are polytopes similar to P. For larger displacements the intersections may be degenerate, or empty; in either case, they are known as *virtual polytopes*, and are *analogous to P*.

In [5], the first and third authors consider dicharacteristic pairs (P, ℓ), where λ is replaced by a homomorphism $\ell \colon T^m \to T^n$. This has the effect of orienting each of the subcircles $\lambda(F_j)$ of T^n, and leads to the construction of an omnioriented quasitoric manifold M; [5, Theorem 3.8] claims that a canonical stably complex structure may then be chosen for M. The proof, however, has two flaws. Firstly, it fails to provide a sufficiently detailed explanation of how a certain complexified neighbourhood of P may be framed, and secondly, it requires an orientation of M (and hence of P) for the stably complex structure to be uniquely defined. The latter issue has already been raised in [3, Section 5.3], but amended proofs have not appeared. One of our aims is to show that analogous polytopes offer a natural setting for several of the missing details.

The main application of [5, Theorem 3.8] is as follows.

Theorem 5.9 [5, Theorem 6.11]**.** *In dimensions > 2, every complex cobordism class contains a quasitoric manifold, necessarily connected, whose stably complex structure is induced by an omniorientation, and is therefore compatible with the action of the torus.*

This result builds upon a construction [4] of a special set of additive generators for the complex cobordism groups Ω_n^U, represented by quasitoric manifolds. The proof proceeds by considering $2n$-dimensional omnioriented quasitoric manifolds

M_1 and M_2, with quotient polytopes P_1 and P_2 respectively, and constructs a third such manifold M, which is complex cobordant to the connected sum $M_1 \# M_2$. For the quotient polytope of M, the authors use the connected sum $P_1 \# P_2$, over which the dicharacteristics naturally extend.

In the light of our preceding observations, we must amend this proof to incorporate the orientations of P_1 and P_2. However, it is not always possible to form $P_1 \# P_2$ in the oriented sense, and simultaneously extend the dicharacteristics. Instead, we replace M_2 with a complex cobordant quasitoric manifold M_2', whose quotient polytope is $I^n \# P_2$, where I^n denotes an appropriately oriented n-cube. It turns out that the resulting gain in geometrical freedom allows us to extend both the orientations and the dicharacteristics; the result is the omnioriented quasitoric manifold $M_1 \# M_2'$ over the polytope

$$P_1 \,\square\, P_2 = P_1 \# I^n \# P_2,$$

which we call the *box sum* of P_1 and P_2. We may then complete the proof of Theorem 5.9 as described in Section 5 below.

In dimension 2, $P_1 \,\square\, P_2$ is combinatorially equivalent to the Minkowski sum $P_1 + P_2$, which is central to the theory of analogous polytopes.

In [5], the authors compare Theorem 5.9 with a famous question of Hirzebruch, who asks for a description of those complex cobordism classes which may be represented by connected algebraic varieties. This is a difficult problem, and remains unsolved; nevertheless, our modification to the proof of Theorem 5.9 adds some value to the comparison, in the following sense.

Given complex cobordism classes $[N_1]$ and $[N_2]$ of the same dimension, suppose that N_1 and N_2 are connected. Then we may form the connected sum $N_1 \# N_2$ in the standard fashion, so that it is also a connected stably complex manifold, and represents $[N_1] + [N_2]$. If, on the other hand, N_1 and N_2 are algebraic varieties, then $N_1 \# N_2$ is not usually algebraic. In these circumstances we might proceed by analogy with the quasitoric case, and look for an alternative representative N_2' such that $N_1 \# N_2'$ is also algebraic.

We now outline the contents of each section, with additional comments where appropriate.

In Section 2 we recall various definitions and notation concerning simple convex polytopes with ordered facets. We introduce the space $\mathbb{R}(P)$ of polytopes analogous to a fixed example P, and consider a linear map $\chi_P \colon V \to \mathbb{R}(P)$, defined on the ambient space V of P. Under χ_P, a point $y \in V$ is mapped to that point of $\mathbb{R}(P)$ which represents the polytope congruent to P obtained by shifting the origin to y. We then interpret the projection from $\mathbb{R}(P)$ to the cokernel of χ_P as mapping a polytope $P' \in \mathbb{R}(P)$ to the vector of distances from a distinguished vertex $v_\star \in P'$ to its $m - n$ opposite facets. This allows us to describe the projection explicitly, as a map $C \colon \mathbb{R}(P) \to \mathbb{R}^{m-n}$.

In Section 3 we summarise the construction of a quasitoric manifold M over a polytope P with m facets [7]. In their work, Davis and Januszkiewicz use an auxiliary T^m-space $\mathcal{Z}_P$, whose quotient by the kernel of a dicharacteristic homomorphism they identify with M. It has become evident that the spaces $\mathcal{Z}_P$ are of great independent interest in toric topology, and they are now known as *moment-*

angle complexes [3]. They arise in homotopy theory as homotopy colimits [12], in symplectic topology as level surfaces for the moment maps of Hamiltonian torus actions [11], and in the theory of arrangements as complements of coordinate subspace arrangements [3, Chapter 8]. Using the matrix of the projection C, we describe $\mathcal{Z}_P$ as a complete intersection of real quadratic hypersurfaces.

In Section 4 we amend the definition of omniorientation so as to include an orientation of M, and recall the stably complex structure which results. In so doing, we frame $\mathcal{Z}_P$ equivariantly in $\mathbb{R}^{2m}$ and consider the quotient framing on P as a submanifold of the positive cone in the space of analogous polytopes.

We review the construction of connected sum for omnioriented quasitoric manifolds in Section 5, by encoding the additional orientations as signs attached to the fixed points. We then explain how to correct [5, Theorem 3.8], and recover Theorem 5.9. Combining the latter with our quadratic description of $\mathcal{Z}_P$ yields the following additional result on representability.

Theorem 5.10. *Every complex cobordism class may be represented by the quotient of a free torus action on a real quadratic complete intersection.*

The importance of the real quadratic viewpoint has been emphasised in recent work of Bosio and Meersseman [2], who consider a specific class of complete intersections of real quadrics in $\mathbb{C}^m$, called *links*. They show that all links (taking products with a circle in odd-dimensional cases) can be endowed with the structure of a non-Kähler complex manifold, generalising the class of Hopf and Calabi–Eckmann manifolds. It is clear from both the results of [2] and our own Section 3 that the class of links coincides with the class of moment-angle complexes $\mathcal{Z}_P$ arising from simple polytopes. This fact provides further connections between toric topology and complex geometry, in which calculations of cohomology rings of moment-angle complexes carried out in [3] feature prominently.

Finally, in Section 6, we discuss the realisation of 4-dimensional complex cobordism classes by omnioriented quasitoric manifolds, and comment on comparable situations in higher dimensions.

Throughout our work we adopt the combinatorial convention that $[n]$ denotes the set of integers $\{1, \ldots, n\}$, for any natural number n. Occasionally, it is convenient to interpret $[0]$ as the empty set. We write $2^{[n]}$ for the Boolean algebra of subsets of $[n]$, ordered by inclusion or its reverse as necessary.

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2. ANALOGOUS POLYTOPES

We work in a real vector space V of dimension n, equipped with a euclidean inner product $\langle \, , \, \rangle$ and an orthonormal basis $e_1, \ldots, e_n$. An *ordered arrangement*

$\mathcal{H}$ of *closed half-spaces* in V is a collection of subsets

$$H_i = \{x \in V : \langle a_i, x \rangle + b_i \geqslant 0\} \quad \text{for } 1 \leqslant i \leqslant m, \tag{2.1}$$

where a_i lies in V and b_i is a real scalar. Unless stated otherwise, we assume that $\mathcal{H}$ has cardinality $m \geqslant n$, and that a_i has unit length for every $1 \leqslant i \leqslant m$. We consider H_i to be a smooth manifold, whose boundary ∂H_i is its *bounding hyperplane*

$$Y_i = \{x \in V : \langle a_i, x \rangle + b_i = 0\} \quad \text{for } 1 \leqslant i \leqslant m, \tag{2.2}$$

with inward pointing normal vector a_i

When the intersection $\bigcap_i H_i$ is bounded, it forms a *convex polytope P*; otherwise, it is a *polyhedron*. We assume that P has maximal dimension n and that $\mathcal{H}$ is irredundant, in the sense that no H_i may be deleted without enlarging P. In these circumstances, $\mathcal{H}$ and P are interchangeable. We may also specify P by a matrix inequality $A_P x + b_P \geqslant 0$, where A_P is the $m \times n$ matrix of row vectors a_i, and b_P is the column vector of scalars b_i in $\mathbb{R}^m$. If we permute the half-spaces (2.1) by an element of the symmetric group Σ_m, we recover P by applying the same permutation to the rows of A_P and the coordinates of b_P.

Examples 2.3. The *standard n-simplex* $\Delta(n)$ is the polytope defined by the half-spaces

$$H_i = \begin{cases} \{x : \langle e_i, x \rangle \geqslant 0\} & \text{for } 1 \leqslant i \leqslant n, \\ \{x : \langle a_{n+1}, x \rangle + 1 \geqslant 0\} & \text{for } i = n + 1 \end{cases} \tag{2.4}$$

in $\mathbb{R}^n$, where $a_{n+1} = (-1, \ldots, -1)$; its vertices are the points $0, e_1, \ldots, e_n$. The *positive cone* $\mathbb{R}^n_{\geqslant}$ is the polyhedron obtained by deleting H_{n+1} from (2.4); it has a single vertex 0, and contains all vectors with non-negative coordinates.

Example 2.5. The *standard n-cube* I^n is the polytope defined by the half-spaces

$$H_i = \begin{cases} \{x : \langle e_i, x \rangle \geqslant 0\} & \text{for } 1 \leqslant i \leqslant n, \\ \{x : -\langle e_i, x \rangle + 1 \geqslant 0\} & \text{for } n + 1 \leqslant i \leqslant 2n \end{cases} \tag{2.6}$$

in $\mathbb{R}^n$; its vertices are the binary sequences $(\delta_1, \ldots, \delta_n)$, where $\delta_j = 0$ or 1 for $1 \leqslant j \leqslant n$.

A *supporting hyperplane* is characterised by the property that P lies within one of its two associated half-spaces. A *proper face* of P is defined by its intersection with any supporting hyperplane, and forms a convex polytope of lower dimension. We regard P as an n-dimensional face of itself; the faces of dimension 0, 1, and $n-1$ are known as *vertices*, *edges*, and *facets* respectively. There is one facet $F_i = P \cap Y_i$ for every bounding hyperplane (2.2), so the facets correspond bijectively to the half-spaces (2.1). We deem a vertex v and facet F_i to be *opposite* whenever v lies in the interior of H_i. If the bounding hyperplanes are in general position, then every vertex of P is the intersection of exactly n facets, and has $m - n$ opposite half-spaces. In these circumstances, P is *simple*.

From this point on, we deal only with simple polytopes, and reserve the notation $q = q(P)$ and $m = m(P)$ for the number of vertices and facets respectively. Every face of codimension k may be written uniquely as

$$F_I = F_{i_1} \cap \cdots \cap F_{i_k} \tag{2.7}$$

for some subset $I = \{i_1, \ldots, i_k\}$ of $[m]$, and the F_I may then be ordered lexicographically for each $1 \leqslant k \leqslant n$.

The faces are elements of the *face poset* $\mathcal{L}(P)$, ordered by reverse inclusion. We use the subscripts I to interpret $\mathcal{L}(P)$ as a subposet of the Boolean algebra $2^{[m]}$, ordered by inclusion; so $\mathcal{L}(P)$ is *ranked* [15, p. 99] by the codimension function $\mathrm{cod}(F_I) = |I|$. It has unique minimal element $F_\varnothing = P$, and its maximal elements are the vertices. It fails to be a lattice only because we usually omit the empty face, which would otherwise form a unique maximal element.

Two polytopes are *combinatorially equivalent* whenever their face posets are isomorphic. A combinatorial equivalence class of polytopes is known as a *combinatorial polytope*, and most of our constructions are defined on such classes. Nevertheless, it often helps to keep a representative polytope in mind, rather than the underlying poset. Natural examples of combinatorial polytopes include the *vertex figures* P_v, which are formed by intersecting P with any closed half-space whose interior contains a single vertex v. Because P is simple, P_v is an n-simplex for any v.

By permuting the facets of P if necessary, we may assume that the intersection $F_1 \cap \cdots \cap F_n$ is a vertex $v_\star$. In this case, we describe P as *finely ordered*, and refer to $v_\star$ as the *initial vertex*; it is the first vertex of P with respect to the ordering implied by (2.7). For computational purposes it is often convenient to locate the initial vertex of P at the origin, and use the normal vectors a_i as an orthonormal basis for V. This may be achieved by applying an affine transformation to P, which preserves its combinatorial type.

Given a second finely ordered polytope P' in $\mathbb{R}^{n'}$, we may list the facets of the product $P \times P'$ as

$$F_1 \times P', \ldots, F_m \times P', P \times F_1', \ldots, P \times F_{m'}', \tag{2.8}$$

where F_i and F_j' range over the facets of P and P' respectively. Then $P \times P'$ is finely ordered by shifting the block of facets $P \times F_1', \ldots, P \times F_{n'}'$ to the left, until it occupies positions $n+1, \ldots, n+n'$ in (2.8). The initial vertex is $(v_\star, v_\star')$. Of course, this procedure yields different results for $P \times P'$ and $P' \times P$.

For a fixed arrangement $\mathcal{H}$, we consider the vector $d_\mathcal{H} \in \mathbb{R}^m$, whose ith coordinate is the signed distance from Y_i to the origin O in V, for $1 \leqslant i \leqslant m$. The sign is positive when O lies in the interior of H_i, and negative in the exterior. So long as we maintain our convention that the normal vectors a_i have unit length, $d_\mathcal{H}$ coincides with b_P; otherwise, the distances have to be scaled accordingly. Every vector $d_\mathcal{H} + h$ in $\mathbb{R}^m$ may then be identified with an *analogous arrangement* of half-spaces, defined by translating each H_i by h_i, for $1 \leqslant i \leqslant m$. Some such arrangements determine convex polytopes $P(h)$, and others, dubbed *virtual polytopes*, do not. In either case, they are deemed to be *analogous to* P. We note that $P(h)$ is given by

$$\{x \in V : A_P x + b_P + h \geqslant 0\}, \tag{2.9}$$

and is combinatorially equivalent to P when h is small. In particular, we have that $P(0) = P$.

Examples 2.10. The zero vector $0 \in \mathbb{R}^m$ is identified with the central arrangement $\mathcal{H}_0$, whose bounding hyperplanes contain the origin; the corresponding polytope $P(-b_P) = \{0\}$ is virtual. The basis vector $e_i \in \mathbb{R}^m$ is identified with the

arrangement obtained from $\mathcal{H}_0$ by translating H_i; the corresponding polytope $P(-b_P + e_i) = P_i$ may be virtual, or a simplex.

Example 2.11. Any $y \in V$ defines a vector $A_P y \in \mathbb{R}^m$. Then $A_P y + b_P$ is identified with the arrangement obtained by translating $\mathcal{H}$ by $-y$; the corresponding polytope $P(A_P y)$ is the translate $P - y$, and is congruent to P. As y varies, we obtain an n-parameter family of analogous polytopes, each being congruent to P.

The *Minkowski sum* of subsets P, $Q \subseteq V$ is given by

$$P + Q = \{x + y \colon x \in P,\, y \in Q\} \subseteq V.$$

If P and Q are convex polytopes, so is $P + Q$; moreover, when P is analogous to Q, so is $P + Q$. Under the identification of $b_P + h$ with $P(h)$, vector addition corresponds to Minkowski sum, and scalar multiplication to rescaling. In this context, we denote the m-dimensional vector space of polytopes analogous to P by $\mathbb{R}(P)$, and consider the identification as an isomorphism

$$k \colon \mathbb{R}^m \to \mathbb{R}(P), \quad \text{where } k(b_P + h) = P(h). \tag{2.12}$$

We may interpret the matrix A_P as a linear transformation $V \to \mathbb{R}^m$. Since the points of P are specified by the constraint $A_P x + b_P \geqslant 0$, the intersection of the affine subspace $A_P(V) + b_P$ with the positive cone $\mathbb{R}^m_{\geqslant}$ is a copy of P in $\mathbb{R}^m$. In other words, the formula $i_P(x) = A_P x + b_P$ defines an affine injection

$$i_P \colon V \to \mathbb{R}^m, \tag{2.13}$$

which embeds P as a submanifold of the positive cone. Since i_P maps the half-space H_i to the half-space $\{y \colon y_i \geqslant 0\}$, it embeds each codimension k face of P in a codimension k face of $\mathbb{R}^m_{\geqslant}$.

The composition $\chi_P = k \circ i_P$ restricts to an affine injection $P \to \mathbb{R}(P)$, and Example (2.11) identifies $\chi_P(y)$ as the polytope congruent to P, obtained by translating the origin to y, for all y in P. Of course, $\chi_P(P)$ is a submanifold of the positive cone $\mathbb{R}(P)_{\geqslant}$, and facial codimensions are preserved as before.

When P is finely ordered, the half spaces $H_1 + h_1, \ldots, H_n + h_n$ determine the initial vertex $v_\star(h)$ of $P(h)$ for any shift vector h. For every $1 \leqslant i \leqslant m$, we write $d_i(h)$ for the signed distance between $v_\star(h)$ and the supporting hyperplane $Y_i + h_i$; in other words,

$$d_i(h) = \langle a_i,\, v_\star(h) \rangle + b_i + h_i \quad \text{for all } 1 \leqslant i \leqslant m,$$

and $d_i(h) = 0$ for $1 \leqslant i \leqslant n$. We define a linear transformation $C \colon \mathbb{R}^m \to \mathbb{R}^{m-n}$ by the formula

$$C(b_P + h) = (d_{n+1}(h),\, \ldots,\, d_m(h)). \tag{2.14}$$

Using (2.12), we may interpret C as a transformation $\mathbb{R}(P) \to \mathbb{R}^{m-n}$, which acts by $P(h) \mapsto (d_{n+1}(h),\, \ldots,\, d_m(h))$. Clearly C is epimorphic.

Proposition 2.15. *As a transformation $V \to \mathbb{R}^{m-n}$, the composition $C \cdot A_P$ is zero.*

Proof. The $d_i(h)$ are metric invariants of the polytope $P(h)$, so C takes identical values on congruent polytopes. In particular, it is constant on the translates $P - y$

for all values $y \in V$, and therefore on the affine plane $A_P(V) + b_P$. So $C(A_P(V)) = 0$, as required. $\qquad\square$

Proposition 2.15 determines a short exact sequence of the form

$$0 \to V \xrightarrow{A_P} \mathbb{R}^m \xrightarrow{C} \mathbb{R}^{m-n} \to 0, \tag{2.16}$$

or equivalently, a choice of basis for $\operatorname{coker} A_P$.

In order to construct a matrix $(c_{i,j})$ for C, it is convenient to use the orthonormal basis $a_1, \dots, a_n$, as described above. Then the basis polytopes P_j of (2.10) satisfy

$$d_i(P_j) = \begin{cases} -a_{i,j} & \text{if } 1 \leqslant j \leqslant n, \\ \delta_{i,j} & \text{if } n+1 \leqslant j \leqslant m \end{cases}$$

for all $n+1 \leqslant i \leqslant m$, giving

$$(c_{i,j}) = \begin{pmatrix} -a_{n+1,1} & \dots & -a_{n+1,n} & 1 & 0 & \dots & 0 \\ -a_{n+2,1} & \dots & -a_{n+2,n} & 0 & 1 & \dots & 0 \\ \dotfill & & & & & & \\ -a_{m,1} & \dots & -a_{m,n} & 0 & 0 & \dots & 1 \end{pmatrix}. \tag{2.17}$$

A permutation of the facets produces an alternative basis for $\operatorname{coker} A_P$, and the corresponding matrix is obtained by reordering the columns of C. Any other $(m-n) \times m$ matrix C for which $CA_P = 0$ also provides a basis for $\operatorname{coker} A_P$, so long as it has full rank; it necessarily satisfies the following property.

Lemma 2.18. *Let C' be the $(m-n) \times (m-k)$ matrix obtained from C by deleting columns $c_{j_1}, \dots, c_{j_k}$, for some $1 \leqslant k \leqslant n$; if the intersection $F_{j_1} \cap \cdots \cap F_{j_k}$ is a face of P of codimension k, then C' has rank $m-n$.*

Proof. Let $\iota \colon \mathbb{R}^{m-k} \to \mathbb{R}^m$ be the inclusion of the subspace

$$\{x \colon x_{j_1} = \cdots = x_{j_k} = 0\}$$

and $\kappa \colon \mathbb{R}^m \to \mathbb{R}^k$ the associated quotient map. Then C' is the matrix of the composition $C \cdot \iota$, and the $k \times n$ matrix A' of the composition $\kappa \cdot A_P$ consists of the rows $a_{j_1}, \dots, a_{j_k}$ of A_P. The data implies that A' has rank k, and therefore that $\kappa \cdot A_P$ is an epimorphism; so $C \cdot \iota$ is also an epimorphism, and its matrix has rank $m-n$. $\qquad\square$

3. Quasitoric Manifolds

In this section we include a summary of Davis and Januszkiewicz's construction of quasitoric manifolds M over a simple polytope P. Throughout, we use the methods and notation of [5], and refer readers to [3, Chapter 6]) for further details. We also assume that P is finely ordered; so M has a distinguished fixed point, near which we insist that T^n act as standard. To illustrate these additional requirements we revisit several standard examples.

We denote the ith coordinate subcircle of the standard m-torus T^m by T_i for every $1 \leqslant i \leqslant m$. Given any subset $I \subseteq [m]$, we define the subgroup T_I by

$$\prod_{i \in I} T_i < T^m,$$

so that $T_\varnothing$ is the trivial subgroup $\{1\}$. Every point p of P lies in the interior of a unique face F_{I_p}, where I_p is given by $\{i : p \in F_i\}$; we abbreviate F_{I_p} and T_{I_p} to $F(p)$ and $T(p)$ respectively. If p is a vertex, for example, then $T(p)$ has dimension n (the maximum possible), and if p is an interior point of P, then $T(p)$ is trivial.

We define the *moment-angle complex* $\mathcal{Z}_P$ as the identification space

$$T^m \times P/{\sim}, \tag{3.1}$$

where $(t_1, p) \sim (t_2, p)$ if and only if $t_1^{-1} t_2 \in T(p)$. So $\mathcal{Z}_P$ admits a canonical left T^m-action whose isotropy subgroups are precisely the subgroups $T(p)$. Construction (3.1) may equally well be applied to the positive cone $\mathbb{R}^m_\geqslant$, in which case the result is the complex vector space $\mathbb{C}^m$. Since the embedding i_P of (2.13) respects facial codimensions, there is a pullback diagram

$$\begin{array}{ccc}
\mathcal{Z}_P & \xrightarrow{\;i_Z\;} & \mathbb{C}^m \\
{\scriptstyle \varrho_P}\downarrow & & \downarrow{\scriptstyle \varrho} \\
P & \xrightarrow{\;i_P\;} & \mathbb{R}^m_\geqslant
\end{array} \tag{3.2}$$

of identification spaces. Here $\varrho(z_1, \ldots, z_m)$ is given by $(|z_1|^2, \ldots, |z_m|^2)$, the vertical maps are projections onto the quotients by the T^m-actions, and i_Z is a T^m-equivariant embedding. It is sometimes convenient to rewrite $\mathbb{C}^m$ as $\mathbb{R}^{2m}$, in which case we substitute $q_j + i r_j$ for the jth coordinate z_j, and let T_j act by rotation.

Then Proposition 2.15 and Diagram (3.2) imply that i_Z embeds $\mathcal{Z}_P$ in $\mathbb{R}^{2m}$ as the space of solutions of the $m - n$ quadratic equations

$$\sum_{k=1}^{m} c_{j,k} \left(q_k^2 + r_k^2 - b_k \right) = 0, \quad \text{for } 1 \leqslant j \leqslant m - n. \tag{3.3}$$

In Lemma 4.2, we will confirm that $\mathcal{Z}_P$ is a frameable submanifold of $\mathbb{R}^{2m}$ of dimension $(m + n)$, and therefore smooth.

In order to construct quasitoric manifolds over P, we need one further set of data. This consists of a homomorphism $\ell \colon T^m \to T^n$, satisfying Davis and Januszkiewicz's independence condition, namely

$$F_I \text{ is a face of codimension } k \quad \Longrightarrow \quad \ell \text{ is monic on } T_I. \tag{3.4}$$

Any such ℓ is called a *dicharacteristic* in [5]. Condition (3.4) ensures that the kernel $K(\ell)$ of ℓ is isomorphic to an $(m - n)$-dimensional subtorus of T^m, and features in a short exact sequence

$$1 \to K(\ell) \to T^m \xrightarrow{\;\ell\;} T^n \to 1. \tag{3.5}$$

Wherever possible we abbreviate $K(\ell)$ to K.

We write the subcircle $\ell(T_i) < T^n$ as $T(F_i)$ for any $1 \leqslant i \leqslant m$, and the subgroup $\ell(T_I)$ as $T(F_I)$ for any face F_I. For each point p in P we let $S(p)$ denote the subgroup $T(F(p))$; it is, of course, $\ell(T(p))$. For example, $S(w) = T^n$ for any vertex w, and $S(p) = \{1\}$ for any point p in the interior of P.

When applied to the initial vertex $v_\star$, (3.4) ensures that the restriction of ℓ to $T_1 \times \cdots \times T_n$ is an isomorphism. So we may use the circles $T(F_1), \ldots, T(F_n)$ to

define a basis for the Lie algebra of T^n, and represent the homomorphism induced by ℓ by an $n \times m$ integral matrix of the form

$$
\Lambda = \begin{pmatrix} 1 & 0 & \ldots & 0 & \lambda_{1,n+1} & \ldots & \lambda_{1,m} \\ 0 & 1 & \ldots & 0 & \lambda_{2,n+1} & \ldots & \lambda_{2,m} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ 0 & 0 & \ldots & 1 & \lambda_{n,n+1} & \ldots & \lambda_{n,m} \end{pmatrix} . \tag{3.6}
$$

It is often convenient to partition Λ as $(I_n \mid \Lambda_\star)$, so that $\Lambda_\star$ is an $n \times (m-n)$-matrix. Given any other vertex $F_{j_1} \cap \cdots \cap F_{j_n}$, (3.4) implies that the corresponding columns $\lambda_{j_1}, \ldots, \lambda_{j_n}$ form a basis for $\mathbb{Z}^n$, and have determinant ± 1. We refer to (3.6) as the *refined form*, and call $\Lambda_\star$ the *refined submatrix* of ℓ.

Since K meets every isotropy subgroup $T(p)$ of the T^m-action trivially, it acts freely on $\mathcal{Z}_P$, and the base of the resulting principal K-bundle $\pi_\ell \colon \mathcal{Z}_P \to M$ is a smooth $2n$-dimensional manifold. By construction, M may be expressed as the identification space

$$
T^n \times P/\approx \tag{3.7}
$$

where $(s_1, p) \approx (s_2, p)$ if and only if $s_1^{-1} s_2 \in S(p)$. Furthermore, M admits a canonical T^n-action α, which is locally isomorphic to the standard action on $\mathbb{C}^n$, and has quotient map $\pi \colon M \to P$. Note that $\pi \cdot \pi_\ell = \varrho_P$ as maps $\mathcal{Z}_P \to P$. The fixed points of α project to the vertices of P, so they are also ordered, and we refer to $\pi^{-1}(v_\star)$ as the *initial fixed point* $x_\star$. Then (3.7) identifies a neighbourhood of $x_\star$ with $\mathbb{C}^n$, on which α is standard; its representation at other fixed points may be read off from the corresponding columns of Λ.

The quadruple (M, α, π, P) is an example of a *quasitoric manifold*, as defined by Davis and Januszkiewicz. Any manifold with a similarly well-behaved torus action over P is θ-equivariantly homeomorphic to one of the form (3.7), see [7, Prop. 1.8]. In this sense, M is typical, and we follow the lead of [5] by assuming that every quasitoric manifold is presented in the form (3.7).

Additional structure on M is associated to the *facial submanifold* M_i, defined as the inverse images of the facet F_i under π, for $1 \leqslant i \leqslant m$. It is clear that every M_i has codimension 2, and that its isotropy subgroup is $T(F_i) < T^n$. The quotient map

$$
\mathcal{Z}_P \times_K \mathbb{C}_i \to M \tag{3.8}
$$

defines a canonical complex line-bundle ρ_i, whose restriction to M_i is isomorphic to the normal bundle ν_i of its embedding in M.

The submanifolds M_i are mutually transverse, and we write

$$
M_I = M_{i_1} \cap \cdots \cap M_{i_k} \tag{3.9}
$$

for any non-empty intersection, using I as in (2.7). So M_I is the inverse image of the codimension k face F_I under π. Of course M_I has codimension $2k$, and its isotropy subgroup is $T(F_I)$. The restriction of $\rho_I = \rho_{i_1} \oplus \cdots \oplus \rho_{i_d}$ to M_I is isomorphic to the normal bundle ν_I of its embedding in M, for any face F_I.

As explained in [7], the bundles ρ_i play an important part in understanding the integral cohomology ring of M. If u_i denotes the first Chern class $c_1(\rho_i)$ in $H^2(M)$, then $H^*(M)$ is generated by $u_1, \ldots, u_m$, modulo two sets of relations. The first are linear, and arise from the refined form (3.6) of the dicharacteristic; the second

are monomial, and arise from the Stanley–Reisner ideal of P. The former may be read off from the refined submatrix as

$$u_i = -\lambda_{i,n+1}u_{n+1} - \cdots - \lambda_{i,m}u_m \quad \text{for } 1 \leqslant i \leqslant n, \tag{3.10}$$

and show that $u_{n+1}, \ldots, u_m$ suffice to generate $H^*(M)$ multiplicatively.

In work such as [3], [5], and [6], fine orderings are not considered, so the matrices representing ℓ rarely appear in refined form. In order to rectify this situation systematically, we may begin by choosing an initial vertex. Then we shuffle the facets of P (and therefore the columns of the representing matrix) until $F_{[n]}$ is $v_\star$, and premultiply the resulting matrix by the unique element of $\text{GL}(n; \mathbb{Z})$ that transforms the first n columns into I_n. To illustrate the effects of this procedure, we now revisit three important examples.

Example 3.11. The n-simplex is finely ordered by (2.4), and has initial vertex the origin. Then i_P embeds $\Delta(n)$ in $\mathbb{R}^{n+1}$ by $i_P(x) = (x_1, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i)$, and $\mathcal{Z}_P$ is the unit sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$. The refined submatrix is the column vector $(-1, \ldots, -1)$ in $\mathbb{R}^n$, so the kernel of the dicharacteristic is the diagonal subcircle

$$T_\delta = \{(t, t, \ldots, t)\} < T^{n+1}.$$

It follows that M is the complex projective space $\mathbb{C}P^n$. Then $(t_1, \ldots, t_n) \in T^n$ acts on the point with homogeneous coordinates $[z_1, \ldots, z_{n+1}]$ as multiplication by $(t_1, \ldots, t_n, 1)$, and the initial fixed point is $[0, \ldots, 0, 1]$. Every facial bundle is isomorphic to $\overline{\eta}$, where η is the Hopf line bundle. The cohomology ring of M is generated by elements $u_1, \ldots, u_{n+1}$ in $H^2(M)$, and relations (3.10) give $u_1 = \cdots = u_{n+1}$; the Stanley–Reisner relations reduce to $u_1^{n+1} = 0$.

Example 3.12. The n-cube is finely ordered by (2.6), and has initial vertex the origin. Then i_P embeds I^n in $\mathbb{R}^{2n}$ by $i_P(x) = (x_1, \ldots, x_n, 1-x_1, \ldots, 1-x_n)$, and $\mathcal{Z}_P$ is the product of unit 3-spheres $|z_k|^2 + |z_{n+k}|^2 = 1$ in $\mathbb{C}^{2n}$, where $1 \leqslant k \leqslant n$. The refined submatrix is

$$D = \begin{pmatrix} -1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ d(1,2) & -1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 \\ \hdotsfor{9} \\ d(1,j) & d(2,j) & \ldots & d(j-1,j) & -1 & 0 & \ldots & 0 & 0 \\ \hdotsfor{9} \\ d(1,n) & d(2,n) & \ldots & d(j-1,n) & d(j,n) & d(j+1,n) & \ldots & d(n-1,n) & -1 \end{pmatrix}$$

for any set of $n(n-1)/2$ integers $d(i, j)$, where $1 \leqslant i < j \leqslant n$; so the kernel of the dicharacteristic is the n-torus

$$\{(t_1, t_1^{-d(1,2)}t_2, \ldots, t_1^{-d(1,n)}t_2^{-d(2,n)} \ldots t_{n-1}^{-d(n-1,n)}t_n, t_1, t_2, \ldots, t_n)\} < T^{2n}.$$

It follows that M is the nth stage Q_n of the *Bott tower* defined in [6] and [9], albeit with permuted coordinates. Then $(t_1, \ldots, t_n) \in T^n$ acts on the equivalence class $[z_1, \ldots, z_{2n}]$ as multiplication by $(t_1, \ldots, t_n, 1, \ldots, 1)$, and the initial fixed point is $[0, \ldots, 0, 1, \ldots, 1]$. The facial bundles are the ρ_i of [6], suitably reordered. The cohomology ring of M is generated by $u_1, \ldots, u_{2n}$ in $H^2(M)$, and relations (3.10)

give

$$u_j = -d(1,\, j)u_{n+1} - \cdots - d(j-1,\, j)u_{n+j-1} + u_{n+j} \quad \text{for } 1 \leqslant j \leqslant n.$$

The Stanley–Reisner relations take the form $u_j u_{n+j} = 0$ for all j.

   If the defining integers satisfy

$$d(i,\, j) = \begin{cases} 1 & \text{for } i = j-1, \\ 0 & \text{otherwise} \end{cases}$$

for every $2 \leqslant j \leqslant n$, then Q_n becomes the *bounded flag manifold* B_n of [4]. If $d(i,\, j) = 0$ for all $i,\, j$, then Q_n is the n-fold product $(S^2)^n$.

Example 3.13. For any pair of integers $r < s$, the facets of $R^{r+s-1} = I^r \times \Delta(s-1)$ are finely ordered by combining (2.8) with Examples 3.11 and 3.12. The initial vertex is the origin. Then i_P embeds R^{r+s-1} in $\mathbb{R}^{2r+s}$ by

$$i_P(x) = \left(x_1,\, \ldots,\, x_{r+s-1},\, 1-x_1,\, \ldots,\, 1-x_r,\, 1 - \sum_{i=r+1}^{r+s-1} x_i \right),$$

and $\mathcal{Z}_P$ is a product $S^3 \times \cdots \times S^3 \times S^{2s-1}$ of $r+1$ unit spheres in $\mathbb{C}^{2r+s}$. The refined submatrix (which is $(r+s-1) \times (r+1)$) is

$$E = \begin{pmatrix} J_r & 0 \\ J_r & -1 \\ 0_{r,s} & -1 \end{pmatrix},$$

where J_r is the $r \times r$ matrix whose only non-zero elements are -1s on the diagonal and 1s on the subdiagonal, $0_{r,s}$ is the $(s-r-1) \times r$ zero matrix, and $0,\, -1$ denote column vectors of the appropriate length. So the kernel of the dicharacteristic is the $(r+1)$-dimensional subtorus

$$\{(t_1,\, t_1^{-1}t_2,\, \ldots,\, t_{r-1}^{-1}t_r,\, ut_1,\, ut_1^{-1}t_2,\, \ldots,\, ut_{r-1}^{-1}t_r,\, u,\, \ldots,\, u,\, t_1,\, t_2,\, \ldots,\, t_r,\, u)\}$$

of T^{2r+s}. It follows that M is the $\mathbb{C}P^{s-1}$-bundle $B_{r,s}$ over B_r as defined in [4], albeit with permuted coordinates. Then $(t_1,\, \ldots,\, t_{r+s-1}) \in T^{r+s-1}$ acts on the equivalence class $[z_1,\, \ldots,\, z_{2r+s}]$ as multiplication by $(t_1,\, \ldots,\, t_{r+s-1},\, 1,\, \ldots,\, 1)$, and the initial fixed point is $[0,\, \ldots,\, 0,\, 1,\, \ldots,\, 1]$. The facial bundles are the ρ_i of [6], suitably reordered. The cohomology ring of $B_{r,s}$ is generated by elements $u_1,\, \ldots,\, u_{2r+s}$ in $H^2(M)$, and it is helpful to write $u_{r+s+i-1} = x_i$ for $1 \leqslant i \leqslant r$, and $u_{2r+s} = y$. Then relations (3.10) give

$$u_1 = x_1, \quad u_{r+1} = x_1 + y, \quad u_{2r+1} = \cdots = u_{r+s-1} = y,$$
$$\text{and} \quad u_i = x_i - x_{i-1}, \quad u_{r+i} = x_i - x_{i-1} + y \quad \text{for} \quad 2 \leqslant i \leqslant r.$$

The Stanley–Reisner relations are of the form $u_i x_i = 0$ for $1 \leqslant i \leqslant r$, and $u_{r+1}u_{r+2}\ldots u_{r+s-1}y = 0$. Thus $H^*(B_{r,s})$ is isomorphic to

$$\mathbb{Z}[x_1,\, \ldots,\, x_r,\, y]/J, \quad \text{where } J = \left(x_i^2 - x_i x_{i-1},\, \sum_{j=0}^{r} x_r^j y^{s-j} \right).$$

The same construction works for $r = s$, but $\Lambda_\star$ is more complicated.

4. Stably Complex Structures, Orientations, and Framings

On a smooth manifold N of dimension d, a *stably complex structure* is an equivalence class of real $2k$-plane bundle isomorphisms $\tau(N) \oplus \mathbb{R}^{2k-d} \cong \zeta$. Here ζ denotes a fixed $\mathrm{GL}(k, \mathbb{C})$-bundle, $\mathbb{R}^{2k-d}$ denotes the trivial $(2k-d)$-dimensional bundle with fibre $\mathbb{R}^{2k-d}$ and k is suitably large. Two such isomorphisms are equivalent when they agree up to stabilisation; or, alternatively, when the corresponding lifts to BU of the classifying map of the stable tangent bundle of N are homotopic through lifts. Note that $\mathbb{R}^{2k-d}$ is canonically oriented (and even framed) by choosing the standard basis, which therefore determines an orientation for N.

Now assume that N has an l-dimensional torus action $\alpha \colon T^l \times N \to N$. A stably complex structure on N is T^l-*invariant* whenever the composition

$$\zeta \xrightarrow{\cong} \tau(N) \oplus \mathbb{R}^{2k-d} \xrightarrow{d\alpha(t,\cdot) \oplus 1} \tau(N) \oplus \mathbb{R}^{2k-d} \xrightarrow{\cong} \zeta$$

is an isomorphism of complex bundles for every $t \in T^l$. In this section we show that every quasitoric manifold admits an invariant stably complex structure and identify the geometric data required to induce these structures.

According to [5], an *omniorientation* of a quasitoric manifold M consists of a choice of orientation for each normal bundle ν_i. This coincides with a choice of complex structure for each ρ_i, and is therefore equivalent to a dicharacteristic ℓ. In [3], a choice of orientation for M is also assumed, since none is implied by ℓ. We adopt this convention henceforth, and refer to the constituent data as the dicharacteristic and orientation *associated to* the omniorientation. The orientation corresponds to a fundamental class μ_M in the integral homology group $H_{2n}(M)$.

An interior point of the quotient polytope P admits an open neighborhood U, whose inverse image under the projection π is canonically diffeomorphic to $T^n \times U$ as a subspace of M. Since T^n is oriented by the choice of basis leading to the refined form (3.6) of the matrix of ℓ, orientations of M correspond bijectively to orientations of P. Every pair (P, Λ) therefore determines a $2n$-dimensional omnioriented quasitoric manifold, where P is the combinatorial type of an oriented finely ordered n-dimensional simple polytope, and Λ is a matrix of the form (3.6).

Definition 4.1. We refer to the pair $(P, \Lambda_\star)$ as the *combinatorial data* underlying the omnioriented manifold M.

We may specify the orientation of P on a representative polytope in $\mathbb{R}^n$, or by an equivalence class of orderings of the n edges incident on $v_\star$ in $\mathcal{L}(P)$. The latter is independent of the fine ordering on P (although they may, of course, agree). When it is important to emphasise that the facial submanifolds of M are ordered, and that α is standard at $x_\star$, we also describe M as *refined*.

In order to explain the stably complex structure induced on M, it is convenient to study the embedding i_Z of (3.2) in more detail.

Lemma 4.2. *The embedding $i_Z \colon \mathcal{Z}_P \to \mathbb{R}^{2m}$ is T^m-equivariantly framed by any choice of matrix $C = (c_{i,j})$ for the transformation* (2.14).

Proof. We describe i_Z by the $m - n$ quadratic equations (3.3) over $P \subset \mathbb{R}^m_\geqslant$. At each point $(q_1, r_1, \ldots, q_m, r_m) \in \mathcal{Z}_P$, the $m - n$ associated gradient vectors are

given by

$$2\,(c_{j,1}q_1,\,c_{j,1}r_1,\,\ldots,\,c_{j,m}q_m,\,c_{j,m}r_m) \quad \text{for } 1 \leqslant j \leqslant m-n, \qquad (4.3)$$

and so form the rows of the $(m-n) \times 2m$ matrix $2CR$, where

$$R = \begin{pmatrix} q_1 & r_1 & \ldots & 0 & 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ 0 & 0 & \ldots & q_m & r_m \end{pmatrix}$$

is $m \times 2m$. By definition of i_P, the set of integers $j_1,\ldots,j_k$ with the property that $q_{j_1} = r_{j_1} = \cdots = q_{j_k} = r_{j_k} = 0$ at some point $z \in \mathcal{Z}_P$ corresponds to an intersection $F_{j_1} \cap \cdots \cap F_{j_k}$ of facets forming a face of P of codimension k. Lemma 2.18 then applies to show that the matrix obtained by deleting the columns $c_{j_1}, \ldots, c_{j_k}$ of C has rank $m-n$. It follows that $2CR$ has rank $m-n$, and therefore that the gradient vectors (4.3) are linearly independent at z, and so frame i_Z.

Furthermore, each of the gradient vectors frames the corresponding quadratic hypersurface in $\mathbb{R}^{2m}$, and is T^m-invariant. $\square$

Remark 4.4. Lemma 4.2 provides an alternative to [5, Proposition 3.4], where insufficient detail is given for readers to complete the proof.

It is particularly illuminating to describe the framing of i_Z in terms of analogous polytopes, as follows.

Factoring out by the action of T^m yields a framing of the embedding i_P, and therefore of P in $\mathbb{R}^m_{\geqslant}$; moreover, on each face of P, the framing lies in the ambient face of $\mathbb{R}^m_{\geqslant}$. Under the identification (2.12), the framing vectors may be represented by $m-n$ independent 1-parameter families of polytopes analogous to P. These families are made explicit by applying the differential $d\varrho_P$ to the rows of the matrix $2CR$. At the point $(q_1, r_1, \ldots, q_m, r_m)$ in $\mathcal{Z}_P$, the matrix of $d\varrho_P$ is given by $2R$, so the framing vectors are the rows of the $(m-n) \times m$ matrix $4CRR^t$. When C takes the form (2.17), we may take the jth framing vector to be

$$f_j = (-a_{n+j,1}y_1, \,\ldots,\, -a_{n+j,n}y_n,\, 0,\, \ldots,\, 0,\, y_{n+j},\, 0,\, \ldots,\, 0)$$

at $y = i_P(x)$, for $1 \leqslant j \leqslant m-n$. Applying (2.12), we conclude that the corresponding 1-parameter family of polytopes $P(f_j, t)$ (for $-1 \leqslant t \leqslant 1$) is obtained from P by: retaining the origin at x, rescaling H_k by $-a_{n+j,k}t$ for $1 \leqslant k \leqslant n$, fixing every facet opposite the initial vertex except H_{n+j}, and rescaling the latter by t.

It is possible to reverse this procedure, and begin by framing i_P. The corresponding T^m-equivariant framing of i_Z is then recovered by applying the construction (3.1). Since P is contractible, all framings of i_P are equivalent, and their lifts to i_Z are equivariantly equivalent. In particular, the equivalence class of the framings described in Lemma 4.2 does not depend on the choice of fine ordering on P.

The smoothness of M is assured by Lemma 4.2, and we now return to its tangent bundle $\tau(M)$. Our analysis reduces to a special case of Szczarba's proof of [16, Theorem (1.1)], and supersedes that given in [5, Theorem (3.8)] which ignores the orientation of M.

Proposition 4.5. *Any omnioriented quasitoric manifold admits a canonical stably complex structure, which is invariant under the T^n-action.*

Proof. There is a T^m-equivariant decomposition

$$\tau(\mathcal{Z}_P) \oplus \nu(i_Z) \cong \mathcal{Z}_P \times \mathbb{C}^m,$$

obtained by restricting the tangent bundle $\tau(\mathbb{C}^m)$ to $\mathcal{Z}_P$. Factoring out by the kernel of $\ell \colon T^m \to T^n$ yields

$$\tau(M) \oplus (\xi/K) \oplus (\nu(i_Z)/K) \cong \mathcal{Z}_P \times_K \mathbb{C}^m, \qquad (4.6)$$

where ξ denotes the $(m-n)$-plane bundle of tangents along the fibres of π_ℓ. The right-hand side of (4.6) is isomorphic to $\bigoplus_{i=1}^m \rho_i$ as $\mathrm{GL}(m, \mathbb{C})$-bundles.

Szczarba [16, Corollary 6.2] identifies ξ/K with the adjoint bundle of π_ℓ, which is trivial because K is abelian; and $\nu(i_Z)/K$ is trivial by Lemma 4.2. So (4.6) reduces to an isomorphism

$$\tau(M) \oplus \mathbb{R}^{2(m-n)} \cong \rho_1 \oplus \cdots \oplus \rho_m, \qquad (4.7)$$

although different choices of trivialisations may lead to different isomorphisms. Since M is connected and $\mathrm{GL}(2(m-n), \mathbb{R})$ has two connected components, such isomorphisms are equivalent when and only when the induced orientations agree on $\mathbb{R}^{2(m-n)}$. We choose the orientation which is compatible with those for $\tau(M)$ and $\rho_1 \oplus \cdots \oplus \rho_m$, as given by the omniorientation.

The induced structure is invariant under the action of T^n, because i_Z is T^m-equivariant. $\qquad \square$

The stably complex structures represented by the two choices of orientation differ by sign. The underlying smooth structure is also T^n-invariant, and is identical to that inferred from Lemma 4.2.

The proof of Proposition 4.5 allows us to evaluate the tangential Chern classes of the canonical stably complex structure.

Corollary 4.8. *In $H^{2i}(M)$, the Chern class $c_i(\tau)$ is given by the ith symmetric polynomial in the variables $u_1, \ldots, u_m$, for $1 \leqslant i \leqslant n$.*

Proof. By (4.7), the total Chern class of τ is $c(\tau) = \prod_{i=1}^m (1 + u_i)$ in $H^*(M)$. $\qquad \square$

Davis and Januszkiewicz's quasitoric manifolds are inspired by the non-singular projective toric varieties of algebraic geometry. Every such X is determined by the normal fan of an integral simple polytope $Q \subset \mathbb{R}^n$, whose vertices lie in the lattice $\mathbb{Z}^n$. We may assume that the origin is a distinguished vertex, that its incident facets lie in the respective coordinate hyperplanes, and that the remaining facets $F_{n+1}, \ldots, F_m$ are ordered. Since X is equipped with a canonical complex structure it is also an omnioriented quasitoric manifold, so we study this example in more detail before moving on.

According to Batyrev [3, Section 8.2], X may be identified with the *geometric quotient* of the *coordinate subspace complement*

$$U(Q) = \mathbb{C}^m \setminus \bigcup \{z \colon z_{i_1} = \cdots = z_{i_k} = 0 \text{ if } F_{i_1} \cap \cdots \cap F_{i_k} = \varnothing \text{ in } Q\}$$

by the complexified group

$$K_{\mathbb{C}} = \ker(\ell_{\mathbb{C}} \colon (\mathbb{C}^*)^m \to (\mathbb{C}^*)^n).$$

By definition, there is a canonical embedding $j \colon \mathcal{Z}_Q \xhookrightarrow{\subseteq} U(Q)$ of the compact subset $\mathcal{Z}_Q$, which induces an algebraic isomorphism $\mathcal{Z}_Q/K \to U(Q)/K_{\mathbb{C}}$. In other words,

$\mathcal{Z}_Q$ is the *Kempf–Ness set* for the action of the algebraic torus $K_{\mathbb{C}}$ on the quasiaffine variety $U(Q)$; see [11, Theorem 3.4] for further details. The integral version

$$0 \to V_{\mathbb{Z}} \xrightarrow{A_Q} \mathbb{Z}^m \xrightarrow{C} \mathbb{Z}^{m-n} \to 0$$

of the short exact sequence (2.16) is the sequence of weight lattices for (3.5).

The tangent bundle of $U(Q)$ is trivial, and admits a $\mathrm{GL}(m-n, \mathbb{C})$-subbundle $\xi_{\mathbb{C}}$ of tangents along the fibre of the quotient map $U(Q) \to X$. Applying [16] once more, we deduce that the complex structure on X is compatible with the corresponding isomorphism

$$\tau(X) \oplus (\xi_{\mathbb{C}}/K_{\mathbb{C}}) \cong U(Q) \times_{K_{\mathbb{C}}} \mathbb{C}^m \qquad (4.9)$$

of quotient $\mathrm{GL}(m, \mathbb{C})$-bundles, where $\xi_{\mathbb{C}}/K_{\mathbb{C}}$ is trivial because $K_{\mathbb{C}}$ is abelian.

Example 4.10. For any non-singular projective toric variety X, we let P be the oriented combinatorial type of Q, and the columns of Λ be the primitive integral inward pointing normal vectors to $F_1, \ldots, F_m$ respectively. So $\Lambda = A_Q^t$ in the notation of Section 2 (although the row vectors a_i of A_Q do not necessarily have unit length). We identify the stably complex structure associated to the combinatorial data $(P, \Lambda_\star)$ by comparing isomorphisms (4.7) and (4.9) as follows.

On fibres, j restricts to the inclusion of the subgroup $K < K_{\mathbb{C}}$. So there is an isomorphism $\xi \oplus \nu(i_Z) \cong j^*\xi_{\mathbb{C}}$ over $\mathcal{Z}_Q$, whose quotient is an isomorphism

$$\xi/K \oplus \nu(i_Z)/K \cong \xi_{\mathbb{C}}/K_{\mathbb{C}}$$

of $\mathrm{GL}(2(m-n), \mathbb{R})$-bundles over X. Similarly, there is a quotient isomorphism

$$\mathcal{Z}_P \times_K \mathbb{C}^m \cong U(Q) \times_{K_{\mathbb{C}}} \mathbb{C}^m$$

of $\mathrm{GL}(m, \mathbb{C})$-bundles, which identifies the right hand sides of (4.7) and (4.9). Moreover, the framing chosen for $\xi/K \oplus \nu(i_Z)/K$ in (4.7) induces the same orientation as that of the complex structure on $\xi_{\mathbb{C}}/K_{\mathbb{C}}$, because both are compatible with the natural orientation of X. So the stably complex structure associated to $(P, \Lambda_\star)$ agrees with that induced by the complex varietal structure on X.

Example 4.11. The standard basis for $\mathbb{R}^n$ defines an orientation of $\Delta(n)$; combining this with Example 3.11 yields the combinatorial data $(\Delta(n), -1)$ for $\mathbb{C}P^n$. The data actually arises from the normal fan of $\Delta(n)$, so Example 4.10 applies, and the corresponding omniorientation agrees with that induced by the complex structure on $\mathbb{C}P^n$. The omniorientation may be altered by conjugating the jth facial bundle for any $1 \leqslant j \leqslant n+1$, which has the effect of negating the jth column of Λ. For $j \leqslant n$, restoring the dicharacteristic to refined form involves replacing the refined submatrix by the column vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, where $\epsilon_i = -1$ for $i \neq j$, and $\epsilon_j = 1$. This procedure may be extended to any subset $J \subseteq [n+1]$. We write the result as P_ϵ, to emphasise the omniorientation; the resulting stably complex structure may be described by an isomorphism

$$\tau(P_\epsilon) \oplus \mathbb{C} \cong |J|\eta \oplus (n+1-|J|)\,\overline{\eta}.$$

Example 4.12. The standard basis for $\mathbb{R}^n$ defines an orientation of I^n; combining this with Example 3.12 yields the combinatorial data (I^n, D) for Q_n. The data arises from the normal fan of a polytope combinatorially equivalent to I^n,

so Example 4.10 applies, and the corresponding omniorientation agrees with that induced by the complex structure on Q_n. The omniorientation may be altered by conjugating the jth facial bundle for any $1 \leqslant j \leqslant 2n$, which has the effect of negating the jth column of Λ. For $j \leqslant n$, restoring the dicharacteristic to refined form involves negating the jth row of D. This procedure may be extended to any subset $J \subseteq [2n]$, although many of the resulting stably complex structures coincide, and several bound [6].

The bounding cases are no less natural to topologists than the projective algebraic varieties, and play an important role in complex cobordism theory [14]. For example, when Q_n is the n-fold product $S = (S^2)^n$, the combinatorial data (I^n, I_n) corresponds to the bounding structure given by $\eta \oplus \bar{\eta}$ on each cartesian factor.

Example 4.13. The orientations of Examples 4.11 and 4.12 describe an orientation for R^{r+s-1}; combining this with Example 3.13 yields the combinatorial data (R, E) for $B_{r,s}$. The data arises from the normal fan of a polytope combinatorially equivalent to R, so Example 4.10 applies, and the corresponding omniorientation agrees with that induced by the complex structure on $B_{r,s}$. This stably complex structure ensures that certain linear combinations of the $B_{r,s}$ form multiplicative generators for the complex cobordism ring Ω^U_* [4].

5. Connected Sums

In this section we review the construction of the connected sum for omnioriented quasitoric manifolds M' and M'', as was sketched in [7, 1.11] and realised in [5]. However, the orientations demanded by Proposition 4.5 were omitted in both descriptions, and we deal with them here in terms of signs associated to the vertices of P.

We denote the dicharacteristics associated to the omniorientations of M' and M'' by ℓ' and ℓ'', with refined submatrices $\Lambda'_\star$ and $\Lambda''_\star$ respectively; and assume that the associated orientations are given by orientations of the polytopes P' and P''. In addition, we let P' and P'' be finely ordered by o' and o'', with initial vertices $v'_\star$ and $v''_\star$ respectively.

The *connected sum* $P' \#_{v'_\star, v''_\star} P''$ may be described informally as follows. First construct the polytope Q' by deleting the interior of the vertex figure $P'_{v'_\star}$ from P'; so Q' has one new facet $\Delta(v'_\star)$ (which is an $(n-1)$-simplex), whose incident facets are ordered by o'. Then construct the polytope Q'' from P'' by the same procedure. Finally, glue Q' to Q'' by identifying $\Delta(v'_\star)$ with $\Delta(v''_\star)$, in such a way that the jth facet of Q' combines with the jth facet of Q'' to give a single new facet for each $1 \leqslant j \leqslant n$. The gluing is carried out by applying appropriate projective transformations to Q' and Q''. Precise details are given in [5, Section 6].

The combinatorial type of the connected sum may be changed, for example, by choosing alternative fine orderings on P' and P''. So long as the choices are clear, or their effect on the result is irrelevant, we use the abbreviation $P' \# P''$. The face lattice $\mathfrak{L}_F(P' \# P'')$ is obtained from $\mathfrak{L}_F(P') \cup \mathfrak{L}_F(P'')$ by identifying the jth facets of each, for $1 \leqslant j \leqslant n$. In particular,

$$q(P' \# P'') = q(P') + q(P'') - 2 \quad \text{and} \quad m(P' \# P'') = m(P') + m(P'') - n. \tag{5.1}$$

By definition, the *connected sum* $M' \#_{x'_\star, x''_\star} M''$ is the quasitoric manifold constructed over $P' \# P''$ using the dicharacteristic $\ell_\# \colon T^{m'+m''-n} \to T^n$ associated to the matrix

$$
\Lambda_\# = \begin{pmatrix}
1 & 0 & \dots & 0 & \lambda'_{1,n+1} & \dots & \lambda'_{1,m'} & \lambda''_{1,n+1} & \dots & \lambda''_{1,m''} \\
0 & 1 & \dots & 0 & \lambda'_{2,n+1} & \dots & \lambda'_{2,m'} & \lambda''_{2,n+1} & \dots & \lambda''_{2,m''} \\
\hdotsfor{10} \\
0 & 0 & \dots & 1 & \lambda'_{n,n+1} & \dots & \lambda'_{n,m'} & \lambda''_{n,n+1} & \dots & \lambda''_{n,m''}
\end{pmatrix}. \tag{5.2}
$$

Of course $\Lambda_\#$ is no longer in refined form, since the first n facets of $P' \# P''$ have empty intersection. Nevertheless, we may finely order $P' \# P''$ by choosing the *second* vertex of P' as initial vertex, and applying the procedure described immediately before Example 3.11.

By construction, $M' \# M''$ is equivariantly diffeomorphic to the equivariant connected sum of M' and M'' at their initial fixed points. If M' and M'' are omnioriented, the only possible obstruction to defining a compatible omniorientation of $M' \# M''$ involves the associated orientations. We deal with this issue in Proposition 5.3 below.

We write $p' \colon M' \# M'' \to M'$ and $p'' \colon M' \# M'' \to M''$ for the maps collapsing the connected sum onto its constituent manifolds.

We recall from [10] that an omniorientation attaches a *sign* $\sigma(w)$ to every vertex w of the quotient polytope P (or, equivalently, to every fixed point of M). By definition, $\sigma(w) = \pm 1$ measures the difference between the orientations induced on the tangent space at w by the dicharacteristic and the fundamental class μ_M respectively. When w is $F_{i_1} \cap \cdots \cap F_{i_n}$, the former is given by the Chern class $c_n(\rho_{i_1} \oplus \cdots \oplus \rho_{i_n})$. So we have that

$$
\sigma(w) = \langle u_{i_1} \cdots u_{i_n}, \mu_M \rangle.
$$

Proposition 5.3. *The connected sum $M' \#_{x'_\star, x''_\star} M''$ admits an orientation compatible with those of M' and M'' if and only if $\sigma(v'_\star) = -\sigma(v''_\star)$.*

Proof. The facets of $P' \# P''$ give rise to complex line bundles ξ_i, ξ'_j and ξ''_k over $M' \# M''$, corresponding to the columns of (5.2). We denote their first Chern classes by

$$
c_1(\xi_i) = w_i, \quad c_1(\xi'_j) = w'_j, \quad \text{and} \quad c_1(\xi''_k) = w''_k
$$

in $H^2(M' \# M'')$, for

$$
1 \leqslant i \leqslant n, \quad n+1 \leqslant j \leqslant m', \quad \text{and} \quad n+1 \leqslant k \leqslant m''
$$

respectively. The relations (3.10) become

$$
w_i = -\lambda'_{i,n+1} w'_{n+1} - \cdots - \lambda'_{i,m'} w'_{m'} - \lambda''_{i,n+1} w''_{n+1} - \cdots - \lambda''_{i,m''} w''_{m''},
$$

which imply that

$$
w_i = p'^* u'_i + p''^* u''_i \quad \text{for } 1 \leqslant i \leqslant n.
$$

Since the first n facets of $P' \# P''$ do not define a vertex, it follows that $w_1 \cdots w_n = 0$ in $H^{2n}(M' \# M'')$, and

$$
(p'^* u'_1 + p''^* u''_1) \cdots (p'^* u'_n + p''^* u''_n) = p'^*(u'_1 \cdots u'_n) + p''^*(u''_1 \cdots u''_n) = 0.
$$

For any choice of fundamental class in $H_{2n}(M' \# M'')$, we deduce that

$$\langle u_1' \cdots u_n', \, p_*' \mu_{M' \# M''} \rangle + \langle u_1'' \cdots u_n'', \, p_*'' \mu_{M' \# M''} \rangle = 0.$$

But the corresponding orientation of $M' \# M''$ is compatible with those of M' and M'' if and only if $p_*' \mu_{M' \# M''} = \mu_{M'}$ and $p_*'' \mu_{M' \# M''} = \mu_{M''}$; that is, if and only if

$$\sigma(v_\star') + \sigma(v_\star'') = 0,$$

as required. $\qquad \square$

Corollary 5.4. *Let M' and M'' be omnioriented quasitoric manifolds over finely ordered polytopes P' and P'' respectively, with $\sigma(v_\star') = -\sigma(v_\star'')$; then the stably complex structure induced on $M' \#_{x_\star', x_\star''} M''$ by Proposition 4.5 and Proposition 5.3 is equivalent to the connected sum of those induced on M' and M''. Moreover, the associated complex cobordism classes satisfy*

$$[M' \# M''] = [M'] + [M''].$$

Proof. The stably complex structures on M' and M'' combine to give an isomorphism

$$\tau(M' \# M'') \oplus \mathbb{R}^{2(m'+m''-n)} \cong \xi_1 \oplus \cdots \oplus \xi_n \oplus \xi_{n+1}' \oplus \cdots \oplus \xi_{m'}' \oplus \xi_{n+1}'' \oplus \cdots \oplus \xi_{m''}''. \quad (5.5)$$

As explained in [5, Theorem 6.9], the isomorphism (5.5) belongs to one of the two equivalence classes specified by Proposition 4.5 over $M' \# M''$. The choice of orientation is then provided by Proposition 5.3.

The equation of cobordism classes follows immediately, because the connected sum is cobordant to the disjoint union. $\qquad \square$

Proposition 5.3 implies that we cannot always form the connected sum of two omnioriented quasitoric manifolds. If the sign of every vertex of P is positive, for example, then it is impossible to construct $M \# M$ directly; we illustrate this situation in Example 5.6 below.

Corollary 5.4 confirms that the complex cobordism class $[M' \# M'']$ is independent of the fine orderings o' and o'', and therefore of the initial vertices.

Example 5.6. For any non-singular projective toric variety, it follows from Example 4.10 that the dicharacteristic and orientation both arise from the complex structure on X. So they are compatible, and every vertex of P has sign $+1$.

Example 5.7. Example 4.12 exhibits an omniorientation of S, defined by the combinatorial data (I^n, I_n), which induces a bounding stably complex structure. The signs of the vertices of I^n (in the notation of Example 2.5) are given by

$$\sigma(\delta_1, \ldots, \delta_n) = (-1)^{\delta_1} \ldots (-1)^{\delta_n}.$$

So adjacent vertices have opposite sign, and both occur with frequency n.

We are now in a position to prove Lemma 5.8, which emphasises an important principle; however unsuitable a quasitoric manifold M may be for the formation of

connected sums, a good alternative representative always exists within the complex
cobordism class $[M]$.

Lemma 5.8. *Let M be an omnioriented quasitoric manifold of dimension > 2
over a finely ordered polytope P; then there exists an omnioriented M' over a finely
ordered polytope P', such that $[M'] = [M]$ and P' has at least two vertices of
opposite sign.*

Proof. Suppose that $v_\star$ is the initial vertex of P. Let S be the omnioriented product
of 2-spheres of Example 5.7, with initial vertex $w_\star$.

If $\sigma(v_\star) = -1$, define M' to be $S \#_{w_\star,v_\star} M$ over $P' = I^n \#_{w_\star,v_\star} P$. Then
$[M'] = [M]$, because S bounds; moreover, adjacent pairs of non-initial vertices of
I^n have opposites signs, which survive under the formation of P', as sought. If
$\sigma(v_\star) = +1$, we make the same construction using the opposite orientation of I^n
(and therefore of S). Since $-S$ also bounds, the same conclusions hold. In either
case, P' may be finely ordered as described above; its initial vertex corresponds to
$(0, \ldots, 0, 1)$ in I^n. $\square$

We may now complete the proof of our amended [5, Theorem 6.11].

Theorem 5.9. *In dimensions > 2, every complex cobordism class contains a qua-
sitoric manifold, necessarily connected, whose stably complex structure is induced
by an omniorientation, and is therefore compatible with the action of the torus.*

Proof. Following [5], we consider cobordism classes $[M_1]$ and $[M_2]$ in Ω_n^U, repre-
sented by omnioriented quasitoric manifolds over quotient polytopes P_1 and P_2
respectively. It then suffices to construct a third such manifold M such that
$[M] = [M_1] + [M_2]$, because a set of quasitoric additive generators for Ω_n^U is given
by [4] for all $n > 0$.

Firstly, we follow Lemma 5.8 and replace M_2 by M_2' over $P_2' = I^n \# P_2$. Then
we finely order P_2' so as to ensure that its initial vertex has opposite sign to that
of P_1, thereby guaranteeing the construction of $M_1 \# M_2'$ over $P_1 \# P_2'$. The re-
sulting omniorientation defines the required cobordism class, by Corollary 5.4 and
Lemma 5.8. $\square$

We refer to the polytope $P_1 \# I^n \# P_2$ of Theorem 5.9 as the *box sum* $P_1 \square P_2$ of P_1
and P_2, because of the intermediate cube. The fact that we have replaced $P_1 \# P_2$
by $P_1 \square P_2$ in the proof of Theorem 5.9 does not affect the following observation
of [5]: for any complex cobordism class, the quotient polytope of a representing
quasitoric manifold may be chosen to be a connected sum of products of simplices.

Combining Theorem 5.9 with the details of Lemma 4.2 and the quadratic de-
scription (3.3) of $\mathcal{Z}_P$ leads to the following interesting conclusion.

Theorem 5.10. *Every complex cobordism class may be represented by the quotient
of a free torus action on a real quadratic complete intersection.*

One further deduction from Theorem 5.9 is the result of [14], that every complex
cobordism class contains a representative whose stable tangent bundle is a sum of
line bundles.

6. Examples and Concluding Remarks

We were taught the importance of adding an orientation to the original definition of omniorientation by certain 4-dimensional examples of Feldman [8]. In this section we describe and develop his examples (noting that 4 is the smallest dimension to which Proposition 5.3 is relevant). They lead to our concluding remarks concerning higher dimensions.

We shall use a result of [10], which identifies the top Chern number of any $2n$-dimensional omnioriented quasitoric manifold as

$$c_n(M) = \sum_w \sigma(w). \tag{6.1}$$

For any quotient polytope P, it is also convenient to refine the notation of (5.1) by writing

$$q(P) = q_+(M) + q_-(M),$$

where $q_\pm(M)$ denotes the number of vertices with sign ± 1 respectively. These numbers are preserved by any θ-equivariant diffeomorphism which respects omniorientations.

When $n = 2$, the complex cobordism class $[\mathbb{C}P^2]$ of the standard complex structure of Example 3.11 is an additive generator of the cobordism group $\Omega_4^U \cong \mathbb{Z}^2$, with $c_2(\mathbb{C}P^2) = 3$ and $q_-(\mathbb{C}P^2) = 0$. Each of the other three omniorientations of Example 4.11 represents the class $[\mathbb{C}P^2] - 4[\mathbb{C}P^1]^2$ (which is an independent additive generator), and $q_-(P_\epsilon)$ is given by the number of negative entries in the relevant ϵ; in other words, it is 1, 1, or 2.

The question then arises of representing $2[\mathbb{C}P^2]$ by an omnioriented quasitoric manifold M. We cannot expect to use $\mathbb{C}P^2 \# \mathbb{C}P^2$ for M, because no vertices of sign -1 are available in $\Delta(2)$, as required by Proposition 5.3. Moreover, M must satisfy $c_2(M) = 6$, by additivity, so the quotient polytope P has 6 or more vertices; as observed by Feldman, it follows that P cannot be $\Delta(2) \# \Delta(2)$, which is a square! So we proceed by appealing to Lemma 5.8, and replace the second copy of $\mathbb{C}P^2$ by the omnioriented quasitoric manifold $(-S) \# \mathbb{C}P^2$ over $P' = I^2 \# \Delta(2)$. Of course $(-S) \# \mathbb{C}P^2$ is cobordant to $\mathbb{C}P^2$, and P' is a pentagon. These observations lead naturally to our second example.

Example 6.2. The omnioriented quasitoric manifold $\mathbb{C}P^2 \#(-S)\#\mathbb{C}P^2$ represents $2[\mathbb{C}P^2]$, and lies over the box sum $\Delta(2) \,\square\, \Delta(2)$, which is a hexagon. Figure 1 illustrates the procedure diagramatically, in terms of dicharacteristics and orientations. Every vertex of the hexagon has sign 1.

On the other hand, $[\mathbb{C}P^1]^2$ is also a generator of Ω_4^U. It is represented by $(\mathbb{C}P^1)^2$ with the standard complex structure, which has second Chern number 4 and may certainly be realised over the square.

Our third example shows a related 4-dimensional situation in which the connected sum of the quotient polytopes *does* support a suitable orientation.

Example 6.3. Let $\overline{\mathbb{C}P^2}$ denote the quasitoric manifold determined by the combinatorial data $(\overline{\Delta}(2), -1)$, whose quotient polytope is the standard 2-simplex with
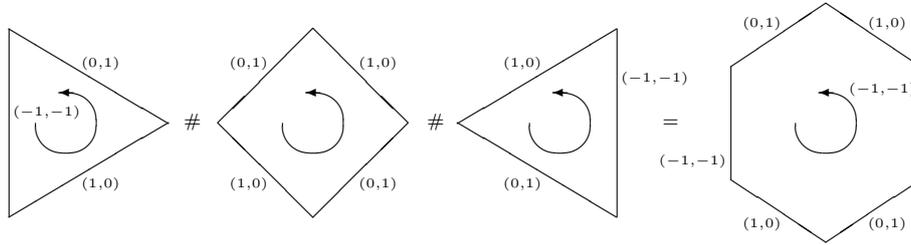
FIGURE 1. The omnioriented connected sum $\mathbb{C}P^2 \# (-S) \# \mathbb{C}P^2$

opposite orientation. Every vertex has sign -1, and we may construct $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ as an omnioriented quasitoric manifold over $\Delta(2) \# \Delta(2)$. Figure 2 illustrates the procedure diagramatically, in terms of dicharacteristics and orientations.
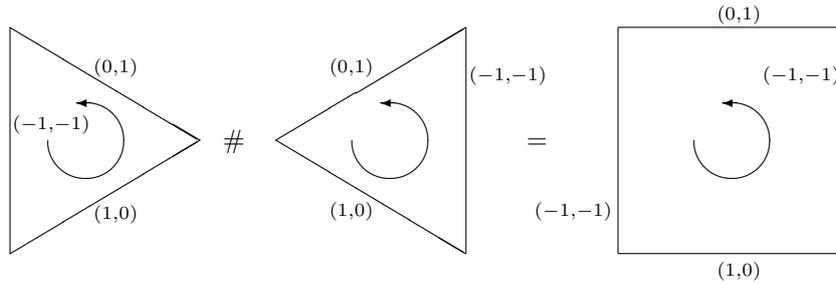
FIGURE 2. The omnioriented connected sum $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

Of course $[\overline{\mathbb{C}P^2}] = -[\mathbb{C}P^2]$. So $[\mathbb{C}P^2] + [\overline{\mathbb{C}P^2}] = 0$ in Ω_4^U, and the resulting manifold bounds by Proposition 5.4.

One other observation on 2-dimensional box sums is also worth making. Given k'- and k''-gons P' and P'' in $\mathbb{R}^2$, it follows from (5.1) that

$$q(P' \square P'') = q(P') + q(P'') \quad \text{and} \quad m(P' \square P'') = m(P') + m(P'').$$

Thus $q(P' \square P'') = m(P' \square P'') = k' + k''$. So $P' \square P''$ is a $(k' + k'')$-gon, and is combinatorially equivalent to the Minkowski sum $P' + P''$ whenever P' and P'' are in general position.

A situation similar to that of Example 6.2 arises in higher dimensions, when we consider the problem of representing complex cobordism classes by non-singular projective toric varieties. For any such V, the top Chern number coincides with the Euler characteristic, and is therefore equal to the number of vertices of the quotient polytope P; so $q_-(V) = 0$, by (6.1). Moreover, the Todd genus satisfies $Td(V) = 1$.

Omnioriented quasitoric manifolds with $q_-(V) = 0$ form an interesting generalisation of non-singular projective toric varieties, as shown by Example 6.5.

Remark 6.4. Suppose that smooth projective toric varieties V_1 and V_2 are of dimension $\geqslant 4$, and have quotient polytopes P_1 and P_2 respectively. Then $c_n(V_1) = q(P_1)$ and $c_n(V_2) = q(P_2)$, yet $q(P_1 \# P_2) = q(P_1) + q(P_2) - 2$, from (5.1). Since c_n is additive, no omnioriented quasitoric manifold over $P_1 \# P_2$ can possibly represent $[V_1] + [V_2]$. This objection vanishes for $P_1 \,\square\, P_2$, because it enjoys an additional $2^n - 2$ vertices.

The fact that no smooth projective toric variety can represent $[V_1] + [V_2]$ follows immediately from the Todd genus.

Example 6.5. For any non-negative integers r and s such that $r + s > 0$, the cobordism class $r[\mathbb{C}P^2] + s[\mathbb{C}P^1]^2$ is represented by an omnioriented quasitoric manifold $M(r, s)$. Its quotient polytope is the iterated box sum

$$P(r, s) = (\square^r \Delta(2)) \,\square\, (\square^s I^2),$$

and $q_-(M(r, s)) = 0$. Applying the Todd genus once more, we deduce that $M(r, s)$ cannot be cobordant to any smooth toric variety, so long as $(r, s) \neq (1, 0)$ or $(0, 1)$.

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