On application of the Hamilton formalism in fibred manifolds to field theory

G. Sardanashvily and O. Zakharov

Department of Theoretical Physics, Moscow State University, Moscow, Russia

Communicated by D. Krupka
Received 28 November 1991
Revised 7 September 1992


Abstract: The Hamiltonian formalism in fibred manifolds is formulated intrinsically in the terms of Hamiltonian jet fields. It is applied to degenerate field systems and physical systems described by composite fibred manifolds.

Keywords: Multisymplectic structure, multimomentum Hamiltonian formalism, jet manifold, gauge theory.

MS classification: 53C80, 58A20.

1. Introduction

The Hamilton formalism in fibred manifolds is the multimomentum generalization of the familiar Hamilton formalism to the case of a fibred manifold $\pi : E \rightarrow X$ over a $n$-dimensional manifold $X$ [1, 2]. It was generalized to degenerate Lagrangian systems [8]. In this case, multimomentum Hamiltonian forms are not derived from Lagrangians. To define them, we have introduced the generalized Liouville form and the canonical splitting of a multimomentum Hamiltonian form by means of a connection on $E$ induced by this form [5]. In this article, we develop the intrinsic formulation of the above-mentioned Hamilton formalism in the terms of Hamiltonian jet fields which play the same role as Hamiltonian vector fields in the familiar Hamilton formalism.

By a bundle, we mean a locally trivial fibred manifold. The tangent bundle $TM$ and the cotangent bundle $T^*M$ over a manifold $M$ are provided with the induced bundle coordinates. By $VE$ and $V^*E$, we denote the vertical tangent and cotangent bundles over a fibred manifold $E$. For the sake of simplicity, the pullback bundles $\pi^*(TX)$ and $\pi^*(T^*X)$ over $E$ are denoted by $TX$ and $T^*X$.

Given a fibred manifold $E$, let $J^1E$ be the first order jet manifold associated with $E$. By $E^{01}$ and $E^1$, we denote the the jet bundle $\pi_{01} : J^1E \rightarrow E$ and the fibred manifold
The Legendre bundle associated with $E$ is
$$
\pi_\Pi : \Pi = \bigwedge^n T^* X \otimes T X \otimes V^* E \rightarrow E.
$$

By $\Pi_1$, we denote the fibred manifold $\pi_{1X} : \Pi \rightarrow X$.

A fibred manifold $E$ is provided with an atlas of fibred coordinates $(x^\lambda, y^i)$. Correspondingly, $(x^\lambda, y^i, y^i_\lambda)$ are affine fibred coordinates of $J^1 E$ and $(x^\lambda, y^i, p^i_\lambda)$ are adapted fibred coordinate of $\Pi$.

Sections of $E^{01}$ are called jet fields on $E$. Global jet fields are connections on $E$.

The generalized Liouville form on the Legendre bundle $\Pi$ is defined to be the canonical bundle monomorphism
$$
\theta : \Pi \rightarrow \bigwedge^{n+1} T^* E \otimes T X,
$$

$$
\theta = -p^i_\lambda dy^i \wedge \omega \otimes \partial_\lambda, \quad \omega = dx^1 \wedge \ldots \wedge dx^n. \tag{1}
$$

The corresponding symplectic form on $\Pi$ is the vertical differential

$$
\Omega = -dv \theta = dp^i_\lambda \wedge dy^i \wedge \omega \otimes \partial_\lambda. \tag{2}
$$

Let $\widetilde{\Gamma}$ be a jet field on $\Pi_1$. Our formulation of the Hamilton formalism in fibred manifolds is based on the notions of a Hamiltonian jet field $\widetilde{\Gamma}$ and a multimomentum Hamiltonian form $H$ which satisfy the relation $\widetilde{\Gamma} \cup \Omega = dH$.

If sections of a fibred manifold $E$ describe classical fields, one can apply the Hamilton formalism in fibred manifolds to field theory [6, 7]. In this work, we develop the covariant Hamiltonian formulation of gauge theory of principal connections on a principal bundle $P$ with a structure group $G$. They are represented by sections of the bundle $J^1 P/G$. The crucial point consists in constructing a connection on this bundle. The Hamilton formalism in fibred manifolds enables us to consider Hamiltonian systems on composite fibred manifolds. In analytical mechanics these are systems with variable parameters. In field theory, composite fibred manifolds describe spontaneous symmetry breaking [4], e.g., in gravitation theory [6].

2. Technical preliminaries

We assume that all morphisms are of class $C^\infty$ and manifolds are real, Hausdorff, finite-dimensional, second-countable and connected.

Transition functions between coordinate charts of $J^1 E$ and $\Pi$ read

$$
x^{i\lambda} = f^\lambda(x^\mu), \quad y^i = f^i(x^\mu, y^j),
$$

$$
y^i_\lambda = (y^i_j \partial_j f^i + \partial_\mu f^i) \partial_\lambda f^\mu,
$$

$$
p^i_\lambda \partial_j f^i = Jp^i_\lambda \partial_\mu f^\lambda, \quad J^{-1} = \det(\partial_\mu f^\lambda).
$$

One says that a fibred manifold $E$ admits the vertical splitting if $VE$ is isomorphic to the pullback bundle
$$
\pi^*(E) = E \times \tilde{E} \tag{3}
$$
where $\mathcal{E}$ is a vector bundle over $X$. An affine bundle $E$ modelled on a vector bundle $\mathcal{E}$ admits the canonical vertical splitting (3).

By forms on a manifold $M$, we mean differential forms

$$\phi : M \rightarrow \bigwedge^r T^*M \otimes B$$

taking their values in a vector bundle $B$ over $M$. If $B = TM$, they are termed the tangent-valued forms. If $M = E$ is a fibred manifold, we consider the horizontal forms

$$\phi : E \rightarrow \bigwedge^r T^*E \otimes B.$$ 

If $r = n$, they are called the horizontal densities. A vertical-valued horizontal 1-form $(B = \mathcal{E})$ is called a soldering form on $E$.

Let $E'$ be a vector bundle over $X$. Every bundle morphism

$$\phi : E \rightarrow \bigwedge^m T^*E \otimes E'$$

(4)

can be represented by a $\pi^* (E')$-valued horizontal $m$-form on $E$ which we call a pullback-valued form and denote by the same symbol $\phi$. Conversely, every pullback-valued form on $E$ corresponds to some morphism (4).

Given $J^1E$, there exist the canonical morphisms

$$\theta_1 : E^{01} \rightarrow T^*X \otimes TE, \quad \theta_1 = dx^\lambda \otimes \partial_\lambda = dx^\lambda \otimes (\partial_\lambda + y_\lambda^i \partial_i),$$

$$\theta_2 : E^{01} \rightarrow T^*E \otimes VE, \quad \theta_2 = dy^i \otimes \partial_\lambda = (dy_i^j - y_\lambda^j dx^\lambda) \otimes \partial_i.$$ 

They define the bundle monomorphisms

$$\hat{\theta}_1 : \pi_1^* (TX) \cong \partial_\lambda \rightarrow \hat{\partial}_\lambda = \partial_\lambda \perp \theta_1 \in \pi_{01}(TE),$$

$$\hat{\theta}_2 : \pi_1^* (VE) \cong dy^i \rightarrow \hat{dy}^i = \theta_2 \perp dy_i^j \in \pi_{01}^* (T^*E).$$

(5)

In particular, there exists the horizontal splitting of the exterior differential

$$d = d_H + d_V = dx^\lambda \partial_\lambda + \hat{dy}^i \partial_i$$

over $J^1E$. Using this splitting and the monomorphisms (5), one can define the vertical differential $d_V$ of the pullback-valued forms (4). If $E$ admits the vertical splitting (3), then $d_V \phi$ is a pullback-valued form

$$d_V \phi : E \rightarrow E^* \wedge \bigwedge^m T^*E \otimes E'.$$

We consider the repeated jet manifold $J^1J^1E = J^1E^1$, its semiholonomic submanifold $\tilde{J}^2E$ and the 2-order jet manifold $J^2E$. They are provided with the adapted coordinates $(x^\lambda, y^i, y_i^j, y_j^i_\lambda, y_{\lambda\mu})$, $(x^\lambda, y^i, y_i^j = y_i^j(\lambda), y_j^i_\lambda)$ and $(x^\lambda, y^i, y_i^j, y_j^i_\lambda = y_j^i(\lambda))$ respectively. We have the canonical splitting

$$\tilde{J}^2E = J^2E \oplus (\bigwedge^2 T^*E \otimes VE).$$
A jet field $\Gamma$ on $E$ can be represented by the horizontal form $\theta_1 \circ \Gamma$ on $E$ which we denote by the same symbol

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma_\lambda^\mu \partial_\mu).$$

If $\Gamma$ is a connection and $\sigma$ is a soldering form on $E$, then $\Gamma + \sigma$ is a connection on $E$. If $\Gamma$ and $\Gamma'$ are connections on $E$, then $\Gamma' - \Gamma$ is a soldering form on $E$.

A connection $\Gamma$ on $E$ generates the connection

$$V \Gamma : VE \to VJ^1 E = J^1 VE,$$

$$V \Gamma = dx^\lambda \otimes \left( \partial_\lambda + \Gamma_\lambda^i \frac{\partial}{\partial y^i} + \partial_j \Gamma_\lambda^i \frac{\partial}{\partial y_j^i} \right),$$

on the fibred manifold $VE \to X$ where $V \Gamma$ is the vertical tangent morphism to $\Gamma$. We have the induced connection

$$V^* \Gamma = dx^\lambda \otimes \left( \partial_\lambda + \Gamma_\lambda^i \frac{\partial}{\partial y^i} - \partial_j \Gamma_\lambda^i \frac{\partial}{\partial y_j^i} \right)$$

on the fibred manifold $V^* E \to X$ due to the fact that the vector bundle $J^1 V^* E \to J^1 E$ is dual to the vector bundle $J^1 VE \to J^1 E$.

Let $\pi : P \to X$ be a principal bundle with a structure finite-dimensional Lie group $G$. It possesses the canonical trivial vertical splitting:

$$\alpha : VP \to P \times g_l,$$

$$\text{pr}_2 \circ \alpha \circ \tau_m = J_m, \quad (6)$$

where $g_l$ is the left Lie algebra of the group $G$, $\{J_m\}$ is a basis for $g_l$ and $\tau_m$ denote the corresponding fundamental (left invariant) vector fields on $P$. Let $\{z_\alpha\}$ be an atlas of $P$ and $(a^m)$ be group parameters of $G$. The bundle $P$ is provided with the bundle coordinates

$$p^m(p) = a^m(g_p), \quad p = z_\alpha(\pi_p(p))g_p, \quad p \in P, \quad g_p \in G, \quad (7)$$

which are adapted to the vertical splitting (6).

A principal connection $A$ on $P$ is a section of the bundle $P^{01}$ which is equivariant under the canonical action of $G$ on $P$ on the right. In the coordinates (7), it is given by expressions

$$A = dx^\lambda \otimes (\partial_\lambda + A_\lambda^\mu(p)\tau_m),$$

$$A_\lambda^m(x^\mu, p^m)\tau_m = A_\lambda^m(x^\mu, 0) \text{ad} g_p^{-1}(\tau_m).$$

The corresponding principal connection form and the local connection 1-form read

$$\bar{A} = (\tau^m - A_\lambda^m(p)dx^\lambda)J_m,$$

$$\bar{A}_\kappa = -A_\lambda^m(x)J_m dx^\lambda, \quad A_\lambda^\kappa(x) = A_\lambda^\kappa(x, 0). \quad (8)$$
Let $E$ be a $P$-associated bundle with a standard fibre $V$. A principal connection $A$ on $P$ induces the associated principal connection $\Gamma$ on $E$. With respect to associated atlases of $P$ and $E$, $\Gamma$ takes the form

$$\Gamma^\lambda_i(y) = A^\lambda_i(x)L_m(y^i)$$

where $L_m$ are generators of the group $G$ acting on $V$ on the left.

Let $E$ be the composite fibred manifold

$$\pi_{EX} \circ \pi_{E\Sigma} : E \rightarrow \Sigma \rightarrow X$$

where $\Sigma$ is a bundle over $X$ and $E \rightarrow \Sigma$ is a bundle denoted by $E_\Sigma$. Its fibred coordinates are bundle coordinates

$$(x^\lambda, \sigma^m, y^i)$$

of $E_\Sigma$ where $(x^\lambda, \sigma^m)$ are bundle coordinates of $\Sigma$. The corresponding fibred coordinates of the jet manifolds $J^1 \Sigma$, $J^1 E_\Sigma$, and $J^1 E$ are $(x^\lambda, \sigma^m, \sigma^m_\lambda)$, $(x^\lambda, \sigma^m, y^i, \tilde{y}^i, \sigma^m_\lambda, y^i_\lambda)$ and $(x^\lambda, \sigma^m, y^i, \sigma^m_\lambda, y^i_\lambda)$ respectively.

Given a section $h$ of $\Sigma$ and a section $\phi_\Sigma$ of $E_\Sigma$, their composite

$$\phi = \phi_\Sigma \circ h$$

is a section of $E$. Conversely, all sections $\phi$ of $E$ are represented by composites (11).

There is the following canonical morphism of jet manifolds

$$\rho : J^1 \Sigma \times \Sigma \rightarrow J^1 E,$$

$$\rho(j^1 h, j^1 (\phi_\Sigma)) = j^1 (\phi_\Sigma \circ h),$$

$$y^i_\lambda \circ \rho = y^i_m \sigma^m_\lambda + \tilde{y}^i_\lambda.$$  \hfill (12)

It follows that, given connections

$$\Gamma = dx^\lambda \otimes (\partial_\lambda + \Gamma^\lambda_\mu \partial_\mu),$$

$$A_\Sigma = dx^\lambda \otimes (\partial_\lambda + \tilde{A}^\lambda_\mu \partial_\mu) + d\sigma^m \otimes (\partial_\mu + A_\mu^i \partial_i)$$

on $\Sigma$ and $E_\Sigma$, one can construct the connection

$$A = dx^\lambda \otimes [\partial_\lambda + \Gamma^\lambda_\mu \partial_\mu + (A_\mu^i \Gamma^\mu_\lambda + \tilde{A}^\lambda_\mu) \partial_i]$$

on $E$. We call it a Berry type connection because of the term

$$A_\mu^i \Gamma^\mu_\lambda \ dx^\lambda \otimes \partial_i.$$  \hfill (13)

3. Hamilton formalism in fibred manifolds

For the sake of simplicity, we further identify forms which differ from each other in the natural contraction.
Definition 3.1. The multimomentum Liouville form is defined to be the canonical bundle monomorphism (1) or the associated form
\[ \theta = \hat{\theta}_i(\theta) = -p^i dx^i \wedge \omega \otimes \hat{\theta}_\lambda = \pi^i_\lambda dy^i \wedge \omega \lambda - p^i_\lambda y^i_\lambda \omega, \quad \omega \lambda = \partial_\lambda \wedge \omega, \]
on the jet manifold \( J^1 \Pi_1 \) provided with the adapted coordinates \((x^\lambda, y^i_\lambda, \theta(y_\mu)), \pi^i_\mu \).

Definition 3.2. The multimomentum symplectic form is defined to be the \( \pi^i_\mu (TX) \)-valued form (2) or the associated horizontal form on \( J^1 \Pi_1 \):
\[ \Omega = \hat{\theta}_i(\Omega) = d \pi^i_\lambda \wedge dy^i \wedge \omega \lambda + \pi^i_\lambda dy^i \wedge \omega - \pi^i_\lambda dp^i_\lambda \wedge \omega = d\omega \wedge \theta. \]

Definition 3.3. Let \( \bar{\Gamma} \) be a jet field (resp. connection) on \( \Pi_1 \). We say that
\[ \bar{\Gamma} = dx^\lambda \otimes (\partial_\lambda + \bar{\Gamma}_i(\lambda) \partial_i + \bar{\Gamma}_\mu(\lambda) \partial_\mu) \]
is a Hamiltonian jet field (resp. Hamiltonian connection) if the form \( \hat{\Omega} \circ \bar{\Gamma} \) is exact.

Definition 3.4. A global exterior form \( H \) on the Legendre bundle \( \Pi \) is called a multimomentum Hamiltonian form if, on a neighborhood of each point of \( \Pi \), there exists a Hamiltonian jet field satisfying the relation
\[ \hat{\Omega} \circ \bar{\Gamma} = \bar{\Gamma} \wedge \Omega = dp^i_\lambda \wedge dy^i \wedge \omega \lambda + \pi^i_\lambda dy^i \wedge \omega - \pi^i_\lambda dp^i_\lambda \wedge \omega = dH. \quad (15) \]

Proposition 3.5. Hamiltonian jet fields for the same multimomentum Hamiltonian form \( H \) differ from each other in soldering forms \( \hat{\sigma} \) which obey the relations
\[ \hat{\sigma} \wedge \Omega = 0, \quad \hat{\sigma}_i(\lambda) = 0, \quad \hat{\sigma}_\mu(\lambda) = 0. \quad (16) \]
Conversely, for each exterior horizontal density \( \tilde{H} = \tilde{H}\omega \) on \( \Pi_1 \), the form \( H - \tilde{H} \) is a multimomentum Hamiltonian form.

Outline of proof. Let \( \tilde{H} \) be an exterior horizontal density. The algebraic equations
\[ \hat{\sigma} \wedge \Omega = (\hat{\sigma}_i(\lambda) dy^i - \hat{\sigma}_\mu(\lambda) dp^i_\mu) \wedge \omega = -d\tilde{H} \]
for the soldering form \( \hat{\sigma} \) always have a local solution \( \hat{\sigma}_i(\lambda) \). They have a global solution if \( E \) admits the vertical splitting.

Proposition 3.6. Multimomentum Hamiltonian forms and Hamiltonian connections exist on the Legendre manifold.

Outline of proof. Indeed, if \( \Gamma \) is a connection on \( E \), the associated form
\[ H_\Gamma = (\Gamma \circ \pi_\Pi) \wedge \theta = p^i_\lambda dy^i \wedge \omega \lambda - p^i_\lambda \Gamma^i_\lambda \omega \]
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is a multimomentum Hamiltonian form. The associated Hamiltonian connection is

$$\tilde{\Gamma} = dx^\lambda \otimes \left( \partial_\lambda + \Gamma^\lambda_{\mu} \partial_\mu + \left( -\partial_j \Gamma^i_{\lambda j} p_\mu^i - A^\mu_{\nu \lambda} p_\mu^\nu + A^\sigma_{\alpha \lambda} p_\mu^\sigma \partial_\mu \right) \right).$$

This is a connection on $\Pi_1$ induced by the connection $V^*\Gamma$ on $V^*E$ and a linear symmetric connection $A$ on $TX$:

$$A^\mu_{\nu \lambda} = -A^\mu_{\nu \lambda}(x)x^\nu, \quad A^\nu_{\mu \lambda}(x) = A^\mu_{\nu \lambda}(x).$$

In view of Propositions 3.5 and 3.6, we further restrict our consideration to multimomentum Hamiltonian forms

$$H = H_\Gamma - \tilde{H} = p_i^\lambda dy^i \wedge \omega_\lambda - \mathcal{H} \omega = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Gamma^j_{\lambda j} \omega - \tilde{\mathcal{H}} \omega$$

where $\Gamma$ is a connection on $E$. Given another connection $\Gamma' = \Gamma + \sigma$, we have

$$\tilde{H}' = \tilde{H} - p_i^\lambda \sigma^j_{\lambda j} \omega.$$

The vertical differential of a multimomentum Hamiltonian form $H$ defines the associated momentum morphism

$$\tilde{H} : \Pi \rightarrow E^{01},$$

$$(x^\lambda, y^i, y_\lambda^i) \circ \tilde{H} = (x^\lambda, y^i, \partial_\lambda \mathcal{H}),$$

and the associated connection $\Gamma_H = \tilde{H} \circ \tilde{\sigma}$ on $E$ where $\tilde{\sigma}$ is the global zero section of $\Pi$. As a consequence, there is the canonical splitting

$$H = H_\Gamma - \tilde{H}.$$

Conversely, every bundle morphism $\Phi : \Pi \rightarrow E^{01}$ over $\text{Id}E$ defines the associated multimomentum Hamiltonian form

$$H_\Phi = (\mathcal{H} \circ \Phi) \circ \tilde{\theta} = p_i^\lambda dy^i \wedge \omega_\lambda - p_i^\lambda \Phi^j_{\lambda j} \omega.$$

For instance, if a multimomentum Hamiltonian form $H$ satisfies the condition $H\tilde{\Gamma} = H$, then $\tilde{H} = \Gamma \circ \pi_\Pi$ and $H = H_\Gamma$ for some connection $\Gamma$ on $E$.

Given a multimomentum Hamiltonian form $H$, we introduce the associated Hamilton operator

$$\mathcal{E}_H = dH - \tilde{\Omega} : \Pi^{01} \rightarrow \wedge^{n+1} T^*\Pi_1,$$

$$\mathcal{E}_H = (y_\lambda^i - \partial^i_{\lambda} \mathcal{H}) dp_i^\lambda \wedge \omega - (p_i^\lambda + \partial_\lambda \mathcal{H}) dy^i \wedge \omega.$$

The relation (15) then takes the form of the Hamilton equations

$$\mathcal{E}_H \circ \tilde{\Gamma} = dH - \tilde{\Gamma} \cdot \Omega = 0,$$

$$\tilde{\Gamma}^i_{\lambda} = -\partial_\lambda \mathcal{H}, \quad \tilde{\Gamma}^i_{(\lambda)} = \partial^i_{\lambda} \mathcal{H}. \quad (17)$$
It follows that a Hamiltonian jet field has the form
\[ \tilde{\Gamma} = dx^\lambda \otimes (\partial_\lambda + \partial_i^\lambda \partial_t + \Gamma_i^\mu_\alpha \partial_t^\mu). \]  
(18)
In particular, if \( \tilde{\Gamma} \) is a Hamiltonian connection, we have \( j^r \pi_\Pi \circ \tilde{\Gamma} = \tilde{\Pi} \).

Let a Hamiltonian jet field \( \tilde{\Gamma} \) for a multimomentum Hamiltonian form \( H \) has an integral section \( r \) of \( \Pi_1 \), that is, \( \Gamma \circ r = j^r r \) where \( j^r r \) denotes the first order jet prolongation of \( r \). Then, \( r \) is a solution of the Hamilton equations
\[ r^*(u \cup dH) = 0 \]  
(19)
for every vertical vector field \( u \) on \( \Pi_1 \).

**Proposition 3.7.** Let \( r \) be a local solution of the Hamilton equations (19) on a closed subset \( N \subset X \). Then, there exists a local Hamiltonian jet field which has the integral section \( r \).

**Outline of proof.** Since local sections of \( \text{Ker} \, \mathcal{E}_H \) exist and differ from each other in soldering forms satisfying the relations (16), \( \text{Ker} \, \mathcal{E}_H \) is locally an affine subbundle of \( \Pi_1^0 \). Then, a section \( r(N) \rightarrow j^1 r(N) \) of \( \text{Ker} \, \mathcal{E}_H \) on \( r(N) \) can be extended to a local Hamiltonian jet field on an open subset of \( \Pi_1 \).

Let us consider multimomentum Hamiltonian forms associated with Lagrangians.

Given a fibred manifold \( E \), a first order Lagrangian \( L \) is defined to be the bundle morphism
\[ L : E^{01} \rightarrow \Lambda^n T^* X, \quad L = \mathcal{L} \omega. \]
The vertical differential of \( L \) generates the corresponding Legendre morphism
\[ \hat{L} : E^{01} \rightarrow \Pi, \]
\[ (x^\lambda, y^i, p_i^\lambda) \circ \hat{L} = (x^\lambda, y^i, \pi_i^\lambda), \quad \pi_i^\lambda = \partial_i^\lambda L. \]
Given \( L \), the following forms associated with \( L \) are defined on \( E^{01} \):
(i) the pullback of the multimomentum Liouville form \( \theta \) by \( \hat{L} \):
\[ \theta_L = \theta \circ \hat{L} = -\pi_i^\lambda dy^i \wedge \omega \otimes \partial_\lambda, \]
(ii) the symplectic form \( \Omega_L = -d_V \theta_L = \Omega \circ \hat{L}, \)
(iii) The Poincaré-Cartan form
\[ \Xi_L = \theta_1 \cup \theta_L + L = \pi_i^\lambda dy^i \wedge \omega_\lambda - \pi_i^\lambda y^i \omega + L \omega. \]

Using the pullback of these forms by the projection \( \hat{J}^2 E \rightarrow J^1 E \), one can construct the Euler-Lagrange operator for \( L \):
\[ \mathcal{E}_L = d\Xi_L - \theta_1 \cup \Omega_L : \hat{J}^2 E \rightarrow \Lambda^{n+1} T^* E, \]
\[ \mathcal{E}_L = (\partial_t \circ \partial_i^\lambda \partial_\lambda) \mathcal{L} dy^i \wedge \omega, \]
\[ \theta_1 = dx^\lambda \otimes \partial_\lambda, \quad \partial_\lambda = \partial_\lambda + y^i \partial_i + y^i \partial_\lambda. \]
A jet field on $E^1$ is termed semiholonomic (resp. holonomic) if it takes its values in $J^2E$ (resp. $J^2E$). Given an Euler-Lagrange operator $\mathcal{E}_L$, a semiholonomic jet field

$$\Gamma = dx^\lambda \otimes (\partial_x + y^i_x \partial_i + \Gamma^i_{\mu,\lambda} \partial^\mu)$$

is called the Lagrangian jet field for $L$ if it satisfies the equation $\mathcal{E}_L \circ \Gamma = 0$.

Let $s$ be a local section of $E^1$ such that $j^1s$ is a holonomic jet field. Then, $s = j^1e$ for some local section $e$ of $E$. If $j^1s$ is a Lagrangian jet field for $L$, the section $e$ is a solution of the Euler-Lagrange equations

$$(j^2e)^* (u \cup \mathcal{E}_L) = 0$$

for any vertical vector field $u$ on $E$.

**Definition 3.8.** We say that a multimomentum Hamiltonian form $H$ is associated with a Lagrangian $L$ if it satisfies the relations

$$\widehat{\mathcal{L}} \circ \widehat{H}|_Q = \text{Id}Q, \quad Q = \widehat{\mathcal{L}}(J^1E),$$

$$H = H_{\hat{L}} + L \circ \hat{H}.$$ 

**Proposition 3.9.** Given a multimomentum Hamiltonian form $H$ associated with a Lagrangian $L$, we have

$$H|_Q = \Xi_L \circ \hat{H}|_Q.$$ 

The following theorems relate the multimomentum Hamiltonian formalism and the Lagrangian formalism for semiregular Lagrangians $L$, that is, $Q$ is a subbundle of $\Pi$ and $\hat{L}^{-1}(p), p \in Q$, is a connected submanifold of $J^1E$.

**Proposition 3.10.** Let a Lagrangian $L$ be semiregular and $H$ be an associated multimomentum Hamiltonian form. The pullback of the Hamilton operator $\mathcal{E}_H$ by the Legendre morphism $\hat{L}$ defines the Euler-Lagrange operator $\mathcal{E}_L$ for $L$:

$$\mathcal{E}_L = \mathcal{E}_H \circ j^1\hat{L}|_{J^2E} : \wedge^n T^*E \to \wedge^n T^*E.$$ 

**Outline of proof.** All multimomentum Hamiltonian forms associated with a semiregular Lagrangian $L$ are equal to each other on the constraint space $Q$ and $\Xi_L = H \circ \hat{L}$ [5,8]. It follows that

$$(y^i_j - \partial_i \mathcal{H}(y^j, \pi^\mu_j) - \partial_j \mathcal{H}(y^i, \pi^\mu_i)) \omega - \partial_i \mathcal{L} + \partial_j \mathcal{H}(y^i, \pi^\mu_i)) = 0.$$ 

Recall that

$$(y^i_j, \pi^\mu_j) \circ j^1\hat{L}|_{J^2E} = (y^i_j, \hat{\partial}_\mu \pi^\lambda_j).$$

**Proposition 3.11.** Let $H$ be a multimomentum Hamiltonian form and $\hat{L}$ be the associated Legendre morphism. Every Hamiltonian jet field $\Gamma \subset J^1Q$ for $H$ defines a Lagrangian jet field for $\mathcal{E}_L$ on $Q' = (\hat{H} \circ \hat{L})(J^1E)$. 
Outline of proof. Given a Hamiltonian jet field $\Gamma$ for $H$, let us consider the local morphism
\[ j^1 \hat{H} \circ \hat{\Gamma} \circ \hat{L} : J^1 E_1 \to J^1 E_1. \] (21)
This morphism fails to be a jet field on $E_1$ since it is projected to $\hat{H} \circ \hat{L} \neq 1d E_1$. Being restricted to the subspace $Q'$, the morphism (21) defines a jet field $\Gamma$ on $Q'$. Using the relation (18), one can verify that $\Gamma$ is a semiholonomic jet field. Then, we have
\[ \mathcal{E}_L \circ \Gamma = \mathcal{E}_H \circ j^1 \hat{L} \circ (j^1 \hat{H} \circ \hat{\Gamma} \circ \hat{L})|_{Q'} = 0. \]

**Corollary 3.12.** Let a section $r$ of $\Pi_1$ takes its values in the image $Q$ of the Legendre morphism for a semiregular Lagrangian $L$. Let $r$ be a solution of the Hamilton equations (19) for some multimomentum Hamiltonian form $H$ associated with $L$. Then, the section $e = \pi_\Pi \circ r$ of $E$ satisfies the Euler-Lagrange equations (20) and $r = \hat{L}(j^1 e)$.

Outline of proof. If $r$ is a solution of the Hamilton equations, there is a Hamiltonian jet field $\Gamma$ such that $\hat{r} \circ r = j^1 r$. Using the relation (18), we have
\[ \hat{H} \circ r = j^1 \pi_\Pi \circ \hat{\Gamma} \circ \hat{r} = j^1 \pi_\Pi \circ j^1 r = j^1 e \]
where $e = \pi_\Pi \circ r$ is a section of $E$. If $r \subset Q$, we obtain the corollary of Proposition 3.11.

**Corollary 3.13.** Given a semiregular Lagrangian $L$, let $s$ be a section of $E_1$ such that $j^1 s$ is a Lagrangian jet field for $L$. Let $H$ be a multimomentum Hamiltonian form associated with $L$ so that
\[ (\hat{H} \circ \hat{L}) \circ s = s. \] (22)
Then, the section $\hat{L} \circ s$ of $\Pi_1$ satisfies the Hamilton equations (19).

One can show that, if the degeneracy rank of $\hat{L}$ is constant, there is a local momentum morphism $\hat{H}$ obeying the condition (22) [5, 8].

4. The $n = 1$ case and applications to analytical mechanics

If $X = \mathbb{R}^1$, the familiar Hamiltonian formalism can be reproduced. In this case, there are isomorphisms
\[ E = X \times F, \quad \Pi = X \times T^* F. \] (23)
We endow $E$ and $\Pi$ with the corresponding coordinates $(x, y)$ and $(x, y^i, p_i = \dot{y}_i)$. Let $x = t$ be the coordinate of $X$ which is compatible with the bundle isomorphism $T^* X = X \times \mathbb{R}$. Then, every soldering form $\sigma$ on $\Pi_1$ can be given by the expression
\[ \sigma = JY \otimes dx = J(Y^i \partial_i + Y_i \partial^i) \otimes dx, \quad J = \frac{dt}{dx}. \]
where $Y$ is some vertical vector field on $\Pi$. Let $\Gamma_0$ be the trivial connection on $E$ corresponding to the trivialization (23). Let 

$$H = H_{\Gamma_0} - \tilde{H} = \dot{y}_i dy^i - \dot{\lambda} dt$$

be a multimomentum Hamiltonian form on $\Pi$. In the coordinates $(t, y^i, \dot{y}_i)$, the Hamiltonian equations (17) are reduced to the familiar Hamilton equation 

$$Y \wedge \Omega = -dV \tilde{H}, \quad \Omega = d\dot{y}_i \wedge dy^i,$$

for a Hamiltonian vector field $Y$ on $\mathbb{R} \times T^* F$. Here, $\tilde{\Omega}$ is the pullback of the symplectic form on $T^* X$ by the projection $\mathbb{R} \times T^* F \to T^* F$.

Let us examine the model of one-dimensional oscillator whose frequency is a fixed function $\sigma(t)$ of time $t \in \mathbb{R}$. It can be described by the composite bundle (9) where 

$$X = \mathbb{R}, \quad \Sigma = \mathbb{R} \times \mathbb{R}^+.$$

Let $\Gamma$ be a linear connection on the bundle $\Sigma$. It cannot be brought into zero by coordinate transformations since the coordinate $\sigma$ on $\Sigma$ is fixed by the function $\sigma(t)$. To define a connection $A_{\Sigma}$ on $E_{\Sigma}$, we follow the adiabatic hypothesis. It postulates that there exists a coordinate $y$ such that the above-mentioned oscillator differ from the standard oscillator only in the kinetic energy. In the coordinates $(t, \sigma, y)$, we therefore can choose $A_{\Sigma} = 0$, and the connection (13) on the composite bundle $E$ then reads 

$$A = dt \otimes (\partial_t + \Gamma_{\sigma} \partial_\sigma).$$

Let $\Pi$ be the corresponding Legendre bundle endowed with the adapted coordinates $(t, \sigma, y, p_\sigma, p_y)$. In accordance with the adiabatic hypothesis, we describe the above-mentioned oscillator by the Hamiltonian form 

$$H = p_\sigma d\sigma + p_y dy - \left( p_\sigma \Gamma_{\sigma}^{\sigma} + \frac{1}{2} \sigma^2 p_y^2 + \frac{1}{2} y'^2 \right) dt.$$ 

(25)

Let us consider the canonical transformation 

$$y = \sigma y', \quad p_y = \frac{1}{\sigma} p_y', \quad p_\sigma = p_\sigma' - \frac{1}{\sigma} y' p_y'$$

$$\Gamma_{\sigma}^{\sigma} = y' + \Gamma_{\sigma}^{\sigma} \sigma = 0, \quad \Gamma_{\sigma}^{\sigma} = \Gamma_{\sigma}^{\sigma}.$$ 

On the quantum level, it fails to be the transformation of physical equivalence. In the coordinates $(t, \sigma, y', p_\sigma, p_y')$, the connection (24) and the Hamiltonian form (25) read 

$$A = dt \otimes \left( \partial_t + \Gamma_{\sigma} \partial_\sigma - \Gamma_{\sigma}^{y'} \partial_{y'} \right),$$

$$H = p_\sigma' d\sigma + p_y' dy' - \left( p_\sigma' \Gamma_{\sigma}^{\sigma} - \frac{1}{\sigma} p_y' y' \Gamma_{\sigma}^{\sigma} + \frac{1}{2} p_y'^2 + \frac{1}{2} \sigma^2 y'^2 \right) dt.$$
The corresponding Hamilton equations take the form
\begin{align}
\partial_t p'_y &= \frac{1}{\sigma} p'_y - \sigma^2 y', \\
\partial_t y' &= -\frac{1}{\sigma} y' \Gamma_i^\sigma + p'_y, \\
\partial_t \sigma &= \Gamma_i^\sigma
\end{align}
(26a)  
(26b)  
(26c)
plus the equation for $p'_\sigma$. We require that the frequency function $\sigma(t)$ be a solution of these equations. Then, substituting (26c) into (26a) and (26b), we obtain that the equations (26a) and (26b) are the Hamilton equations corresponding to the Hamiltonian
\[ \mathcal{H} = -\frac{1}{\sigma} \partial_t \sigma y'_\Gamma + \frac{1}{2} p'_y^2 + \frac{1}{2} \sigma^2 y'^2. \]
This is the classical Hamiltonian of the well-known Berry’s oscillator and the term (14) is the Berry connection
\[ \tilde{A}_i^y = -\frac{1}{\sigma} \partial_t \sigma y'. \]
(27)

5. Yang-Mills theory

Let $P \to X$ be a principal bundle with a structure $\text{finite-dimensional Lie group } G$. There is 1:1 correspondence between principal connections $A$ on $P$ and global sections $A^C$ of the affine bundle $C = P^{01}/G$ modelled on the vector bundle
\[ \bar{C} = T^*X \otimes V^G P, \quad V^G P = VP/G. \]
There is the canonical vertical splitting $VC = C \times C$. The bundle $C$ is provided with the fibred coordinates $(x^\mu, k^m_\mu)$. A section $A^C$ of the bundle $C$ then has the coordinate expression
\[ (k^m_\mu \circ A^C)(x) = A^m_\mu(x) \]
where $A^m_\mu(x)$ are coefficients of the local connection 1-form (8). In gauge theory, sections $A^C$ are treated as gauge potentials.
A configuration space of gauge potentials is the jet manifold $J^1 C$. The affine bundle $C^{01}$ is modelled on the vector bundle
\[ T^*_C(C \times T^*X \otimes V^G P). \]
There exists the canonical splitting [3]
\[ J^1 C = C_+ \oplus C_- = (J^2 P/G) \oplus (\Lambda^2 T^*X \otimes V^G P) \]
(28)
where $C_+ \to C$ is the affine bundle modelled on the vector bundle
\[ \bar{C}_+ = \sqrt{2} T^*X \otimes V^G P. \]
The jet manifold $J^1C$ is provided with the adapted coordinates

\[(x^\mu, k^m_\mu, s^m_\mu, F^m_{\lambda\mu}) = (x^\mu, k^m_\mu, k^m_\lambda + \kappa^m_\lambda k^l_\mu, k^m_\lambda + \kappa^m_\lambda k^l_\mu - \epsilon^m_{\mu\lambda} k^l_\mu k^l_\mu)\]  

(29)

where $\epsilon^m_{\mu\lambda}$ are the structure constants of the right Lie algebra $g_r$. These coordinates are compatible with the splitting (28). In the coordinates (29), the conventional Yang-Mills Lagrangian $L(A)$ of gauge potentials is given by the expression

\[L(A) = \frac{1}{4\epsilon^2} a^G_{mn} g^{\beta\nu} F^m_{\lambda\beta} F^m_{\mu\nu} \sqrt{|g|} \omega, \quad g = \det g_{\mu\nu},\]  

(30)

where $a^G$ is a $G$-invariant metric in the Lie algebra $g$ and $g$ is a fibre metric in $T^*X$.

For gauge potentials, we have the Legendre manifold

\[\Pi = \Lambda^n T^*X \otimes TX \otimes V^*C = \Lambda^n T^*X \otimes [C \times C]^* \]  

\[= \Lambda^n T^*X \otimes [C^*_+ \oplus C^*_-_C]\]

provided with the adapted coordinates

\[(x^\mu, k^m_\mu, p^{(\mu\lambda)}_m) = \frac{1}{2}(p^\mu_\lambda + \kappa^m_\lambda k^l_\mu, p^\mu_\lambda = \frac{1}{2}(p^\mu_\lambda - p^\lambda_\mu)).\]

The Legendre morphism associated with the Lagrangian (30) is

\[\tilde{L}_{(A)}: J^1C \to Q = \Lambda^n T^*X \otimes C^*_C \subset \Pi,\]

\[p^{(\mu\lambda)}_m = 0, \quad (31a)\]

\[p^\mu_\lambda = \frac{1}{2} a^G_{mn} g^{\lambda\alpha} g_{\alpha\beta} F^m_{\alpha\beta} \sqrt{|g|}. \quad (31b)\]

The multimomentum Hamiltonian forms associated with the Lagrangian (30) read

\[H_B = p^{(\mu\lambda)}_m k^m_\mu \wedge \omega_\lambda - p^\mu_\lambda \Gamma^m_{\mu\lambda} \omega - \tilde{H} \omega, \]

\[\tilde{H} = \frac{1}{4\epsilon^2} a^G_{mn} g_{\mu\beta} p^\lambda_\alpha p^\mu_\beta \sqrt{|g|}^{-1/2}, \]

\[\Gamma^m_{\mu\lambda} = \frac{1}{2}[\epsilon^m_{\mu\lambda} k^l_\mu + \partial_\lambda B^m_\mu + \partial_\mu B^m_\lambda - \epsilon^m_{\mu\lambda} (k^l_\mu B^l_\lambda + k^l_\lambda B^l_\mu)] - \Gamma^\beta_{\mu\lambda} (B^m_\beta - k^m_\beta), \]

(32)

where $B$ is some section of $C$, $\tilde{\Gamma}$ is a connection on $C$ and $\Gamma^\beta_{\mu\lambda}$ are Christoffel symbols of a world metric $g$. We have

\[H_B |_Q = p^{(\mu\lambda)}_m k^m_\mu \wedge \omega_\lambda - \frac{1}{2} p^\mu_\lambda [\epsilon^m_{\mu\lambda} k^l_\mu k^l_\mu - \tilde{H} \omega, \]

\[\tilde{H} B \circ \tilde{L}_{(A)}: (x^\mu, k^m_\mu, s^m_\mu, F^m_{\lambda\mu}) \to (x^\mu, k^m_\mu, s^m_\mu, (x^\nu, k^m_\nu, F^m_{\lambda\nu})). \]

(33)

The Hamilton equations corresponding to the multimomentum Hamiltonian form (32) read

\[\partial_\lambda p^{(\mu\lambda)}_m = -\epsilon^m_{\mu\lambda} k^l_\mu p^\mu_\lambda + \epsilon^m_{\mu\lambda} B^l_\mu p^{(\mu\lambda)}_n - \Gamma^\mu_{\lambda\lambda} p^\mu_\lambda, \]

\[\partial_\lambda k^m_\mu + \partial_\mu k^m_\lambda = 2 \Gamma^m_{(\lambda\mu)} \]

(34)  

(35)
plus the equations (31b). The equations (31b) and (34) are the familiar Yang-Mills equations. The equation (35) plays the role of gauge condition. On the other hand, it shows that, given a solution \( k(x) \) of the Yang-Mills equations corresponding to the Lagrangian (30), there is the multimomentum Hamiltonian form (32) with \( B(x) = k(x) \) and the associated global morphism \( \tilde{H}_B \) such that the morphism (33) satisfies the condition (22) and \( k(x) \) is a solution of the corresponding Hamilton equations.

6. Spontaneous symmetry breaking

In classical field theory, spontaneous symmetry breaking is modelled by a classical Higgs field. Let \( P \) be a principal bundle with a structure Lie group \( G \) reducible to its closed subgroup \( K \). A classical Higgs field is represented by a global section \( h \) of the quotient bundle \( \Sigma = P/K \) with the standard fibre \( G/K \) on which the group \( G \) acts on the left. There is a correspondence between the Higgs fields \( h \) and the reduced \( K \)-subbundles \( P_h \) of \( P \). There are different types of spontaneous symmetry breaking. We examine the case of matter fields possessing only exact symmetries.

We shall say that sections \( \phi_h \) of a vector bundle \( E_h \) with a standard fibre \( V \) describe matter fields in the presence of a Higgs field \( h \) if \( E_h \) is associated with the reduced subbundle \( P_h \) of \( P \), that is,

\[
E_h = (P_h \times V)/K. 
\]  
(36)

A connection on \( E_h \) is assumed to be associated with a principal connection on \( P_h \).

Matter fields in the presence of different Higgs fields \( h \) and \( h' \) are described by sections of the matter bundles \( E_h \) and \( E_{h'} \) associated with the different reduced bundles \( P_h \) and \( P_{h'} \). There is no canonical isomorphism between \( E_h \) and \( E_{h'} \). Moreover, a principal connection \( A_h \) on \( P_h \) is extended to a principal connection \( \Gamma_h \) on \( P \) such that \( h \) is parallel with respect to \( \Gamma_h \), but \( \Gamma_h \) fails to induce a connection on \( P_{h'} \). It follows that matter fields and gauge potentials possessing only exact symmetries must be regarded only in a pair with a certain Higgs field.

We describe these pairs by sections of the composite fibred manifold (9) where

\[
E_\Sigma = (P \times V)/K
\]
is the bundle associated with the \( K \)-principal bundle \( P_\Sigma \) [4]. Indeed, there exists the canonical isomorphism of the bundle (36) to the portion \( \pi_{E_h}^{-1}(h(X)) \) of \( E_\Sigma \) over \( h(X) \), and this is inclusion of \( E_h \) into \( E \). As a consequence, there is 1:1 correspondence between sections of \( E_h \) and sections (11) of \( E_\Sigma \) such that \( \pi_{E_\Sigma} \circ \phi = h \). In the bundle coordinates of \( E_h \) induced by the bundle coordinates (10) of \( E_\Sigma \), we have \( \phi_h = \phi_\Sigma \circ h \).

Note that \( E \) can be provided with the structure of a \( P \)-associated bundle. Its standard fibre is the bundle \( \Lambda = (G \times V)/K \) associated with the \( K \)-principal bundle \( G \rightarrow G/K \). The structure group \( G \) of \( P \) acts on \( \Lambda \) by the law

\[
G \ni g : (G \times V)/K \rightarrow (gG \times V)/K.
\]
We however do not consider this structure because, in general, there is no canonical inclusion of the standard fibre $V$ of the bundles $E_h$ into $\Lambda$. For instance, the fibre coordinates (10) of $E$ are not compatible with the structure of $\Lambda$ as a $P$-associated bundle. As a consequence, we cannot use connections on $E$ associated with principal connections on $P$ in order to describe the above-mentioned pairs of matter and Higgs fields. At the same time, one can consider the Berry type connections (13) on $E$ if $\Gamma$ and $A_\Sigma$ are associated with principal connections on $P$ and $P_\Sigma$ respectively.

Let $\Pi$ be the Legendre bundle over $E$, it is provided with the adapted coordinates $(x^\lambda, \sigma^m, y^i, p^\lambda_m, p^i_l)$. Given a Berry type connection $A$ on $E$, one can construct a multimomentum Hamiltonian form

$$H_{(h)} = (p^\lambda_m dy^i + p^\lambda_m d\sigma^m) \wedge \omega^\lambda - (p^\lambda_m A^1_\lambda + p^\lambda_m \Gamma^m_\lambda) \omega = \tilde{H} \omega$$

on $\Pi$. In gauge theory of internal symmetries, a Higgs field $h$ fails to be a dynamic variable. It is regarded as a background field. Therefore, the Hamiltonian density $\tilde{H}$ in $H_{(h)}$ is independent on the momenta $p^\lambda_m$. Then, the Hamilton equations with respect to $p^\lambda_m$ read

$$\partial_\lambda h^m = \Gamma^m_\lambda. \quad (37)$$

Substituting this relation into the expression (13), we obtain the connection

$$A = dx^\lambda \otimes [(\partial_\lambda + \partial_\lambda h^m \partial_m) + (A^i_m \partial_\lambda h^m + \tilde{A^i}_\lambda) \partial_i] \quad (38)$$
on $E$ with the term (14)

$$A^i_m \partial_\lambda h^m dx^\lambda \otimes \partial_i \quad (39)$$

similar to the Berry connection (27). The connection (38) takes its values in $J^1 E_h \subset J^1 E$ and so is reducible to the connection

$$A_h = dx^\lambda \otimes [\partial_\lambda + (A^i_m \partial_\lambda h^m + \tilde{A^i}_\lambda) \partial_i] \quad (40)$$
on $E_h$. This connection is associated with the principal connection on $P_h$ induced by the principal connection $A_\Sigma$ on $P_\Sigma$. Moreover, one can always choose the bundle coordinates of $\Sigma$ so that a solution of the equation (37) takes the form $h = \text{const}$, $\Gamma = 0$. In these coordinates, the term (39) in the connections (38) and (40) disappears. At the same time, deviations from a background Higgs field are represented by sections of the bundle $\Sigma^1$ which are not the jet prolongations of sections of $\Sigma$. In this case, the term (39) takes the form $\sigma^m_\lambda A^i_m$. It describes interaction between Higgs field deviations and matter fields.

7. Gravitation theory

By $P$, we further mean the $GL^+(4, \mathbb{R})$-principal linear frame bundle over an oriented world manifold $X^4$, and $\Sigma$ is the Higgs bundle $\Sigma = P/SO(3, 1)$. Global sections of $\Sigma$
are tetrad gravitational fields. In the gauge gravitation theory, dynamic variables are pairs of tetrad gravitational fields \( h \) and gauge gravitational potentials \( g \).

There is 1:1 correspondence between the reduced \( SO(3,1) \)-subbundles \( P_h \) of \( P \) and tetrad gravitational fields \( h \) identified with global sections of the Higgs bundle \( \Sigma \). The gauge potentials are identified with sections \( A_h \) of the bundle \( P^0_1 / SO(3,1) \).

Let \( \Psi^h = \{ z^h_\kappa \} \) be an atlas of \( P \) such that sections \( z^h_\kappa \) take their values in the reduced \( SO(3,1) \)-subbundle \( P_h \). Let \( \Psi^T \) be a holonomic atlas of \( P \). With respect to these atlases, the tetrad field \( h \) is represented by tetrad functions

\[
h_\kappa(x) = (\psi^T_\kappa \circ z^h_\kappa)(x),
\]

\[
(\psi^T_\kappa(x))^{-1} t_a = (\psi^T_\kappa(x))^{-1} h^\mu_\alpha(x) t_\mu = h^\mu_\alpha(x) \partial_\mu,
\]

where \( \{ t \} \) is a fixed basis for the standard fibre \( \mathbb{R}^4 \) of \( TX \).

A gauge gravitational potential \( g \) is uniquely extended to a principal connection \( A \) on \( P \). With respect to an atlas \( \Psi^h \), \( A \) is represented by coefficients of the local connection form (8):

\[
A^a_{\mu}(x) = -A^a_{\mu}(x) = A^a_{\mu}(x) I_a = -\frac{1}{2} A^a_{\mu}(x)(I_{cd}) a^b
\]

where \( (I_{cd})^a_b \) are generators of \( SO(3,1) \) acting on \( \mathbb{R}^4 \).

To define configuration and phase spaces of gravitational fields, let us consider the composite fibred manifold

\[
P \to \Sigma \to X^4
\]

(41)
denoted by \( \bar{P} [6] \). This is not a principal bundle. The canonical action of \( GL^+(4, \mathbb{R}) \) on \( \bar{P} \) on the right is not compatible with the fibration (41) whereas the action of \( SO(3,1) \) on \( P \) is not free. The pairs \( (h, A_h) \) of tetrad gravitational fields and gauge gravitational potentials can be represented by sections of the composite fibred manifold

\[
\bar{P}^0_1 / SO(3,1) \to \Sigma \to X^4
\]

which we denote by \( C_K \). It follows that, in the gauge gravitation theory, the configuration space is the jet manifold \( J^1C_K \) and the phase space is the Legendre bundle

\[
\Pi = \wedge^1 T^*X^4 \otimes TX^4 \otimes V^*C_K.
\]

(42)

Given an atlas \( \{ z^\Sigma_\kappa \} \) of \( P_\Sigma \) and a holonomic atlas \( \{ \psi^T_\kappa \} \) of \( P \), the fibred manifold \( C_K \) is endowed with the local fibred coordinates

\[
(x^\mu, \sigma^\lambda_\alpha, k^{ab}_\lambda, -k^{ba}_\lambda, \sigma_{ab}^\lambda)
\]

(43)

where \( \sigma^\lambda_\alpha(\sigma), \sigma \in \Sigma \), are the fibred coordinates of \( \Sigma \) which are matrix components of the element \( (\psi^T_\kappa \circ z^\Sigma_\kappa)(\sigma) \in G \) acting on \( \mathbb{R}^4 \). There is the relation (12)

\[
k^{ab}_\lambda = k^{ab}_\mu + k^{abc}_\mu \sigma_{c\lambda}^\mu
\]
between the coordinates $k^{ab}_\lambda$ of $C_K$ and the bundle coordinates $\tilde{k}^{ab}_\lambda$, $k^{ab}_\mu$ of the bundle $P^{SO(3,1)}_\Sigma$. One can use this relation in order to obtain transition functions between the coordinate charts (43). Since the bundle

$$GL^+(4,\mathbb{R}) \to GL^+(4,\mathbb{R})/SO(3,1)$$

is trivial, we can restrict ourselves to atlases $\{z^\mu_\lambda\}$ with transition functions being constant on fibres of the bundle $\Sigma$. In this case, the coordinates $\sigma^\lambda_\alpha$ and $k^{ab}_\lambda$ of $C_K$ are transformed independently of each other. Let $s$ be a section of $C_K$. In the coordinates (43), we have

$$(\sigma^\lambda_\alpha \circ s)(x) = h^\lambda_\alpha(x), \quad (k^{ab}_\lambda \circ s)(x) = A^{ab}_\lambda(x).$$

It follows that sections of $C_K$ describe the above-mentioned pairs of tetrad fields and gauge gravitational potentials and that we can use the jet manifold $J^1C_K$ as the configuration space of gravity. It is provided with the fibred coordinates

$$x^\mu, \sigma^\lambda_\alpha, k^{ab}_\lambda = -k^{ba}_\lambda, \sigma^\lambda_\alpha, \sigma^\lambda_{\alpha(\mu)}, s^{ab}_\mu\lambda, F^{ab}_\mu\lambda, \sigma^\lambda_\alpha,$$

$$s^{ab}_\mu\lambda = k^{ab}_\mu\lambda + k^{ab}_\mu\lambda - k^a_{\mu\rho}k^{b\rho}_\mu + k^a_{\mu\rho}k^{b\rho}_\mu,$$

$$F^{ab}_\mu\lambda = k^{ab}_\mu\lambda - k^{ab}_\mu\lambda + k^a_{\mu\rho}k^{b\rho}_\mu - k^a_{\mu\rho}k^{b\rho}_\mu,$$

(44)

where $(x^\mu, \sigma^\lambda_\alpha, \sigma^\lambda_{\alpha(\mu)}, \sigma^\lambda_{alpha})$ are the coordinates of the repeated jet manifold $J^1\Sigma^1$. For the sake of simplicity, we restrict our consideration to its semiholonomic submanifold $\sigma^\lambda_{alpha} = \sigma^\lambda_{a(\mu)}$. The adapted coordinates of the Legendre bundle (42) are $(x^\mu, \sigma^\lambda_\alpha, k^{ab}_\lambda, \sigma^\lambda_{\alpha(\mu)}, p^\lambda_\mu, p^\lambda_{a(\mu)}, p^\lambda_{\mu(\alpha)})$ where $(x^\mu, \sigma^\lambda_\alpha, p^\lambda_\mu)$ are the adapted coordinates of the Legendre bundle

$$\vee^4 T^*X \otimes TX \otimes V^*\Sigma.$$  

To define them, one can use the vertical splitting (6) and the local morphism

$$\alpha \circ V^\Sigma_\alpha : V\Sigma \to V P\Sigma \to P\Sigma \times g_l$$

where $g_l$ is the left Lie algebra of $SO(3,1)$. In the coordinates (44), Lagrangians of gravitation theory are constructed by means of the curvature $F^{ab}_\mu\lambda$ and the torsion

$$\Omega^\lambda_{bc} = \sigma^a_{\lambda(\mu)}\sigma^b_\mu\sigma^c_\nu - \sigma^a_{\nu(\mu)}\sigma^b_\mu\sigma^c_\nu$$

(45)

where $\sigma^a_\mu$ are components of the inverse matrix.

We here restrict our consideration to the Hilbert-Einstein Lagrangian of classical gravity because of its feature in comparison with Lagrangians quadratic in curvature. In the coordinates (44), this Lagrangian reads

$$L_{HE} = -\frac{1}{2\kappa} F^{ab}_\mu\lambda\sigma^a_\mu\sigma^b_\lambda  \omega^{-1}, \quad \omega = \det(\sigma^a_\mu).$$

(46)

The corresponding Legendre morphism $\hat{L}_{HE}$ is given by the expressions

$$p^{[a\mu]} = \pi^{[a\mu]} = \frac{-1}{\kappa\sigma^b_\mu} \sigma^{[a\mu]}_b, \quad p^{(a\mu)} = 0, \quad p^{a\lambda}_\mu = \frac{\partial L_{HE}}{\partial k^{ab}_\mu},$$

$$p^a_\lambda = 0, \quad p^{a\mu}_\lambda = 0.$$

(47)
The following multimomentum Hamiltonian forms are associated with the Lagrangian (46):

\[
H_{HE} = (p_{ab}^{\lambda \mu} k_{ab}^{\lambda} + p_{a}^{\mu} d\sigma_{a}^{\lambda} + p_{\lambda}^{\nu \mu} d\sigma_{a\nu}^{\lambda}) \wedge \omega_{\mu} - \mathcal{H}_{HE} \omega, 
\]

\[
\mathcal{H}_{HE} = (p_{ab}^{\lambda \mu} \Gamma_{ab}^{\lambda \mu} + p_{a}^{\mu} \Gamma_{a}^{\lambda \mu} + p_{\lambda}^{\nu \mu} \Gamma_{a\nu}^{\lambda \mu} + N_{ab}^{\lambda \mu}(p_{ab}^{\lambda \mu} - p_{a}^{\mu} \pi_{ab}^{\lambda \mu})),
\]

\[
\Gamma_{a\mu}^{\lambda} = B_{a\mu}^{\lambda} - \Gamma_{\nu\mu}^{\lambda} \sigma_{a\nu}^{\lambda},
\]

\[
\Gamma_{ab}^{\lambda \mu} = \frac{1}{2}[\kappa_{a\lambda} k_{b}^{\mu} - k_{a}^{\mu} \kappa_{b}^{\lambda} + \partial_{\lambda} B_{ab}^{\mu} + \partial_{\mu} B_{ab}^{\lambda} + B_{ab}^{\nu} k_{\nu a}^{\lambda} - B_{b\lambda}^{a} k_{\mu a}^{\lambda} - B_{a\lambda}^{b} k_{\mu b}^{\lambda} - B_{a}^{\nu b} k_{\mu b}^{\lambda} - B_{b}^{\nu a} k_{\mu a}^{\lambda} + \Gamma_{\nu\lambda}^{\mu} k_{ab}^{\nu} - \Gamma_{\nu\mu}^{\lambda} B_{ab}^{\nu}].
\]

\[
\Gamma_{a\nu\mu}^{\lambda} = \partial_{\nu} B_{a\mu}^{\lambda} - B_{d\mu}^{\nu} \sigma_{a\nu}^{\lambda} + B_{d\nu}^{\mu} \sigma_{a\mu}^{\lambda} - \partial_{\mu} \Gamma_{\beta\nu}^{\lambda} \sigma_{a\beta}^{\alpha} - \Gamma_{\mu\nu}^{\lambda} \beta_{\mu\nu}^{\beta} + B_{d\mu}^{\nu} (\sigma_{a\nu}^{\lambda} - \Gamma_{d\nu}^{\lambda}) + \Gamma_{\nu\mu}^{\lambda} (\sigma_{a\beta}^{\lambda} - \Gamma_{\alpha\beta}^{\lambda}) - \Gamma_{\mu\nu}^{\lambda} (\sigma_{a\mu}^{\lambda} - \Gamma_{a\mu}^{\lambda}),
\]

where $\Gamma$ is a principal connection on $P$, $(\Gamma, B)$ is some composite connection on $\tilde{P}$, $\Gamma$ is a connection on $C_{K}$ and $N_{ab}^{\lambda \mu} = -N_{ab}^{\mu \lambda}$ is a soldering form on $C_{K}$. We have

\[
H_{HE}|_{Q} = p_{ab}^{\lambda \mu} k_{ab}^{\lambda} \wedge \omega_{\mu} - \frac{1}{2} p_{ab}^{\lambda \mu} (k_{a\lambda} k_{b}^{\mu} - k_{a}^{\mu} k_{b\lambda}^{\nu} \omega_{\nu}),
\]

\[
\hat{H}_{HE} \circ \hat{L}_{HE} : (x, \sigma_{b}^{a}, k_{ab}^{\lambda}, \sigma_{a\mu}^{\lambda}, \sigma_{a\nu}^{\lambda}, \sigma_{a\mu}^{\lambda}, F_{ab}^{\lambda \mu}, \sigma_{a\nu}^{\lambda})
\]

\[
\rightarrow (x, \sigma_{b}^{a}, k_{ab}^{\lambda}, \tilde{\Gamma}_{a\mu}^{\lambda}, \hat{\Gamma}_{a\nu}^{ab}(\mu), 2 N_{a\mu}^{\lambda}, \tilde{\Gamma}_{a\nu}^{ab}(\mu)) \quad (48)
\]

The Hamilton equations corresponding to $H_{HE}$ read

\[
F^{a}_{b\mu} = 2 N_{ab}^{\mu \lambda},
\]

\[
\partial_{\mu} k_{ab}^{\lambda} + \partial_{\lambda} k_{ab}^{\mu} = 2 \Gamma_{ab}^{\beta}(\mu \lambda),
\]

\[
\partial_{\mu} \sigma_{a}^{\lambda} = \Gamma_{a\mu}^{\lambda},
\]

\[
\partial_{\mu} \sigma_{a\nu}^{\lambda} = \Gamma_{a\nu\mu}^{\lambda},
\]

\[
\partial_{\mu} p_{a}^{\lambda \mu} = - \frac{\partial H_{HE}}{\partial k_{ab}^{\lambda}},
\]

\[
\partial_{\mu} p_{a\nu}^{\lambda \mu} = - \frac{\partial H_{HE}}{\partial \sigma_{a\nu}^{\lambda}},
\]

plus the equations which are reduced to the identities on the constraint space (47). On this space, the equations (49e) and (49f) take the form

\[
\partial_{\mu} p_{a}^{\lambda \mu} = 2 k_{c\mu} \sigma_{ab}^{\lambda} + \pi_{ac}^{\beta} \Gamma_{bc}^{\lambda \beta} = 0.
\]

Substituting the equation (49a) into the equation (50b), we obtain the Einstein equations. The equations (49c) and (49d) have the solution $\sigma_{a\mu}^{\lambda} = \sigma_{a\nu}^{\lambda} = \Gamma_{a\mu}^{\lambda}$. It follows that the composite connection $B$ belongs to the type (38). Then, the equation (49c) means that, given a tetrad field $\sigma(x)$, the connection $B$ reduced on $P$ coincides with $\Gamma$. The equation (49b) plays the role of the gauge condition. Its solution is $B = k$. 
In this case, the morphism (48) satisfies the condition (22). Moreover, the equation (50a) shows that $k$ is the Levi-Civita connection for the tetrad field $\sigma(x)$.

**Acknowledgement**

The authors wish to thank L. Mangiarotti for fruitful discussions and critical remarks.

**References**


