Self-calibrating tomography for angular Schmidt modes in spontaneous parametric down-conversion

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We report an experimental self-calibrating tomography scheme for entanglement characterization in high-dimensional quantum systems using Schmidt decomposition techniques. The self-tomography technique based on maximal likelihood estimation was developed for characterizing nonideal measurements in the Schmidt basis, allowing us to infer both Schmidt eigenvalues and detecting efficiencies.

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I. INTRODUCTION

Building an experimental setup for performing sophisticated measurements (for example, the ones required for performing quantum tomography), one generally needs to calibrate it. For the signals on the single-particle level the task is quite challenging, especially if one cannot easily use precalibrated etalon detectors and/or signal sources for that purpose. More than 30 years ago Klyshko outlined an efficient way to solve this problem using the fact that quantum features of the signal (for example, the character of the photon number distribution) can be used as a precise measurement tool for a calibration. In particular, Klyshko suggested using photon pair creation in the process of down-conversion for performing an “absolute calibration” of the detecting scheme [1]. Detection of one photon of the pair in one arm of the absolute-calibration scheme means that there is a second photon of the pair going through the other arm. Thus, the ratio of registered counts in both arms gives an experimentalist efficiency of the detecting setup installed in the second arm of the scheme without any precalibrated detector or source. This idea was actively developed and implemented (see, for example, Refs. [2–5]). Recently, this idea has also given rise to the more general concept of “self-calibration” as simultaneous inference of parameters of both the measurement scheme and the signal [6–8]. Very recently, the first experimental realization of the self-calibration scheme was presented using polarization-encoded one- and two-photon states [9]. There, the unknown rotation angle of the measurement basis was recovered together with the density matrix of the signal.

Here we present an example of experimental self-calibrating tomography when a set of parameters describing efficiency of the detecting scheme is actually inferred together with the parameters of the spatial state of entangled photon pairs generated in the process of spontaneous parametric down-conversion (SPDC). Spatial entanglement in SPDC was the subject of intense research during the last decade. Besides fundamental issues, spatial states of biphoton pairs offer a platform for the high-dimensional quantum-state engineering motivating this interest. One can distinguish two complementary approaches to spatial qudit engineering with biphotons: one using “pixel entanglement” and similar schemes [10–13] and another one based on using high-order coherent (usually Laguerre-Gaussian) modes [14–25]. In both approaches achievable dimensionality and collection efficiency are figures of merit. Dimensionality of effective Hilbert space is limited by the degree of spatial entanglement. In pixel entanglement schemes, for example, the pixel size should be made smaller than the coherence radius of the pump in the far zone, and since the pump is always divergent, even a plane wave, selected by pointlike aperture, would be correlated to a whole set of plane-wave modes in the conjugate beam. The same holds, in general, for other possible choices of modes.

It is remarkable that there is a “preferred” basis among the multitude of possible coherent spatial modes, which consists only of pairwise correlated modes. It is a set of Schmidt modes. Since it was used by Law and Eberly [26], it has become a common tool for entanglement analysis of infinite-dimensional systems in general and of spatial states of photons in particular. A direct experimental attempt to address spatial entanglement of SPDC biphotons in the Schmidt basis was made in our recent work [27]. The technique of projective measurements used in that work suffers from poor quality of the spatial mode transformations, resulting in a nonideal measurement scheme. Here we use self-calibration to account for this nonideality. We present a self-consistent analysis of the data collected by measuring approximate Schmidt modes via a setup similar to that in Ref. [27] and demonstrate that self-calibrating tomography is a feasible and practical way to update both the information about the measurement scheme and the Schmidt eigenvalues, starting from very general assumptions about them.

This paper is organized as follows: in Sec. II we briefly describe the main features of the SPDC angular spectrum with emphasis on spatial entanglement and Schmidt decomposition. Section III describes the general concept of the self-calibrating tomography scheme and its particular realization for inferring Schmidt eigenvalues. Section IV gives a detailed description of our experiments with spatial Schmidt modes. Section IV describes the practical implementation of the self-calibration scheme.
II. ANALYZING SPATIAL ENTANGLEMENT OF SPDC BIPHOTONS WITH SCHMIDT DECOMPOSITION

Biphons generated in the SPDC process have a continuous frequency and angular spectrum. Let us consider its structure in some detail. SPDC can be phenomenologically described using the following effective interaction Hamiltonian [28]:

\[ H = \int d^3 \overline{r} X_{\overline{r}}^{(2)}(\overline{r}) E_{\overline{r}}^{(-)}(\overline{r}) E^{(+)(\overline{r})} + \text{H.c.} \]  

(1)

Here \( E_{\overline{r}} \) is the classical amplitude of the pump field, and \( E \) is the scattered field operator. Considering pump to be monochromatic, the first order of perturbation theory gives the following expression for the state of the scattered field:

\[ |\Psi\rangle = |\text{vac}\rangle + \int d\overline{k}_1 d\overline{k}_2 \Psi(\overline{k}_1, \overline{k}_2) |1\rangle_k_1 |1\rangle_k_2, \]

(2)

where \( \overline{\Delta} = \overline{k}_1 + \overline{k}_2 - \overline{k}_p \), \( \omega_1 + \omega_2 = \omega_p \). In the case of collinear phase matching and under wide crystal approximation one can obtain the following biphon field amplitude [29–31]:

\[ \Psi(\overline{k}_1, \overline{k}_2) = \mathcal{F}(\overline{k}_1 - \overline{k}_2), \]

(3)

where \( \mathcal{F}(\overline{k}_1 - \overline{k}_2) \) stands for the angular spectrum of the pump and \( \mathcal{F}(\overline{k}_1 - \overline{k}_2) \) is a geometrical factor determined by phase-matching conditions.

The authors of [30,31] give the following expression for \( \mathcal{F} \):

\[ \Psi(\overline{k}_1, \overline{k}_2) = \mathcal{N}\mathcal{E}_p(\overline{k}_1 + \overline{k}_2) \mathcal{F}(\overline{k}_1 - \overline{k}_2), \]

(4)

with \( \mathcal{N} \) being the crystal length and \( \mathcal{E}_p \) being a normalization constant. Although this expression is strictly valid only in the case of small pump divergence [13,32], it appropriately describes the SPDC spectrum in our experiments.

The most developed approach to quantitative analysis of spatial (and frequency) entanglement of SPDC biphon states is based on using coherent mode decomposition. Biphon spatial state space is “discretized” by switching from continuous distributions in the plane-wave basis of the previous section to discrete distributions in a chosen basis of spatial mode functions \( \xi_i(\tilde{k}_{1,2,\perp}) \). For an arbitrary choice of mode functions in the decomposition of the spatial state for each of the photons, the biphon amplitude takes the following form:

\[ \Psi(\overline{k}_{1,\perp}, \overline{k}_{2,\perp}) = \sum_{i,j=0}^{\infty} C_{ij} \xi_i(\overline{k}_{1,\perp}) \xi_j(\overline{k}_{2,\perp}). \]

(5)

It turns out that with the appropriate choice of the basis mode functions one can transform expression (5) into a single-sum form,

\[ \Psi(\overline{k}_{1,\perp}, \overline{k}_{2,\perp}) = \sum_{i=0}^{\infty} \sqrt{\lambda_i} \psi_i(\overline{k}_{1,\perp}) \psi_i(\overline{k}_{2,\perp}), \]

(6)

which is called Schmidt decomposition. In this case the basis functions \( \psi_i(\overline{k}_{1,\perp}) \) should be eigenfunctions of the single-photon density matrix \( \rho_{1,2}(\overline{k}_{1,\perp}, \overline{k}_{1,\perp}) \), and \( \lambda_i \) are the corresponding eigenvalues. This means that the appropriate mode functions may be found by solving the following integral equation:

\[ \int \rho_{1,2}(\overline{k}_{1,\perp}, \overline{k}_{1,\perp}) \psi_i(\overline{k}_{1,\perp}) d\overline{k}_{1,\perp} = \lambda_i \psi_i(\overline{k}_{1,\perp}). \]

(7)

The average number of Schmidt modes determined by Schmidt number,

\[ K = \frac{1}{\sum_{i=0}^{\infty} \lambda_i^2}, \]

(8)

is widely used as an operational measure of entanglement [26].

The degree of spatial entanglement of SPDC biphons described by wave function (4) was analyzed by Law and Eberly in [26]. The pump was assumed to be Gaussian \( \mathcal{E}_p = \exp[-\overline{|k}_1 + \overline{k}_2|/\sigma^2] \). Law and Eberly derived an analytical expression for the Schmidt number by approximating the \( \mathcal{F}(\overline{k}_{1,\perp} - \overline{k}_{2,\perp}) \) function of (4) by a Gaussian function:

\[ K_g = \frac{1}{4} \left( \frac{b\sigma + 1/b\sigma}{2} \right)^2, \]

(9)

where \( b = \sqrt{\mathcal{E}_p/(4\sigma)} \) is the waist of the Gaussian function modeling \( \mathcal{F}(\overline{k}_{1,\perp} - \overline{k}_{2,\perp}) \). In the following we will call this procedure a double-Gaussian approximation. The value of \( K_g \) is determined by a single parameter \( b\sigma \) and may reach very high values for \( b\sigma \gg 1 \) and \( b\sigma \ll 1 \). Numerical calculations performed for

\[ \mathcal{F}(\overline{k}_{1,\perp} - \overline{k}_{2,\perp}) = \text{sinc}[b(\overline{k}_{1,\perp} - \overline{k}_{2,\perp})^2] \]

(10)

showed that real value of \( K \) is larger than \( K_g \) for all values of \( b\sigma \).

The choice of mode functions set in (5) is quite arbitrary and may be determined by convenience in analyzing a particular physical situation. If the situation corresponds to a beam propagating in free space, it is natural to choose the solutions of a paraxial wave equation, i.e., Hermite-Gaussian or Laguerre-Gaussian modes, as a set of mode functions. Moreover it turns out that these functions form the Schmidt decomposition for SPDC with a Gaussian pump of moderate divergence.

Let us choose the set of Hermite-Gaussian modes as a basis for decomposition:

\[ H_{nm}(k_x, k_y) \]

\[ \propto H_n \left( \frac{k_x^2}{(\Delta k_y)^2} \right) H_m \left( \frac{k_y^2}{(\Delta k_x)^2} \right) \exp \left( -\frac{k_x^2 + k_y^2}{2(\Delta k)^2} \right), \]

(11)

where \( H_n(x) \) are Hermite polynomials and \( k_x, k_y \) are transverse wave-vector components. One can get rid of two indices in decomposition (11) and transform it to a form of the Schmidt decomposition using a double-Gaussian approximation [26,33]. For small \( \sigma \) we can make a substitution \( \sin(\frac{\gamma}{2}) \rightarrow \exp(-\gamma^2/4) \), where \( \gamma \) is a coefficient chosen to make both functions “close” to each other. A good approximation is provided by choosing a value of \( \gamma = 0.86 \) [34]. The biphon wave function now takes the following form:

\[ \Psi(\overline{k}_{1,\perp}, \overline{k}_{2,\perp}) \propto \exp \left( -\frac{(k_{1,\perp} + k_{2,\perp})^2}{2a_x^2} \right) \exp \left( -\frac{(k_{1,\perp} - k_{2,\perp})^2}{2b_y^2} \right), \]

(12)
where \( a \) determines the angular bandwidth of the pump and 
\( b = \sqrt{4k_p/\gamma L} \) is the phase-matching bandwidth. Since the 
wave function is a product of functions depending on only 
k_{1,2\alpha} and \( k_{1,2\beta} \), it is sufficient to consider the problem in one dimension:

\[
\Psi(k_{1,2\alpha}, k_{1,2\beta}) = \sqrt{\frac{2}{\pi ab}} \exp\left(-\frac{(k_{1\alpha} + k_{2\beta})^2}{2a^2}\right) \exp\left(-\frac{(k_{1\beta} - k_{2\alpha})^2}{2b^2}\right).
\]  

(13)

One can show that solutions of (7) for such a wave function 
have the form [33]

\[
\psi_n(k_{1,2\alpha}) = \left(\frac{2}{ab}\right)^{1/4} \phi_n \left(\sqrt{\frac{2}{ab}} k_{1,2\alpha}\right),
\]

(14)

where \( \phi_n(x) = (2^n n! \sqrt{\pi})^{-1/2} e^{-x^2/2} H_n(x) \). For the corresponding eigenvalues and Schmidt number we obtain

\[
\lambda_n = 4ab \frac{(a - b)^2}{(a + b)^{2n+1}}, \quad K_x = \frac{a^2 + b^2}{2ab}.
\]  

(15)

Therefore we have the following form of the Schmidt decomposition for a SPDC biphoton state under the double-Gaussian approximation:

\[
\Psi(k_1, k_2) = \sum_{nm} \sqrt{\lambda_n \lambda_m} \psi_n(k_{1\alpha}) \psi_m(k_{1\beta}) \psi_n(k_{2\beta}) \psi_m(k_{2\alpha}).
\]  

(16)

The degree of entanglement for this two-dimensional wave packet is given by the Schmidt number:

\[
K = K_x K_y = (a^2 + b^2)^2/4a^2b^2.
\]  

(17)

Let us note that (16) is not the only possible form of Schmidt decomposition. It may also be described in terms of Laguerre-Gaussian modes as in [26,35]. The value of the Schmidt number is, of course, basis independent, as was explicitly shown in a recent work by Miatto et al. [36]. In fact, the difference between these two representations corresponds to the choice of polar or Cartesian coordinates on the plane of transverse momentum components.

### III. SELF-CALIBRATING TOMOGRAPHY SCHEME FOR INFERRING SCHMIDT EIGENVALUES

Here we describe briefly the concept of self-calibration as simultaneous updating of information about the state and the measurement device as it was formulated in a recent work [8]. Also, we discuss the application of the self-calibration scheme for the tomography of angular Schmidt modes in SPDC.

Generally, the possibility of self-calibration follows naturally from the Born rule,

\[
p(\rho, X) = \text{Tr}(\hat{\Pi}(X)\rho),
\]

(18)

which gives one the probability of getting the particular measurement results \( p(\rho, X) \), which is linear on coefficients of the representation of the signal density matrix \( \rho \) and the elements of the positive operator-valued measure (POVM) \( \hat{\Pi}(X) \) in an arbitrary basis. Obviously, \textit{a priori} knowledge of some parts of the signal density matrix can be traded for getting knowledge of some parts of the POVM. Even for the probability \( p(\rho, X) \), which is dependent nonlinearly on parameters of the POVM and the signal density matrix, these parameters can still be estimated. For that a measure of estimation success should be strictly convex with respect to all the parameters and coefficients to be found. For this measure one can take, for example, the Kullback-Leibler divergence between the expected inferred set of probabilities \( \tilde{p} = \left\{ p_{ij}(\rho, X) \right\} \) and the actually measured frequencies \( \tilde{f} = \left\{ \tilde{f}_{ij} \right\} [8] \).

\[
D(\tilde{p}, \tilde{f}) \propto \ln(L) = \sum_{i,j} f_{ij} \ln \left( \frac{p_{ij}(\rho, X)}{P} \right),
\]

(19)

Minimizing this divergence is equivalent to performing a maximum-likelihood (ML) estimation [37], i.e., maximizing the likelihood \( L \) of the model.

Building the self-calibration statistical estimation procedure for our case is greatly simplified, first, by the available \textit{a priori} knowledge about the signal state [represented by the form of the state (21)] and, second, by the character of measurements (projections on Schmidt modes). Actually, one needs to estimate diagonal elements of the density matrix (6) containing terms corresponding to no more than single photons. This allows one to implement a highly efficient iterative expectation-maximization algorithm for performing the ML estimation of the diagonal elements of the density matrix, preserving positivity at each step of the iteration procedure [38] (for practical implementation of the diagonal elements estimation, see, for example, Refs. [6,39]). Also, as will be seen below, the measurement is done by performing rather accurate (albeit still significantly nonideal) projection on the chosen Schmidt mode approximated as the Gaussian function (see the previous section). Local measurements in one of the directions are sufficient for the inference.

Therefore the reduced local signal density matrix describing one photon of the pair is

\[
\rho(\tilde{k}, \tilde{k}) = \sum_{n,n'} \lambda_{nm} \psi_{nm}(\tilde{k}) \psi_{nm}(\tilde{k}'),
\]

(20)

where coefficients \( \lambda_{nm} = \lambda_m \lambda_n \) are Schmidt numbers corresponding to the mode \( \psi_{nm} \). The POVM elements for the local projective measurement of the mode \( \psi_{nm} \) can be written as

\[
\Pi_{nm}(\tilde{k}, \tilde{k}') = \sum_{i,j} \mu_{ij}^{(nm)} \psi_{ij}(\tilde{k}) \psi_{ij}(\tilde{k}'),
\]

(21)

where parameters \( \mu_{ij}^{(nm)} \) describe the losses and efficiency of the projection. Knowledge about these parameters should be updated via the self-calibration procedure. This update can be done via the already standard maximal-likelihood iterative estimation technique [6,39,40], leading in this case to expectation-maximization iterative procedure:

\[
\lambda_{nm}^{(k+1)} = \lambda_{nm}^{(k)} \sum_{i,j} \tilde{f}_{ij}^{(k)} \frac{\mu_{ij}^{(nm)}}{\mu_{ij}^{(nm)}}
\]

(22)
and probabilities \( p^{(k)}_{ij} \) are estimated via the Born rule [Eq. (18)] on the \( k \)th iteration of the procedure. For the initial approximation \( \lambda^{(0)}_{nm} \), one can choose, for example, the set of equal numbers (the choice of \( \lambda^{(0)}_{nm} \) is not really important provided that they are nonzero [39]). An important feature of procedure (22) specific to our case is that the result of the estimation should be factorable, \( \lambda^{(k)}_{nm} = \gamma^{(0)}_{nm} \lambda^{(k)}_0 \). Practically, it can be done by replacing the result of each iteration by the closest factorable matrix. As we shall see below, procedure (22) provides for fast and efficient estimation of Schmidt eigenvalues.

The self-calibration procedure can be imagined as follows: one assumes some values of parameters \( \mu^{(nm)}_{ij} \), estimates \( \lambda_{nm} \), and probabilities \( p_{ij} \) and calculates the value of the Kullback-Leibler divergence (19). Then one repeats the whole procedure for another set of \( \mu^{(nm)}_{ij} \). If the Kullback-Leibler divergence is convex in the chosen region of parameters \( \mu^{(nm)}_{ij} \), one chooses the values of \( \mu^{(nm)}_{ij} \) corresponding to the minimum divergence as the ones allowing the closest fit of the experimental results.

### IV. EXPERIMENTAL REALIZATION OF MEASUREMENTS IN THE SCHMIDT BASIS

Here we describe local measurement performed by projection to the approximated Gaussian Schmidt modes. We used a 2-mm beta barium borate (BBO) crystal pumped by a cw He-Cd laser with \( \lambda_p = 325 \) nm wavelength. The crystal was cut for collinear frequency-degenerate type-I phase matching. The angular bandwidth of phase matching in such a crystal (neglecting the pump divergence), \( b \), in (12), is \( \frac{\lambda_p}{\pi} b = 0.033 \), where \( \lambda = 2\lambda_p = 650 \) nm is the wavelength of down-converted photons. It was convenient for our purposes to select the value of pump divergence corresponding to a moderate Schmidt number. We focused the pump inside the crystal with a 150-mm quartz lens and measured the divergence, \( a \) in (12), to be \( \frac{\lambda}{2a} = (5.8 \pm 0.1) \times 10^{-3} \) [41].

To ensure the applicability of double-Gaussian approximation we calculated eigenvalues and eigenfunctions for the reduced single-photon density matrix, corresponding to the precise SPDC wave function (4) numerically. The calculation was performed as follows: Hermite-Gaussian modes corresponding to the approximate function (13) were chosen as a basis; we have restricted ourselves to ten lower-order modes (giving the Schmidt number with three-decimal-digit precision) and calculated the matrix elements of the precise density matrix in this basis. Diagonalizing the calculated matrix, we obtained eigenvalues and eigenfunctions. The results are in reasonable correspondence with a simplified double-Gaussian model; at least, we should expect that phase holograms for Schmidt modes should be close to those of Hermite-Gaussian modes of appropriate divergence. We should note that the measured waist size of the pump beam in the focal plane of the lens was \( w_p = (25 \pm 1) \mu m \), corresponding to \( M^2 = 1.4 \). That means the pump beam is aberrated and is not really Gaussian, which may cause some deviations from the Hermite-Gaussian shape of the Schmidt modes as well.

We have used a Liquid-Crystal-on-Silicon (LCoS) Spatial Light Modulator with Vertically Aligned Nematic (VAN) matrix produced by Cambridge Correlators. The matrix has 1027 \( \times \) 768 pixels of 10-\( \mu m \) size each. It is an eight-bit device, capable of introducing a phase shift of up to 0.87\( \pi \). Since larger phase shifts are required for our holograms, we used a double-reflection scheme. We used two polymer film polarizers in front of and behind the SLM to reduce the unwanted polarization rotations by an additional dielectric mirror necessary in such a scheme (see insets in Figs. 1 and 5).

To estimate the quality of mode transformation with this device we used the setup sketched in Fig. 1. We used an attenuated 650-nm diode laser as a source. The beam was mode filtered with single-mode fiber and focused with a 20\( \times \) microscope objective to obtain divergence similar to that of an HG\( \text{nm} \) Schmidt mode and the waist at the position of the crystal. Thus we obtained a single-mode Gaussian beam modeling the zeroth-order Schmidt mode of a SPDC beam. The beam was collimated with a 145-mm lens and after reflection from SLM was focused with an 8\( \times \) microscope objective to a single-mode fiber followed by a single-photon counter (Perkin Elmer). The focused beam waist exactly coincided with the mode size of the fiber (4 \( \mu m \)). We should stress that we paid special attention to mode matching, and optics were chosen in such a way that the detection mode exactly coincided with the calculated HG\( \text{nm} \) Schmidt mode. The parameters of the phase holograms were adjusted to minimize the detector counting rate, i.e., to ensure orthogonality of transformed modes to a fundamental Gaussian one. We have actually adjusted three parameters: the position of the phase step for HG\( \text{100} \) modes, which is determined by the beam position at the SLM (in the horizontal and vertical directions, respectively), and the distance between the phase steps for HG\( \text{20} \) modes determined by beam size at the SLM plane. These parameters define the shape of the holograms for other modes in a unique way. If we define “visibility” for mode transformations as the ratio of counting rates with holograms for HG\( \text{nm} \) modes to that for the untransformed Gaussian mode, 

\[
V = \frac{(R_{\text{nm}} - R_{\text{00}})}{(R_{\text{00}} + R_{\text{nm}})},
\]

then for almost all of the modes with \( 0 \leq m, n \leq 4 \), it exceeds 97\%, corresponding to a reasonably high quality of mode transformations. The histogram of counting rates for various modes is shown in Fig. 2. Notice that counts rate for the supposedly symmetrical modes HG\( \text{01} \) and HG\( \text{10} \) are visibly
different. This is a consequence of different qualities of projections for these modes. Such nonsymmetry should be accounted for by the corresponding POVM elements (namely, by the parameters $\mu_{ij}^{(nm)}$). This hardly controllable nonsymmetry and other artifacts of nonperfect mode transformations inevitable with phase-only holograms are the main reason for applying the self-calibrating reconstruction procedure for this case.

To check whether the spatial structure of the transformed modes is at least close to Hermite-Gaussian, we scanned the fiber tip in the focal plane of the O3 objective. The counting rate dependence on fiber position is determined by the convolution of a corresponding Hermite-Gaussian function and a fundamental Gaussian mode of the fiber:

\[
R(x) \propto \left| \int_{-\infty}^{\infty} H_{nm}(\sqrt{2} \tilde{x}/w) \exp \left( -\frac{\tilde{x}^2}{w^2} \right) \times \exp \left( -\frac{(x - \tilde{x})^2}{w^2} \right) d\tilde{x} \right|^2, \tag{23}
\]

where $w$ is the Gaussian mode waist. Experimental dependencies are shown in Fig. 3 and have the characteristic shape of double-peaked curves. The distance between maxima depends on the mode number and is shown in Fig. 4 for “horizontal” $HG_{n0}$ and “vertical” $HG_{0m}$ modes, together with theoretical predictions for Hermite-Gaussian modes. To plot the theoretical predictions correctly we estimated the waist size by fitting the convolution for $HG_{00}$ mode with a Gaussian curve, obtaining $w = (3.87 \pm 0.07) \mu m$.

When the attenuated laser beam is substituted with SPDC radiation, the described scheme realizes projective measurements in an approximately Hermite-Gaussian basis. The full scheme of the experimental setup is shown in Fig. 5. The pump was focused to a 2-mm BBO crystal with a 150-mm quartz lens (L1); a second lens (L2) with $F = 145$ mm was set confocal with L1 to collimate the beam. Pump radiation was cut off with a UV mirror (UVM), and SPDC radiation was frequency filtered with an interference filter (IF). We used filters with a central wavelength of 650 nm and bandwidths of 40 and 10 nm and did not observe any significant improvement of
visibility with a narrower filter. All the following results were obtained with the wide 40-nm filter. Photon pairs were split with a 50:50 nonpolarizing beam splitter. An SLM was placed in the transmitted channel, and after reflection the radiation was focused into a single-mode fiber placed in the focal plane of an 8× microscope objective. In the reflected channel the beam was focused into a similar single-mode fiber with an identical objective. Signals in both channels were detected by single-photon counters connected to a coincidence circuit.

Experimental evidence of the similarity of Schmidt modes to Hermite-Gaussian ones may be obtained by analyzing the dependencies of single counts and coincidences on the fiber tip position in the focal plane of the focusing microscope objective. We expect the dependence for coincidences to be described by (23). Experimental curves for the case when the fiber in the transmitted channel is scanned are shown in Fig. 6. The distance between maxima behaves analogously to the case of an attenuated laser beam, as shown in Fig. 7.

We obtained the same dependencies of coincidence counting rate when the fiber tip was scanned in the reflected channel (see Fig. 8). In this case single counts, obviously, do not depend at all on the mode selected in the conjugate channel. This effect is a straightforward consequence of intermodal correlations in SPDC and may be thought of as a sort of “ghost interference” [42]. We should note that the almost zero coincidence counting rate in the central position of the fiber is an interference effect demonstrating the spatial coherence of detected modes. Therefore this result may be considered an experimental demonstration of one of the main features of Schmidt modes: their spatial coherence.

Note that the double-peak structure characteristic of Hermite-Gaussian modes appears only in coincidence dependence, while single counts behave monotonously, as is expected for spatially multimode radiation. The maximal value of the single counting rate, however, decreases with an increasing value of mode indexes. This is clear from the form of the single-photon density matrix, described by (20). With the fiber placed in the central position, the detection scheme in the transmitted arm of the setup realizes projections described by (21); single count rates in this case correspond to frequencies \( f_{ij} \) in (22) and are the data used for statistical inference of Schmidt eigenvalues.

![FIG. 5. (Color online) Experimental setup. L1, 150-mm quartz lens; L2, 145-mm lens; BBO, 2-mm BBO crystal placed in the joint focus of L1 and L2; UVM, UV mirror cutting off the pump; IF, interference filter for 650 nm with 40-nm bandwidth; BS, nonpolarizing 50:50 beam-splitter; O1, 2, 8× microscope objectives; PM, spatial light modulator (shown as a transmitting mask for simplicity; real alignment is shown in the inset); PM2, phase mask made of thin glass plates; SMF, single-mode fiber; SMF/MMF, single- or multimode fiber depending on the experiment (see text for details); D1, 2, single-photon counters (Perkin Elmer). A 200-μm vertical slit S was used in “ghost” imaging experiments.](https://example.com/fig5)

![FIG. 6. (Color online) (a) Coincidence and (b) single count rate dependence on the fiber tip position in the focal plane of the microscope objective in the channel with SLM for different modes. The fiber is scanned in the horizontal direction.](https://example.com/fig6)

![FIG. 7. (Color online) Dependence of maxima positions for the coincidence distributions of Fig. 6 on the mode number [red (light gray) bars]. The calculated dependence for Hermite-Gaussian modes with \( w = (3.0 \pm 0.1) \mu m \) corresponding to a HG00 waist size (dark gray bars) is provided for comparison.](https://example.com/fig7)
FIG. 8. (Color online) Coincidence counting rate dependence on the fiber tip position in the reflected channel (without SLM) for different modes. The fiber tip is scanned in the horizontal direction.

V. PERFORMING SELF-CALIBRATION FOR INFERENCE OF SCHMIDT EIGENVALUES

The essential features of the measurement procedure described in the previous section can be captured by writing down the parameters of POVM (21) in the following form:

$$\mu_{ij}^{(nm)} \approx \mu_{tot} \eta_{nm} \gamma_{ij}^{(nm)}.$$  

(24)

In Eq. (24) the parameter $\mu_{tot}$ represents total losses (detection efficiency) that are equal for all measured modes. Since it contributes to normalization of the estimated signal density matrix only, it is irrelevant. The matrix $\eta_{nm}$ represents asymmetry of losses. Thus, we assume $\eta_{nm} \equiv 1$ for $n \leq m$. The parameters $\eta_{nm}$ for $n \geq m$ are to be determined via the self-calibration procedure. Parameters $\gamma_{ij}^{(nm)}$ describe the quality of the projection. Since it was demonstrated that the projection is of a reasonably good quality, we assume

$$\gamma_{ij}^{(nm)} \approx \Delta_{nm} \delta_{ij} \delta_{nm} + d_{ij}^{(nm)},$$  

(25)

where all the parameters on the right-hand side of formula (25) are taken to be non-negative and for each $m, n$ the quantity

$$\Delta_{nm} \gg \sum_{i,j} d_{ij}^{(nm)}.$$  

(26)

Also, for the sake of simplicity we assume that inefficiency of the projection occurs solely from the contributions of other modes, so $\sum_{i,j} \gamma_{ij}^{(nm)} = 1$. Parameters $\Delta_{nm}$ can be chosen from the results of the projection quality measurements described in the previous section. Thus, we have taken $\Delta_{nm} = 0.97$ for all $n, m$ apart from $\Delta_{01} = \Delta_{11} = 0.9$. For simulations small parameters $d_{ij}^{(nm)}$ were sampled randomly from the homogeneous distribution.

To model asymmetry of losses, we have introduced two parameters, $\eta_1$ and $\eta_2$. We assume that elements of the matrix $\eta_{nm}$ closest to the main diagonal are equal to $\eta_1$. Other elements of $\eta_{nm}$ for $n \geq m$ are taken to be equal to $\eta_2$. In Fig. 9 the Kullback-Leibler divergence (19) is shown for different values of parameters $\eta_{1,2}$; for calculations Eq. (19) is recast in the standard form,

$$D(\tilde{p}, \tilde{f}) = \sum_{i,j} f_{ij} \ln \left( \frac{f_{ij}}{p_{ij}(\rho, X)} \right),$$  

(27)

where both sets of estimated probabilities $\tilde{p}$ and measured frequencies $\tilde{f}$ are normalized to unity. The calculation is done for an experimentally obtained set of frequencies shown in Fig. 10(a) using the iterating ML estimation procedure described in Sec. III and given by Eq. (22). One can see that the Kullback-Leibler divergence is obviously convex for the chosen range of parameters $\eta_{1,2}$. The minimum is reached for $\eta_1 \approx 1.3, \eta_2 \approx 1.125$.

FIG. 9. (Color online) The Kullback-Leibler divergence (27) for the experimentally obtained frequencies.

FIG. 10. (Color online) (a) The registered frequencies of photocounts for different modes $HG_{nm}$. (b) Variations of the estimated minimal Kullback-Leibler divergence for different realizations of the randomly chosen parameters $d_{ij}^{(nm)}$; $N_{runs}$ denotes the number of the particular realization. (c) The reconstructed signal density matrix elements $\lambda_{nm}$ for asymmetry parameters, $\eta_1 = 1.3$ and $\eta_2 = 1.125$. (d) The absolute value of the differences between experimentally measured frequencies and the estimated probabilities for values of parameters in (c).
The result of the signal state estimation with the inferred parameters of the detecting scheme is given in Fig. 10(c). It is symmetric, i.e., $\lambda_{nm} = \lambda_{mn}$ and it is rather different from the experimentally found frequencies shown in Fig. 10(a)]. However, the reconstructed set of $\lambda_{nm}$ gives a set of probabilities rather close to the measured frequencies [Fig. 10(d)]. The reconstruction procedure is robust with respect to small imperfections in performing projections. It is seen [see Fig. 10(b)] that as long as condition (26) holds, the result of the estimation changes rather weakly for different realizations of random variables $\vartheta_{ij}^{(nm)}$.

Thus, we have established that the self-calibration procedure allows one to infer with rather high precision the quantum state entering the detection scheme and parameters of this detection scheme. Now let us consider how the inferred state agrees with the double-Gaussian model developed in Sec. II.

In Fig. 11 one can compare estimated values of $\lambda_{nm}$ and ones modeled according to Eq. (15) (light gray bars). Errors bars represent the deviation estimated using the Fisher matrix method (28). The inset shows the estimated (dark gray bars) and the modeled (light gray bars) Schmidt eigenvalues corresponding to the parameters $\lambda_{\pi a} = 5.8 \times 10^{-4}$, $\lambda_{\pi b} = 0.0334$.

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[34] We have tried several criteria for choosing $\gamma$, for example, equality of FWHM for both functions, which is used in [33] and gives $\gamma = 0.249$. However, $\gamma = 0.86$ better describes experimental data, which is not completely understood but may be caused by the limited angular detection aperture.
[41] One should pay attention to the fact that all formulas in previous sections do not take into account the refraction on the crystal surface, so to compare with experiment all angular variables for the pump should be divided and for SPDC multiplied by $n_s(\lambda_s) = n_s(\lambda_o) = 1.667$.