Regular Dynamics of a Hula-hoop

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Hula-hoop is a popular toy – a thin hoop that is twirled around the waist, limbs or neck. To twirl a hula-hoop the waist of a gymnast carries out a periodic motion in the horizontal plane. For the sake of simplicity we assume that the waist of the gymnast is round and its center moves in the horizontal plane along an elliptic trajectory $x = a \sin \omega t$, $y = b \cos \omega t$ close to a circle $(a \approx b)$ with frequency ω, semi-major axis a, and semi-minor axis b, see Fig. 1. Then in new time $\tau = \omega t$ we have the equation for angle φ that determines the position of the hula-hoop center with respect to the waist

$$\ddot{\varphi} + \gamma \dot{\varphi} + \mu \cos(\varphi - \tau) = \varepsilon \cos(\varphi + \tau) \tag{1}$$

along with the condition for the hula-hoop not to separate from the waist during its motion

$$\dot{\varphi}^2 - 2\mu \sin(\varphi - \tau) + 2\varepsilon \cos(\varphi + \tau) > 0, \tag{2}$$

with dimensionless parameters $\gamma = \frac{k}{2mR^2\omega}$, $\mu = \frac{a+b}{4(R-r)} > 0$, $\varepsilon = \frac{a-b}{4(R-r)} \geq 0$, where k is the coefficient of viscous friction, r is the radius of the waist, R and m are the radius and mass of the hula-hoop.

In [1] the periodic motion of the gymnast's waist along only one axis was considered $(b = 0)$. In the present study, we consider the hula-hoop excitation along two axes corresponding to an elliptic trajectory as in [2]. But in contrast to previous works [1, 2] we do not require that all parameters of excitation μ and dissipation γ are small. We assume only ε to be small like in our work [3] which we extend here.

When the waist moves along a circle ($a = b$, i.e. $\varepsilon = 0$) we have exact solutions of (1)

$$\varphi = \tau + \varphi_0, \quad \varphi_0 = \pm \arccos(-\gamma/\mu) \mod 2\pi \tag{3}$$

corresponding to the hula-hoop rotation with a constant angular velocity equal to the excitation frequency, provided that $|\gamma| \leq \mu$. According to the *Lyapunov's theorem on* *the stability based on a linear approximation solution* with the use of Routh-Hurwitz criterion we obtain asymptotic stability conditions $\gamma > 0$ and $\sin\varphi_0 < 0$ from which the inequalities $0 < \gamma < \mu$ follow.

Inseparability condition (2) takes the form $1 - 2\mu\sin\varphi_0 > 0$. Hence, rotation (3) with $\varphi_0 = -\arccos(-\gamma/\mu)$ is asymptotically stable and inseparable, while that of with $\varphi_0 = \arccos(-\gamma/\mu)$ is unstable and inseparable only if $\mu < \sqrt{1/4+\gamma^2}$.

When the waist center trajectory has small ellipticity ($a > b$, i.e. $\varepsilon > 0$) we use simple perturbation method assuming that solution of (1) can be expressed in series $\varphi = \tau+\varphi_0+\varepsilon\varphi_1(\tau)+\ldots$ of small parameter ε. After substitution of this series in (1) and grouping the terms by powers of ε we obtain φ_0 like in (3). Taking $\varphi_0 = -\arccos(-\gamma/\mu)$ corresponding to the stable solution (3) of the unperturbed system we derive

$$\varphi_1(\tau) = \frac{2\gamma\sin(\varphi_0+2\tau) - \left(4-\sqrt{\mu^2-\gamma^2}\right)\cos(\varphi_0+2\tau)}{3\gamma^2+\mu^2-8\sqrt{\mu^2-\gamma^2}+16}.$$

Inseparability condition (2) leads in the first approximation to the following inequality

$$\varepsilon < \frac{1+2\sqrt{\mu^2-\gamma^2}}{2}\sqrt{\frac{\mu^2+3\gamma^2-8\sqrt{\mu^2-\gamma^2}+16}{\mu^2+8\gamma^2-12\sqrt{\mu^2-\gamma^2}+36}}.$$

The hula-hoop can rotate in both direction only if all parameters μ, γ, and ε are small since the coexistence condition for both direct and inverse rotations has the form $0 < \gamma < \min\{\varepsilon, \mu\}$. In physical variables the coexistence condition takes the form

$$0 < 2k\frac{R-r}{R^2\omega m} < a - |b|$$

meaning that the trajectory of the waist should be sufficiently prolate.

References

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