Dynamics and Stability of the Simplest Skateboard Model

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Abstract Analysis and simulation are performed for a simplest model of a skateboard in the absence of rider control. Equations of motion of the model are derived. The problem of integrability of the obtained equations is investigated. The influence of various parameters of the model on its dynamics and stability is studied.

1 Introduction

Skateboarding is one of the most popular extreme sports of today. However, despite of the growing number of participants, skateboarding is poorly represented in the scientific literature. At the late 70^{th} – early 80^{th} of the last century Hubbard [5, 6] proposed several mathematical models describing the motion of the rider on a skateboard. Hubbard considered the motion of the skateboard in assumption, that the tilt of the board is small as well as the steering angles. In our paper we give the further development of the models offered by Hubbard and assume that the tilt of the board and the steering angles can be finite.

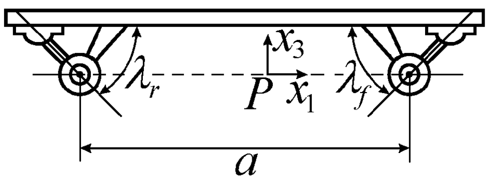

Fig. 1 The Skateboard Side View

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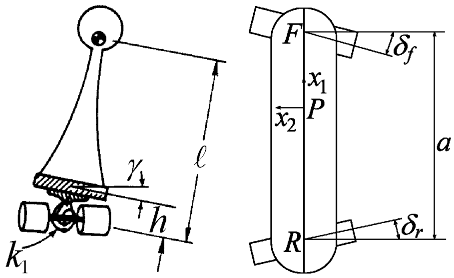

Fig. 2 The Skateboard Rear and Top View

The skateboard typically consists of the board, a set of two trucks and four wheels (Fig. 1). The modern board is generally from 78-83 cm long, 17-21 cm wide and 1-2 cm thick [5]. The most essential elements of a skateboard are the trucks, connecting the axles to the board. Angular motion of both the front and rear axles is constrained to be about their respective nonhorizontal pivot axes (Fig. 1), thus causing a steering angle of the wheels whenever the axles are not parallel to the plane of the board (Fig. 2). The vehicle is steered by making use of this static relationship between steering angles and tilt of the board. The specific construction of the trucks distinguishes the skateboard from the other types of boards, in particular, from the snakeboard studied in many papers (see e.g. [9]). In addition, there is a torsional spring, which exerts a restoring torque between the wheelset and the board proportional to the tilt of the board with respect to the wheelset. We denote the stiffness of this spring by k_1 (Fig. 2).

2 The Problem Formulation. Equations of Motion.

We assume that the rider, modeled as a rigid body, remains perpendicular with respect to the board. Therefore, when the board tilts through γ, the rider tilts through the same angle relative to the vertical. Let us introduce an inertial coordinate system $OXYZ$ in the ground plane. Let $FR = a$ is a distance between the two axle centers F and R of a skateboard. The position of line FR with respect to the $OXYZ$-system is defined by X and Y coordinates of its centre G and by the angle θ between this line and the OX-axis (Fig. 3).

The tilt of the board is accompanied by rotation of the front wheels clockwise through δ_f and rotation of the rear wheels anticlockwise through δ_r (Fig. 2, 3). The wheels of a skateboard are assumed to roll without lateral sliding. This condition is modeled by constraints, which may be shown to be nonholonomic

$$\begin{aligned} &\dot{Y}\cos\left(\theta-\delta_f\right)-\dot{X}\sin\left(\theta-\delta_f\right)+\frac{a}{2}\dot{\theta}\cos\delta_f=0,\\ &\dot{Y}\cos\left(\theta+\delta_r\right)-\dot{X}\sin\left(\theta+\delta_r\right)-\frac{a}{2}\dot{\theta}\cos\delta_r=0. \end{aligned} \tag{1}$$

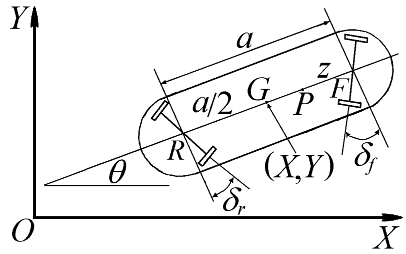

Fig. 3 The Basic Coordinate Systems

Under these conditions velocities of the points F and R will be directed horizontally and perpendicularly to the axles of the wheels and there is a point P on the line FR which has zero lateral velocity. Its forward velocity we denote by u. It may be shown, that (see e.g. [5, 6, 7, 8, 13])

$$u = -\frac{a\dot{\theta}\cos\delta_f\cos\delta_r}{\sin\left(\delta_f+\delta_r\right)},\ FP = \frac{a\sin\delta_f\cos\delta_r}{\sin\left(\delta_f+\delta_r\right)},\ \dot{\theta} = -\frac{u\sin\left(\delta_f+\delta_r\right)}{a\cos\delta_f\cos\delta_r}. \tag{2}$$

Using results obtained in [7, 13] we conclude that the steering angles δ_f and δ_r are related to the tilt of the board by the following equations

$$\tan\delta_f = \tan\lambda_f\sin\gamma, \quad \tan\delta_r = \tan\lambda_r\sin\gamma, \tag{3}$$

where λ_f and λ_r are the fixed angles which the front and rear axes make with the horizontal (Fig. 1). Using constraints (3) we can rewrite equations (1) as follows:

$$\begin{aligned}\dot{X} &= u\cos\theta + \frac{\left(\tan\lambda_f - \tan\lambda_r\right)}{2}u\sin\gamma\sin\theta,\\ \dot{Y} &= u\sin\theta - \frac{\left(\tan\lambda_f - \tan\lambda_r\right)}{2}u\sin\gamma\cos\theta.\end{aligned} \tag{4}$$

Expressions (2) become

$$FP = \frac{a\tan\lambda_f}{\tan\lambda_f+\tan\lambda_r} = \text{const},\ \dot{\theta} = -\frac{\left(\tan\lambda_f+\tan\lambda_r\right)}{a}u\sin\gamma. \tag{5}$$

Suppose that the board of the skateboard is located a distance h above the line FR. The length of the board is also equal to a. The board's center of mass is located at its center. As to the rider we suppose that the rider's center of mass is not located above the board center of mass, but it is located over the central line of the board a distance d from the front truck. Let l be the height of the rider's center of mass above the point P. Other parameters for the problem are: m_b – the mass of the board; m_r – the mass of the rider; I_{bx}, I_{by}, I_{bz} – the principal central moments of inertia of the board; I_{rx}, I_{ry}, I_{rz} – the principal central moments of inertia of the rider. We introduce also the following parameters:

$$I_x = I_{bx}+I_{rx}, \quad I_y = I_{by}+I_{ry}, \quad I_z = I_{bz}+I_{rz}.$$

It can be proved (see [7]) that the variables u and γ satisfy the following differential equations

$$\begin{aligned}&\left(A+(C-2D)\sin^2\gamma+K\sin^4\gamma\right)\dot{u}+B\left(\ddot{\gamma}\cos\gamma-\dot{\gamma}^2\sin\gamma\right)\sin\gamma+\\&+\left(C-3D+3K\sin^2\gamma\right)u\dot{\gamma}\sin\gamma\cos\gamma=0,\\[1ex]&E\ddot{\gamma}+\left(D-K\sin^2\gamma\right)u^2\sin\gamma\cos\gamma+B\left(\dot{u}\sin\gamma+u\dot{\gamma}\cos\gamma\right)\cos\gamma+\\&+k_1\gamma-(m_bh+m_rl)\,g\sin\gamma=0.\end{aligned}\tag{6}$$

Here $A, \ldots, E, K$ – are functions of the parameters, namely

$$A = m_b+m_r,\; E=I_x+m_bh^2+m_rl^2,$$

$$B = \frac{m_bh}{2}\left(\tan\lambda_f-\tan\lambda_r\right)+\frac{m_rl}{a}\left((a-d)\tan\lambda_f-d\tan\lambda_r\right),$$

$$C = \frac{m_b}{4}\left(\tan\lambda_f-\tan\lambda_r\right)^2+\frac{I_z}{a^2}\left(\tan\lambda_f+\tan\lambda_r\right)^2+\frac{m_r}{a^2}\left((a-d)\tan\lambda_f-d\tan\lambda_r\right)^2,$$

$$D = \frac{\left(\tan\lambda_f+\tan\lambda_r\right)}{a}\left(m_bh+m_rl\right),\; K=\frac{\left(\tan\lambda_f+\tan\lambda_r\right)^2}{a^2}\left(I_y+m_bh^2+m_rl^2-I_z\right).$$

Equations (4)-(6) form the close system for the skateboard motion.

3 Stability of uniform straight-line motion of a skateboard

Equations (6) has the particular solution

$$u=u_0=\text{const},\quad \gamma=0,\tag{7}$$

which corresponds to uniform straight-line motion of a skateboard. Consider the problem of stability of this particular solution.

Setting $u=u_0+\xi$ and keeping for γ its notation we write the equations of the perturbed motion

$$E\ddot{\gamma}+Bu_0\dot{\gamma}+\left(Du_0^2+k_1-(m_bh+m_rl)\,g\right)\gamma=\Gamma,\quad \dot{\xi}=\Xi.\tag{8}$$

Here Γ and Ξ are functions of γ, $\dot{\gamma}$ and ξ, whose development as a series in powers of said variables starts with terms of at least the second order. Moreover, these functions identically vanish with respect to ξ when $\gamma=0$ and $\dot{\gamma}=0$ (this fact can be verified manually):

$$\Gamma(0,0,\xi)=0,\quad \Xi(0,0,\xi)=0.$$

The characteristic equation corresponding to the linearized system (8) has the form:

$$\lambda\left(E\lambda^2+Bu_0\lambda+Du_0^2+k_1-(m_bh+m_rl)\,g\right)=0. \tag{9}$$

When conditions

$$E>0,\quad Bu_0>0,\quad Du_0^2+k_1-(m_bh+m_rl)\,g>0 \tag{10}$$

are fulfilled, equation (9) has one zero-root and two roots in the left half plane. Since the functions Γ and Ξ identically vanish for $\gamma=0$, $\dot\gamma=0$, then under conditions (10) we have the critical case of one-zero root [3, 11] and solution (7) is stable with respect to γ, $\dot\gamma$ and u (asymptotically stable with respect to γ and $\dot\gamma$).

Since a condition $E>0$ is valid for all values of parameters then conditions of stability (10) may be finally written in the form:

$$\left[\frac{m_bh}{2}\left(\tan\lambda_f-\tan\lambda_r\right)+\frac{m_rl}{a}\left((a-d)\tan\lambda_f-d\tan\lambda_r\right)\right]u_0>0, \tag{11}$$

$$k_1+\left(\left(\tan\lambda_f+\tan\lambda_r\right)\frac{u_0^2}{a}-g\right)(m_bh+m_rl)>0. \tag{12}$$

If at least one of the conditions (11)-(12) is not fulfilled then equation (9) has the root in the right-half plane and solution (7) will be unstable.

We can make now simple conclusions about stability of a straight-line motion of the skateboard using condition (11)-(12). Note, first of all, that expression on the left-hand side of inequality (11) contains u_0 as a multiplier. This means that the stability of motion depends on its direction. If one direction of motion is stable the opposite direction is necessary unstable. Such behavior is peculiar to many nonholonomic systems. First of all we can mention here the problem of motion of a rattleback (aka wobblestone or celtic stone, see e.g. [1, 4, 10, 12]). In this problem the stability of permanent rotations of a rattleback also depend on the direction of rotation.

Let us find now conditions of stability of the equilibrium position of a skateboard, i.e. the particular solution

$$u_0=0,\quad \gamma=0.$$

When $u_0=0$ the characteristic equation (9) has one zero-root and two pure imaginary roots under the condition

$$k_1-(m_bh+m_rl)\,g>0. \tag{13}$$

It can be proved (see [5, 6, 7, 13]) that inequality (13) is necessary and sufficient condition for stability of equilibrium position of the skateboard. Thus we can conclude that the equilibrium position of the skateboard will be stable if the torsional spring constant is sufficient to overcome the destabilizing gravity torque.

4 Nonlinear dynamics of the skateboard near the equilibrium position

Assume for the skateboard that we have $u_0 = 0,\ \gamma = 0$, i.e. the skateboard is in the equilibrium position on the plane. According to the previous results, inequality (13) provides the necessary and sufficient condition for stability of this equilibrium. Suppose that this condition is fulfilled.

Solving equations (6) with respect to $\dot{u}$ and $\ddot{\gamma}$ and assuming that u, γ and $\dot{\gamma}$ are small, we can write the equations of the perturbed motion taking into account the terms which are quadratic in u, γ and $\dot{\gamma}$:

$$\dot{u} = \frac{B\Omega^2}{A}\gamma^2, \quad \ddot{\gamma} + \Omega^2\gamma = -\frac{Bu\dot{\gamma}}{E}, \quad \text{where} \quad \Omega^2 = \frac{k_1 - (m_b h + m_r l)\, g}{E} \tag{14}$$

Note, that the linear terms in the second equation of the system (14) have a form which corresponds to a normal oscillations. For investigation of nonlinear system (14) we reduce it to a normal form [2]. To obtain the normal form of the system (14) first of all we make a change of variables and introduce two complex-conjugate variables z_1 and z_2:

$$\gamma = \frac{z_1 - z_2}{2i},\ \dot{\gamma} = \frac{z_1 + z_2}{2}\Omega,\ u = z_3.$$

In variables z_k, $k = 1,2,3$ the linear part of the system (14) has a diagonal form and the derivation of its normal form reduces to separating of resonant terms from the nonlinearities on the right-hand sides of the transformed system (14). Finally, the normal form of the system (14) may be written as follows:

$$\dot{z}_1 = i\Omega z_1 - \frac{B}{2E}z_1 z_3,\ \dot{z}_2 = -i\Omega z_2 - \frac{B}{2E}z_2 z_3,\ \dot{z}_3 = \frac{B\Omega^2}{2A}z_1 z_2.$$

Introducing real polar coordinates according to the formulae

$$z_1 = \rho_1 (\cos\sigma + i\sin\sigma)\,,\ z_2 = \rho_1 (\cos\sigma - i\sin\sigma)\,,\ z_3 = \rho_2$$

we obtain from system (14) the normalized system of equations of perturbed motion which is then split into two independent subsystems:

$$\dot{\rho}_1 = -\frac{B}{2E}\rho_1\rho_2, \quad \dot{\rho}_2 = \frac{B\Omega^2}{2A}\rho_1^2, \tag{15}$$

$$\dot{\sigma} = \Omega. \tag{16}$$

Terms of order higher than the second in (15) and those higher than the first in ρ_k, $k = 1,2$ in (16) have been omitted here.

In the ε-neighborhood of the equilibrium position the right-hand sides of equations (15) and (16) differ from the respective right-hand sides of the exact equations of perturbed motion by quantities of order ε^3 and ε^2 respectively. The solutions of the exact equations are approximated by the solutions of system (15)-(16) with an error of ε^2 for ρ_1, ρ_2 and of order ε for σ in a time interval of order $1/\varepsilon$. Restricting the calculations to this accuracy, we will consider the approximate system (15)-(16) instead of the complete equations of perturbed motion.

Equation (16) is immediately integrable. We obtain

$$\sigma = \Omega t + \sigma_0.$$

System (15) describes the evolution of the amplitude ρ_1 of the board oscillations and also the evolution of the velocity ρ_2 of a straight-line motion of the skateboard. One can see that this system has the first integral

$$E\rho_1^2 + \frac{A}{\Omega^2}\rho_2^2 = An_1^2, \tag{17}$$

where n_1 is a constant, specified by initial conditions. We will use this integral for solving of the system (15) and for finding the variables ρ_1 and ρ_2 as functions of time: $\rho_1 = \rho_1(t)$, $\rho_2 = \rho_2(t)$. Expressing ρ_1^2 from the integral (17) and substitute it to the second equation of the system (15) we get

$$\dot{\rho}_2 = \frac{B}{2E}\left(\Omega^2 n_1^2 - \rho_2^2\right). \tag{18}$$

The general solution of equation (18) has the following form:

$$\rho_2(t) = \frac{\Omega n_1\left(1 - n_2\exp\left(-\frac{B\Omega n_1}{E}t\right)\right)}{\left(1 + n_2\exp\left(-\frac{B\Omega n_1}{E}t\right)\right)}, \tag{19}$$

where n_2 is a nonnegative arbitrary constant. Now, using the integral (17), we can find the explicit form of the function $\rho_1(t)$:

$$\rho_1(t) = 2\sqrt{\frac{An_1^2 n_2}{E}}\frac{\exp\left(-\frac{B\Omega n_1}{2E}t\right)}{1 + n_2\exp\left(-\frac{B\Omega n_1}{E}t\right)}. \tag{20}$$

Let us consider the properties of the solutions (19), (20) of system (15) and their relations with the properties of motion of the skateboard. System (15) has an equilibrium position

$$\rho_1 = 0, \quad \rho_2 = \Omega n_1 \tag{21}$$

(these particular solutions can be obtained from general functions (19)-(20) if we suppose in these functions $n_2 = 0$). The arbitrary constant n_1 can be both positive and negative. The positive values of this constant correspond to straight-line motions of the skateboard with small velocity in the stable direction and the negative ones – in the unstable direction. Indeed, if we linearize equations (15) near the equilibrium position (21) we get

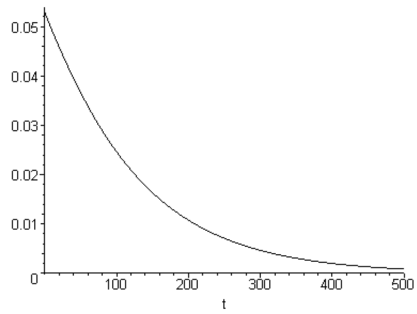

Fig. 4 Evolution of the amplitude ρ_1 of the board oscillations in time for the case $n_1 > 0$, $n_2 \leq 1$.

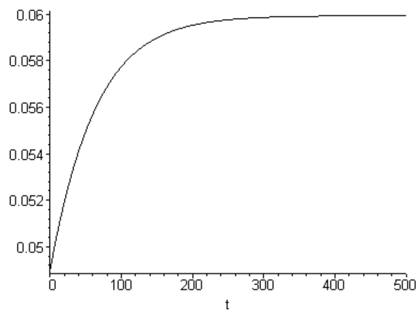

Fig. 5 Evolution of the velocity ρ_2 of the skateboard in time for the case $n_1 > 0$, $n_2 \leq 1$.

$$\dot{\rho}_1 = -\frac{B}{2E}\Omega n_1 \rho_1, \quad \dot{\rho}_2 = 0.$$

Thus, for $n_1 > 0$ the equilibrium position (21) is stable and for $n_1 < 0$ it is unstable.

Evolution of the functions ρ_1 and ρ_2 give the complete description of behavior of a skateboard with small velocities. Let us suppose, that at initial instant the system is near the stable equilibrium position $(n_1 > 0)$ and $\rho_2(0) \geq 0$, i.e. $n_2 \leq 1$ (the case when $n_1 > 0$, $n_2 > 1$ is similar to the case $n_1 < 0$, $n_2 < 1$, which will be investigated below). These initial conditions correspond to the situation when at initial instant the skateboard takes the small velocity

$$\rho_2(0) = \Omega n_1 \frac{1 - n_2}{1 + n_2} \tag{22}$$

in the stable direction. Then in the course of time the amplitude of oscillations of the board ρ_1 decreases monotonically from its initial value

$$\rho_1(0) = \frac{2n_1}{1 + n_2}\sqrt{\frac{An_2}{E}}$$

to zero, while the velocity of the skateboard ρ_2 increases in absolute value. In the limit the skateboard moves in the stable direction with a constant velocity Ωn_1 (see Fig. 4, 5).

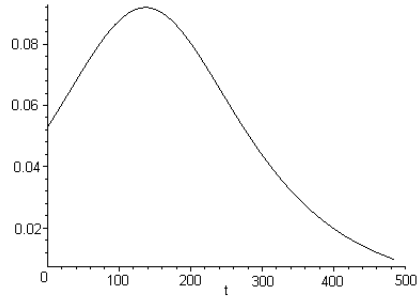

Fig. 6 Evolution of the amplitude ρ_1 of the board oscillations in time for the case $n_1 < 0$, $n_2 \leq 1$.

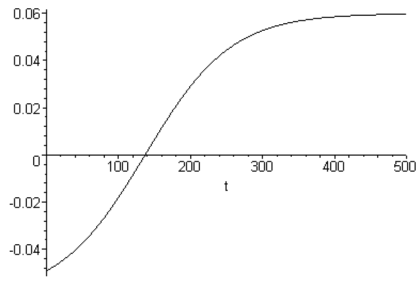

Fig. 7 Evolution of the velocity ρ_2 of the skateboard in time for the case $n_1 < 0$, $n_2 \leq 1$.

Suppose now that at the initial instant the system is near the unstable equilibrium position $n_1 < 0$. Suppose again, that at the initial instant $n_2 < 1$, i.e. $\rho_2(0) < 0$ (the case $n_1 < 0$, $n_2 > 1$ is similar to the case $n_1 > 0$, $n_2 < 1$ which was considered above). These initial conditions correspond the situation when at the initial instant the skateboard takes the small velocity (22) in the unstable direction. In this case the limit of the system motions is the same as when $\rho_2(0) \geq 0$ but the evolution of the motion is entirely different. When

$$0 < t < t_* = \frac{E \ln(n_2)}{B\Omega n_1}$$

the absolute value of the oscillation amplitude ρ_1 increases monotonically and the skateboard moves in the unstable direction with decreasing velocity. At the instant $t = t_*$ the velocity vanishes and the oscillation amplitude ρ_1 reaches its maximum value

$$\rho_1(t_*) = \sqrt{\frac{An_1^2}{E}}.$$

When $t > t_*$ the skateboard already moves in the stable direction with an increasing absolute value of its velocity and the oscillation amplitude decreases monotonically. Thus when $\rho_2(0) < 0$ during the time of evolution of the motion a change in the direction of motion of the skateboard occurs (Fig. 6, 7). Similar nonlinear effects (in particular the change of the direction of motion) were observed earlier in other problems of nonholonomic mechanics (for example in a classical problem of dynamics of a rattleback [1, 4, 10, 12]). Thus, we describe here the basic features of dynamics of the simplest skateboard model, proposed in [5, 6] and developed by us.

5 Conclusions

In this paper the problem of motion of the skateboard with a rider was examined. This problem has many common features with other problems of nonholonomic dynamics. In particular it was shown that the stability of motion of the skateboard depends on the direction of motion. Moreover the system can change its direction of motion. The similar effects have been found earlier in the classical problem of a rattleback dynamics.

Acknowledgements

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