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Mathematical Modeling of the Nonlinear Electrodynamics Effect of Signal Delay in the Magnetic Field of Pulsars

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Abstract—The paper is devoted to mathematical modeling of the nonlinear vacuum electrodynamics effect: the action of the strong magnetic field of a pulsar on the propagation of electromagnetic waves. It is shown that, due to the birefringence of the vacuum, for one normal wave, it takes more time to travel from a pulsar to a detector installed on astrophysical satellites than for the other normal wave. The delay of the pulse carried by the second normal wave relative to pulse carried by the first normal wave from the common point of origin to the satellite is calculated.

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1. INTRODUCTION

Various electromagnetic processes in the vacuum are described by Maxwell's electrodynamics, which is a linear theory. However, experiments conducted on the Stanford linear accelerator [1] has shown that, even in the vacuum, electrodynamics is nonlinear. Therefore, development of various nonlinear mathematical models of electrodynamics in the vacuum is an urgent problem [2–7].

In theoretical and mathematical physics, two models of nonlinear electrodynamics are being actively discussed. The equations of the electromagnetic field in these models coincide with the equations of Maxwell's macroscopic theory in matter:

$$\begin{aligned} \mathrm{curl}\mathbf{H} &= \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}, \quad \mathrm{div}\mathbf{D} = 4\pi\rho, \\ \mathrm{curl}\mathbf{E} &= -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\mathbf{B} = 0, \end{aligned} \tag{1}$$

differing from the latter in the meaning of the vectors **D** and **H**.

In particular, in the Born–Infeld nonlinear electrodynamics [2], these vectors satisfy the following constitutive relations:

$$\begin{aligned} \mathbf{D} &= \frac{\mathbf{E} + a^2(\mathbf{BE})\mathbf{B}}{\sqrt{1 + a^2(\mathbf{B}^2 - \mathbf{E}^2) - a^4(\mathbf{BE})^2}}, \\ \mathbf{H} &= \frac{\mathbf{B} - a^2(\mathbf{BE})\mathbf{E}}{\sqrt{1 + a^2(\mathbf{B}^2 - \mathbf{E}^2) - a^4(\mathbf{BE})^2}}, \end{aligned} \tag{2}$$

where a is a constant such that $1/a \sim 10^{16}$ G.

In the limit of $a \longrightarrow 0$, the Born–Infeld nonlinear electrodynamics passes to Maxwell's linear electrodynamics in the vacuum.

For the fields attainable in laboratories on the Earth, the dimensionless ratios $a^2\mathbf{E}^2$ and $a^2\mathbf{B}^2$ are significantly smaller than unity. In this case, constitutive relations (2) can be expanded [8] in the small parameters $a^2\mathbf{E}^2 \ll 1$ and $a^2\mathbf{B}^2 \ll 1$:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + a^2(\mathbf{BE})\mathbf{B} - \frac{a^2}{2}(\mathbf{B}^2 - \mathbf{E}^2)\mathbf{E}, \\ \mathbf{H} &= \mathbf{B} - a^2(\mathbf{BE})\mathbf{E} - \frac{a^2}{2}(\mathbf{B}^2 - \mathbf{E}^2)\mathbf{B}. \end{aligned} \tag{3}$$

In the Heisenberg–Euler electrodynamics, nonlinearity is an implication of the polarization of the electron–positron vacuum by electromagnetic fields.

This theory employs a characteristic quantity B_q, expressed in terms of the electron mass m, electron charge e, and Planck constant $\hbar$: $B_q = m^2c^3/(e\hbar) = 4.41 \times 10^{13}$ G. If the magnitude of the electromagnetic fields $\mathbf{B}$ and $\mathbf{E}$ is smaller than B_q, then the fields are considered weak and the first two terms in the expansion of the constitutive relations in the vacuum in the small parameters $(\mathbf{B}^2 - \mathbf{E}^2)/B_q^2$ and $(\mathbf{BE})/B_q^2$ have the form [9]

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + \frac{7\alpha\xi}{45\pi}(\mathbf{BE})\mathbf{B} - \frac{2\alpha\xi}{45\pi}(\mathbf{B}^2 - \mathbf{E}^2)\mathbf{E}, \\ \mathbf{H} &= \mathbf{B} - \frac{7\alpha\xi}{45\pi}(\mathbf{BE})\mathbf{E} - \frac{2\alpha\xi}{45\pi}(\mathbf{B}^2 - \mathbf{E}^2)\mathbf{B}, \end{aligned} \tag{4}$$

where $\alpha = e^2/\hbar c \approx 1/137$ is the fine structure constant and $\xi = 1/B_q^2$.

For calculation of various effects of the nonlinear vacuum electrodynamics in the approximation of a weak electromagnetic field ($B < B_q$, $E < B_q$), which is applicable for all models, post-Maxellian formalism was suggested and developed [10–13]. In this formalism, the constitutive relations of any nonlinear vacuum electrodynamics in the case of weak fields are written in the parametric form:

$$\begin{aligned} \mathbf{D} &= \mathbf{E} + 2\xi\{\eta_1(\mathbf{E}^2 - \mathbf{B}^2)\mathbf{E} + 2\eta_2(\mathbf{BE})\mathbf{B}\}, \\ \mathbf{H} &= \mathbf{B} + 2\xi\{\eta_1(\mathbf{E}^2 - \mathbf{B}^2)\mathbf{B} - 2\eta_2(\mathbf{BE})\mathbf{E}\}, \end{aligned} \tag{5}$$

which includes two dimensionless post-Maxwellian parameters η_1 and η_2 with the values depending on the model of the vacuum nonlinear electrodynamics.

Comparing expressions (3)–(5), it is easy to find that, in the Heisenberg–Euler nonlinear electrodynamics, the parameters η_1 and η_2 have different values: $\eta_1 = \alpha/(45\pi) = 5.1 \times 10^{-5}$ and $\eta_2 = 7\alpha/(180\pi) = 9.0 \times 10^{-5}$, whereas, in the Born–Infeld theory, they are equal: $\eta_1 = \eta_2 = a^2 B_q^2/4$.

The model of nonlinear vacuum electrodynamics employing Eqs. (1) and (5) is termed in the literature the parametric post-Maxwellian electrodynamics, by analogy with the parametric post-Newtonian formalism [14] of the gravitation theory.

In [15–17], it was shown that system of equations (1) and (5) has the equation of characteristics

$$G_{(1)}^{im} G_{(2)}^{nk} \left(\frac{\partial S}{\partial x^i}\right)\left(\frac{\partial S}{\partial x^m}\right)\left(\frac{\partial S}{\partial x^n}\right)\left(\frac{\partial S}{\partial x^k}\right) = 0, \tag{6}$$

where the tensors $G_{(1)}^{im}$ and $G_{(2)}^{nk}$ in the post-Maxwellian approximation are expressed via the metric tensor g^{im} of the four-dimensional Riemannian space and the electromagnetic field tensor F^{ip}:

$$\begin{aligned} G_{(1,2)}^{im} &= g^{im} + 4\eta_{1,2}\xi F^{ij}F_j^m, \\ G_{nk}^{(1,2)} &= g_{nk} - 4\eta_{1,2}\xi F_{np}F_k^p. \end{aligned} \tag{7}$$

Equation (6) implies that, for $\eta_1 \neq \eta_2$, any electromagnetic wave in an external electromagnetic field splits into two normal waves, which have mutually orthogonal polarizations and propagate along the rays at different velocities. Expressions (7) do not mean that the Einstein principle of equivalence in electrodynamics [18] is violated: the presence of the second terms in them means that the propagation of the electromagnetic wave is influenced not only by the gravitational field, described by the metric tensor g_{nk}, but by the external electromagnetic field as well. By analogy, it is established that the charged particle motion in the gravitational and electromagnetic fields is different from the motion only in the gravitational field.

Equation (6) can be transformed by the Lagrange–Charpit method [17] to equations of isotropic geodesic lines:

$$\frac{dk^n}{d\sigma} + G^{np}_{(1,2)}\left[\frac{\partial G_{pm}{}^{(1,2)}}{\partial x^i} - \frac{1}{2}\frac{\partial G_{mi}{}^{(1,2)}}{\partial x^p}\right]k^i k^m = 0, \tag{8}$$

where σ a parameter and $k^n = dx^n/d\sigma$ is a four-dimensional vector [19].

System of equations (8) has the first integral

$$G^{(1,2)}_{nk}\frac{dx^n}{d\sigma}\frac{dx^k}{d\sigma} = 0. \tag{9}$$

The primary problem in the development of a parametric model of post-Maxwellian electrodynamics is the experimental determination of the values of the post-Maxwellian parameters η_1 and η_2. To this end, using the system of equations (1) and (5)–(9), it is necessary to calculate a variety of electrodynamic processes in order to find nonlinear effects in the vacuum that could be available for measurements. However, the fields attainable in conditions of the Earth are so small as compared to the field B_q that measuring the nonlinear effects considered in [20–23] is beyond the capabilities of modern technology.

In this connection, it is more promising to consider electrodynamic processes near pulsars [24], i.e., rotating neutron stars with a dipole magnetic field $B \sim 10^{13}$ G on the surface, and near magnetars [25], i.e., rotating neutron stars with $B \sim 10^{16}$ G. The computations performed in [26–29] for the cases when rays of electromagnetic waves lie in the plane of the magnetic meridian or in the plane of the magnetic equator of the dipole magnetic field of a neutron star have shown that, in certain conditions, the nonlinear effect in the dependence of the propagation velocity of a wave on its polarization can be detected experimentally. Therefore, this effect should be studied in the most general case—not only in the magnetic meridian and magnetic equator planes.

The aim of the present work is the mathematical modeling of the nonlinear electrodynamics effect of signal delay in the magnetic field of pulsars in the most general case, when the vector of the magnetic moment of a neutron star is arbitrarily orientated with respect to the ray of an electromagnetic wave, and estimation on this basis the nonlinear effects arising in this case.

2. PROBLEM STATEMENT

Let us consider a neutron star with a magnetic dipole moment **m**, mass M, and radius R_s.

In the magnetosphere of many pulsars and magnetars, bursts of X- and gamma radiation occur. Propagating in space, this radiation reaches the vicinity of the Earth and is detected by equipment installed on astrophysical satellites. When X- and gamma radiation passes through the magnetosphere, it is subjected to the action of strong magnetic and gravitational fields of the neutron star, which leads to birefringence and bending of rays. As in [30, 31], we will neglect the action of plasma found in the magnetosphere on the propagation of X- and gamma radiation, because, in this range of frequencies ω, the refractive index $n = 1 - N/\omega^2$ unessentially differs from unity.

Suppose that, at a point $\mathbf{r} = \mathbf{R}_0$ of pulsar's magnetosphere at a time $\tau = \tau_0$, a burst of X- and gamma radiation has occurred. This point will be the initial point for a bundle of rays along which electromagnetic pulses will propagate in all directions and some of them will reach a detector installed on a satellite located at a distance $R_d \sim 10^{18}$ km from the center of the pulsar. In the magnetic field of the pulsar, due to the nonlinear electrodynamics birefringence, each pulse is carried by two normal ways having different velocities and mutually orthogonal polarizations. These waves, which travel to the satellite along different rays, pass their way for different times.

Let us calculate the delay of one normal wave relative to the other normal way as they travel from the common source to the satellite. As in [31], we will use the spherical coordinates with the axes oriented so as to simplify the further calculations. Place the origin at the center of the pulsar. Take one ray from the bundle of rays of the first normal wave, connecting the point $\mathbf{r} = \mathbf{R}_0$ and the satellite. Draw the tangent line to this ray at the point $\mathbf{r} = \mathbf{R}_0$. Turn the coordinate axes so as this tangent line lie in the plane $\theta = \pi/2$ and the coordinate φ of vector $\mathbf{R}_0$ equal π. Therefore, the plane $\theta = \pi/2$ will touch the plane for the first ray at the point $\mathbf{r} = \mathbf{R}_0$.

Since the rays of the first and second normal waves that connect the X- and gamma radiation source and the satellite are different, the initial conditions for them are imposed in different ways. For the chosen orientation of the axes, the ray of the first normal wave originates at the point $r = R_0$, $\theta = \pi/2$, $\varphi = \pi$, touches ($d\theta/d\varphi = 0$) at this point the plane $\theta = \pi/2$, and has the impact parameter $b > R_s$.

The ray of the second normal wave also originates at the point $r = R_0$, $\theta = \pi/2$, $\varphi = \pi$ and terminates at the point $\mathbf{r} = \mathbf{R}_d$ at which the ray of the first normal wave hits the detector. In addition, all waves are radiated from the point $\tau = \tau_0$ at the same time $\mathbf{r} = \mathbf{R}_0$.

3. ESSENTIAL EQUATIONS AND RELATIONSHIPS OF THE MATHEMATICAL MODEL

In the model under consideration, an appropriate choice for the metric tensor g_{ik} is the Schwarzschield metric tensor [19] in the post-Newtonian approximation [14]. Denoting the gravitational radius of the star by $r_g = 2GM/c^2$, we have

$$g_{00} = 1 - \frac{r_g}{r}, \quad g_{11} = -1 - \frac{r_g}{r}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta. \tag{10}$$

In the spherical coordinates, the nonzero components of the tensor F_{ik} describing the magnetic dipole field of the neutron star has the form

$$\begin{gathered} F_{12} = F_{r\theta} = -\frac{1}{r^2}[m_x \sin\varphi - m_y \cos\varphi], \\ F_{32} = F_{\varphi\theta} = \frac{2\sin\theta}{r}\{m_z \cos\theta + \sin\theta[m_x \cos\varphi + m_y \sin\varphi]\}, \\ F_{13} = F_{r\varphi} = \frac{\sin\theta}{r^2}\{m_z \sin\theta - \cos\theta[m_x \cos\varphi + m_y \sin\varphi]\}. \end{gathered} \tag{11}$$

To simplify further calculations, we introduce auxiliary angles α and β by the relationships

$$m_x = |\mathbf{m}| \sin\alpha \cos\beta, \quad m_y = |\mathbf{m}| \sin\alpha \sin\beta, \quad m_z = |\mathbf{m}| \cos\alpha. \tag{12}$$

Using expressions (10)–(12), we find the components of the tensor $G_{pm}^{(1,2)}$:

$$\begin{gathered} G_{00}^{1,2} = 1 - \frac{r_g}{r}, \quad G_{11}^{(1,2)} = -1 - \frac{r_g}{r} - \frac{4\xi\eta_{1,2}}{r^2\sin^2\theta}[F_{12}^2 \sin^2\theta + F_{13}^2], \\ G_{22}^{(1,2)} = -r^2 - \frac{4\xi\eta_{1,2}}{r^2\sin^2\theta}[r^2 F_{12}^2 \sin^2\theta + F_{23}^2], \quad G_{33}^{(1,2)} = -r^2\sin^2\theta - \frac{4\xi\eta_{1,2}}{r^2}[r^2 F_{13}^2 + F_{23}^2], \\ G_{12}^{(1,2)} = -\frac{4\xi\eta_{1,2} F_{13} F_{23}}{r^2 \sin^2\theta}, \quad G_{13}^{(1,2)} = \frac{4\xi\eta_{1,2} F_{12} F_{23}}{r^2}, \quad G_{23}^{(1,2)} = -4\xi\eta_{1,2} F_{13} F_{12}. \end{gathered} \tag{13}$$

The remaining components of the symmetric tensor $G_{pm}^{(1,2)}$ are zero.

In Eqs. (8) and the first integral (9), we pass from the differentiation with respect to the parameter σ to the differentiation with respect to the angle φ, using the relationship $d/d\sigma = k^3 d/d\varphi$. As a result, we obtain

$$\begin{gathered} \frac{d^2x^n}{d\varphi^2} + \left[G_{(1,2)}^{np} - \frac{dx^n}{d\varphi} G_{(1,2)}^{3p}\right]\left[\frac{\partial G_{pm}^{(1,2)}}{\partial x^i} - \frac{1}{2}\frac{\partial G_{mi}^{(1,2)}}{\partial x^p}\right]\frac{dx^i}{d\varphi}\frac{dx^m}{d\varphi} = 0, \\ G_{nm}^{(1,2)} \frac{dx^n}{d\varphi}\frac{dx^m}{d\varphi} = 0, \end{gathered} \tag{14}$$

where $n = 0, 1, 2$ and $x^0 = c\tau$, $x^1 = r$, $x^2 = \theta$, and $x^3 = \varphi$.

For $n = 3$, Eq. (14) turns into the identity $0 = 0$.

Equations (14) with the initial conditions give the unique solution of the above-formulated problem.

4. INTEGRATION OF THE RAY EQUATIONS

Substitute expression (13) into Eqs. (14). Since, the problem under consideration has two small parameters: r_g/r and $\mathbf{m}^2\xi\eta_{1,2}/r^6$, the solution to Eqs. (14) will be sought by the method of successive approximations.

In the initial approximation, $d\theta/d\varphi = 0$ for all points of the chosen ray and system of equations (14) takes the form

$$\frac{d^2\tau}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)\left(\frac{d\tau}{d\varphi}\right) = 0,$$

$$\frac{d^2 r}{d\varphi^2} - \frac{2}{r}\left(\frac{dr}{d\varphi}\right)^2 - r = 0, \quad c^2\left(\frac{d\tau}{d\varphi}\right)^2 - \left(\frac{dr}{d\varphi}\right)^2 - r^2 = 0.$$

Integrating these equations and taking into account the initial conditions, we obtain in this approximation

$$r(\varphi) = \frac{b}{\sin(\varphi - \chi)}, \quad \tau(\varphi) = \tau_0 + \frac{b\cos\chi}{c\sin\chi} + \frac{b\cos(\varphi - \chi)}{c\sin(\varphi - \chi)}, \quad \theta(\varphi) = \frac{\pi}{2},$$

where χ is an auxiliary angle relating b and R_0 by the formula $\sin\chi = b/R_0$.

The solution to system of equations (14) in the next approximation is conveniently sought in the form

$$\tau_{1,2}(\varphi) = \tau_0 + \frac{b\cos\chi}{c\sin\chi} + \frac{r(\varphi)}{c}\cos(\varphi - \chi) + \frac{r_g}{cb}F_1(\varphi) + \frac{\mathbf{m}^2\xi\eta_{1,2}}{cb^5}F_2(\varphi),$$

$$r_{1,2}(\varphi) = b\left[\sin(\varphi - \chi) + \frac{r_g}{b}F_3(\varphi) + \frac{\mathbf{m}^2\xi\eta_{1,2}}{b^6}F_4(\varphi)\right]^{-1}, \tag{15}$$

$$\theta_{1,2}(\varphi) = \frac{\pi}{2} + \frac{r_g}{b}F_5(\varphi) + \frac{\mathbf{m}^2\xi\eta_{1,2}}{b^6}F_6(\varphi),$$

where $F_1(\varphi)$, $F_2(\varphi)$, $F_3(\varphi)$, $F_4(\varphi)$, $F_5(\varphi)$, and $F_6(\varphi)$ are functions to be determined from Eqs. (14).

Write Eqs. (14) with allowance for expression (15) in the linear approximation with respect to the small parameters. The functions describing the gravitational interaction satisfy the equations

$$\frac{dF_1(\varphi)}{d\varphi} = \frac{\sin(\varphi - \chi)}{2} - \frac{1}{\sin(\varphi - \chi)},$$

$$\frac{d^2F_3(\varphi)}{d\varphi^2} + F_3(\varphi) = \frac{3\sin^2(\varphi - \chi)}{2}, \quad \frac{d^2F_5(\varphi)}{d\varphi^2} + F_5(\varphi) = 0.$$

The solutions of these equations have the form

$$F_1(\varphi) = C_1 - \frac{\cos(\varphi - \chi)}{2} - \log\left|\frac{\sin(\varphi - \chi)}{1 + \cos(\varphi - \chi)}\right|,$$

$$F_3(\varphi) = C_2\cos\varphi + C_3\sin\varphi + 1 - \frac{\sin^2(\varphi - \chi)}{2}, \quad F_5(\varphi) = C_4\cos\varphi + C_5\sin\varphi, \tag{16}$$

where C_1, C_2, C_3, C_4, and C_5 are integration constants.

The functions describing the nonlinear electrodynamics interaction satisfy the equations

$$\frac{dF_2(\varphi)}{d\varphi} = 2\sin^4(\varphi-\chi)\{[5\sin^2(\varphi-\beta)\sin^2(\varphi-\chi)-\sin^2(\varphi-\beta)$$
$$-\sin2(\varphi-\beta)\sin2(\varphi-\chi)-4\sin^2(\varphi-\chi)]\sin^2\alpha-\cos^2\alpha\},$$

$$\frac{d^2F_4(\varphi)}{d\varphi^2}+F_4(\varphi) = 6\sin^4(\varphi-\chi)\{2\sin(\varphi-\chi)+[16\sin^3(\varphi-\chi)\sin^2(\varphi-\beta)-18\sin^2(\varphi-\beta)\sin(\varphi-\chi)$$
$$-7\sin2(\varphi-\beta)\sin^2(\varphi-\chi)\cos(\varphi-\chi)+3\sin2(\varphi-\beta)\cos(\varphi-\chi)-12\sin^3(\varphi-\chi)+14\sin(\varphi-\chi)]\sin^2\alpha\},$$

$$\frac{d^2F_6(\varphi)}{d\varphi^2}+F_6(\varphi) = 6\sin2\alpha\sin^4(\varphi-\chi)[\sin2(\varphi-\chi)\sin(\varphi-\beta)-3\cos^2(\varphi-\chi)\cos(\varphi-\beta)].$$

Solving these equations, we obtain

$$F_2(\varphi) = C_6+\frac{1}{512}\{[9\sin(8\varphi-2\beta-6\chi)+8\sin6(\varphi-\chi)-56\sin(6\varphi-2\beta-4\chi)$$
$$+158\sin(4\varphi-2\beta-2\chi)+2\sin(4\varphi+2\beta-6\chi)-56\sin4(\varphi-\chi)-304\sin2(\varphi-\beta)+232\sin2(\varphi-\chi)$$
$$-40\sin(2\varphi+2\beta-4\chi)+312\varphi\cos2(\beta-\chi)-288\varphi]\sin^2\alpha-384\varphi+256\sin2(\varphi-\chi)-32\sin4(\varphi-\chi)\},$$

$$F_4(\varphi) = C_7\cos\varphi+C_8\sin\varphi+\frac{1}{1024}\{480\sin3(\varphi-\chi)-32\sin5(\varphi-\chi)\cos^2\alpha-3840\varphi\cos(\varphi-\chi)$$
$$-[9\sin(9\varphi-2\beta-7\chi)+8\sin7(\varphi-\chi)-43\sin(7\varphi-2\beta-5\chi)+30\sin(5\varphi-2\beta-3\chi) \qquad (17)$$
$$+2\sin(5\varphi+2\beta-7\chi)-192\sin3(\varphi-\chi)+414\sin(3\varphi-2\beta-\chi)+18\sin(3\varphi+2\beta-5\chi)$$
$$-2184\varphi\cos(\varphi-2\beta+\chi)+2880\varphi\cos(\varphi-\chi)-936\varphi\cos(\varphi+2\beta-3\chi)]\sin^2\alpha\},$$

$$F_6(\varphi) = C_9\cos\varphi+C_{10}\sin\varphi+\frac{\sin2\alpha}{512}\{5\cos(7\varphi-\beta-6\chi)-288\varphi\sin(\varphi-\beta)$$
$$+312\varphi\sin(\varphi+\beta-2\chi)-28\cos(5\varphi-\beta-4\chi)+2\cos(5\varphi+\beta-6\chi)$$
$$+42\cos(3\varphi-\beta-2\chi)+12\cos(3\varphi+\beta-4\chi)\},$$

where C_6, C_7, C_8, C_9, and C_{10} are integration constants.

The initial conditions for the first normal wave have the form $r = R_0$, $\theta = \pi/2$, $\tau = \tau_0$, and $d\theta/d\varphi = 0$ at $\varphi = \pi$ and, $r = b$ at $\varphi = \chi + \pi/2$. Hence, at $\varphi = \pi$, we must satisfy the conditions $F_1(\pi) = F_2(\pi) = F_3(\pi) = F_4(\pi) = F_5(\pi) = F_6(\pi) = 0$, $dF_5(\varphi)/d\varphi|_{\varphi=\pi} = dF_6(\varphi)/d\varphi|_{\varphi=\pi} = 0$ and, at $\varphi = \chi + \pi/2$, the conditions $F_3(\chi + \pi/2) = F_4(\chi + \pi/2) = 0$.

Using these relationships, we find the integration constants:

$$C_1 = -\frac{\cos\chi}{2}+\log\left|\frac{\sin(\chi)}{1-\cos(\chi)}\right|, \quad C_2 = 1-\frac{\sin^2\chi}{2}, \quad C_4 = C_5 = 0, \quad C_3 = \frac{\sin3\chi+5\sin\chi-4}{8\cos\chi},$$

$$C_6 = \frac{1}{512}\{[9\sin(2\beta+6\chi)+8\sin6\chi-56\sin(2\beta+4\chi)+158\sin2(\beta+\chi)$$
$$+2\sin(6\chi-2\beta)-56\sin4\chi-304\sin2\beta+232\sin2\chi+40\sin(2\beta-4\chi)$$
$$-312\pi\cos2(\beta-\chi)+288\pi]\sin^2\alpha+32[12\pi+8\sin2\chi-\sin4\chi]\},$$

$$C_7 = \frac{1}{1024}\{32[15\sin3\chi-\sin5\chi\cos^2\alpha+120\pi\cos\chi]-[8\sin7\chi+9\sin(2\beta+7\chi)$$
$$-43\sin(2\beta+5\chi)+30\sin(2\beta+3\chi)+2\sin(7\chi-2\beta)-192\sin3\chi+414\sin(2\beta+\chi)$$
$$-18\sin(2\beta-5\chi)+2184\pi\cos(2\beta-\chi)-2880\pi\cos\chi+936\pi\cos(2\beta-3\chi)]\sin^2\alpha\},$$

$$C_8 = \frac{1}{2048\cos\chi}\{[2496\chi\sin 2(\chi-\beta) - 936\pi\sin(4\chi-2\beta) + 2880\pi\sin 2\chi - 2184\pi\sin 2\beta$$
$$+ 2\cos(8\chi-2\beta) + 9\cos(2\beta+8\chi) + 16\cos(6\chi-2\beta) - 52\cos(6\chi+2\beta) - 18\cos(4\chi-2\beta)$$
$$+ 73\cos(4\chi+2\beta) - 696\cos(2\beta-2\chi) + 384\cos 2(\beta+\chi) + 8\cos 8\chi - 8\cos 6\chi - 192\cos 4\chi$$
$$+ 192\cos 2\chi + 368 - 414\cos 2\beta]\sin^2\alpha + 32[120\pi\sin 2\chi + (\cos 6\chi - \cos 4\chi + 2)\cos^2\alpha$$
$$- 15\cos 4\chi + 15\cos 2\chi + 30]\}, \tag{18}$$

$$C_9 = \frac{\sin 2\alpha}{512}\{312\pi\sin(2\chi-\beta) - 5\cos(\beta+6\chi) - 288\pi\sin\beta + 28\cos(\beta+4\chi)$$
$$- 2\cos(6\chi-\beta) - 42\cos(\beta+2\chi) - 12\cos(\beta-4\chi)\},$$

$$C_{10} = \frac{\sin 2\alpha}{512}\{140\sin(4\chi+\beta) - 10\sin(6\chi-\beta) - 35\sin(\beta+6\chi) - 36\sin(4\chi-\beta)$$
$$+ 312\sin(2\chi-\beta) - 126\sin(\beta+2\chi) - 288\sin\beta - 312\pi\cos(2\chi-\beta) + 288\pi\cos\beta\}.$$

Now let us find the coordinates R_d, θ_d, φ_d of the point at which the satellite with the detector is located for the chosen orientation of the coordinate axes.

The radial coordinate of the satellite has not changed on the rotation of the spherical coordinates and equals $r = R_d$. Let us determine the angular coordinates φ_d and θ_d of the satellite for the chosen orientation of the spherical coordinates. Taking into account that, due to the nonlinear electrodynamics and gravitational action, rays are bended, we can write the coordinate φ_d in the form

$$\varphi_d = \chi + \delta\varphi_d, \tag{19}$$

where $\delta\varphi_d \ll 1$. Then, from the equation $r_1(\varphi_d) = R_d$, we have

$$\delta\varphi_d = \frac{b}{R_d} - \frac{r_g}{2b}\left\{\cos\chi + 2 + \frac{\cos\chi}{(1+\sin\chi)}\right\} - \frac{\mathbf{m}^2\xi\eta_1}{1024b^6\cos\chi}\{[184\sin\chi - 18\sin(5\chi-2\beta) - 9\sin(7\chi+2\beta)$$
$$- 2\sin(7\chi-2\beta) - 369\sin(3\chi-2\beta) - 30\sin(3\chi+2\beta) - 21\sin(\chi-2\beta) - 414\sin(\chi+2\beta)$$
$$- 8\sin 7\chi + 192\sin 3\chi + 43\sin(5\chi+2\beta) - 1248\pi\cos(3\chi-2\beta) + 936\chi\cos(3\chi-2\beta)$$
$$- 1872\pi\cos(\chi-2\beta) + 2184\chi\cos(\chi-2\beta) + 2880(\pi-\chi)\cos\chi]\sin^2\alpha$$
$$+ 32[15\sin 3\chi + (\sin\chi - \sin 5\chi)\cos^2\alpha + 15\sin\chi + 120(\pi-\chi)\cos\chi]\}. \tag{20}$$

Substituting expressions (19) and (20) into the last expression in (15), we find the angular coordinate θ_d of the satellite for the chosen orientation of the spherical coordinates:

$$\theta_d = \frac{\pi}{2} + \frac{\mathbf{m}^2\xi\eta_1\sin 2\alpha}{512b^6}\{600(\pi-\chi)\sin(\chi-\beta) + 4\cos(7\chi-\beta) + 15\cos(7\chi+\beta) + 6\cos(5\chi-\beta)$$
$$- 76\cos(5\chi+\beta) - 180\cos(3\chi-\beta) + 126\cos(3\chi+\beta) + 45\sin(\chi-\beta) + 60\cos(\chi+\beta)\}. \tag{21}$$

Expressions (15) for the ray along which the pulse carried by the second normal wave propagates takes the form

$$\tau_2(\varphi) = \tau_0 + \frac{b\cos\chi}{c\sin\chi} + \frac{r(\varphi)}{c}\cos(\varphi-\chi) + \frac{r_g}{cb}F_1(\varphi) + \frac{\mathbf{m}^2\xi\eta_2}{cb^5}F_2(\varphi),$$

$$r_2(\varphi) = b\left[\sin(\varphi-\chi) + \frac{r_g}{b}F_3(\varphi) + \frac{\mathbf{m}^2\xi\eta_2}{b^6}F_4(\varphi)\right]^{-1}, \tag{22}$$

$$\theta_2(\varphi) = \frac{\pi}{2} + \frac{r_g}{b}F_5(\varphi) + \frac{\mathbf{m}^2\xi\eta_2}{b^6}F_6(\varphi).$$

The functions $F_1(\varphi)$, $F_2(\varphi)$, $F_3(\varphi)$, $F_4(\varphi)$, $F_5(\varphi)$, and $F_6(\varphi)$, entering into these relationships are determined by expressions (16) and (17), but only some of the integration constants are different, due to different initial conditions.

The ray of the second normal wave at $\varphi = \pi$ and $\tau = \tau_0$ originates at the same point $r = R_0$, $\theta = \pi/2$, as the ray of the first wave. Therefore, the integration constants C_1, C_2, C_4, C_6, C_7, and C_9 are the same as those for the ray of the first normal wave (18). In order to determine the remaining integration constants, we take into account that, from the bundle of rays of the second normal wave, we choose the one hitting the detector, which has the coordinates $\varphi = \varphi_d = \chi + \delta\varphi_d$, $r = R_d$, $\theta = \theta_d$.

Substituting $\varphi = \varphi_d = \chi + \delta\varphi_d$ into the equation $r_2(\varphi_d) = R_d$ and solving it, we obtain the integration constants C_3 and C_8 for the second normal wave. Then we substitute relationship (19) into the equation $\theta_2(\varphi_d) = \theta_d$. Hence we find the integration constants C_5 and C_{10}. Thus, we have all necessary for the analysis of the basic effects of nonlinear vacuum electrodynamics observed on the propagation of electromagnetic waves in the magnetic field of pulsars.

5. ANALYSIS OF RESULTS

From the analysis of (20) and (21) it follows that rays of normal waves are bent in the magnetic and gravitational fields of a pulsar; the gravitational field bends only rays in the tangent plane, and the magnetic field, in other planes. Since the ray bending effect has been thoroughly studied in earlier published works [13, 28, 31], we will not consider this effect here; moreover, due to the large distance ($R_d \gg R_s$) between pulsars and the Earth, this bending is impossible to measure.

Therefore, we will study the effect of delay: the presence of a nonzero difference $\Delta\tau = [\tau_2(\varphi) - \tau_1(\varphi)]|_{\varphi=\varphi_d}$ between the time τ_2 required for a pulse carried by the second normal wave to pass the wave from the origin to the satellite and the time τ_2 required to pass the same way for the pulse carried by the first normal wave.

Using relationships (15), (16)–(18), and (22), we obtain the following expression for the delay $\Delta\tau$:

$$\Delta\tau = \frac{\mathbf{m}^2\xi(\eta_2 - \eta_1)}{cb^5} f(\alpha, \beta, \chi), \tag{23}$$

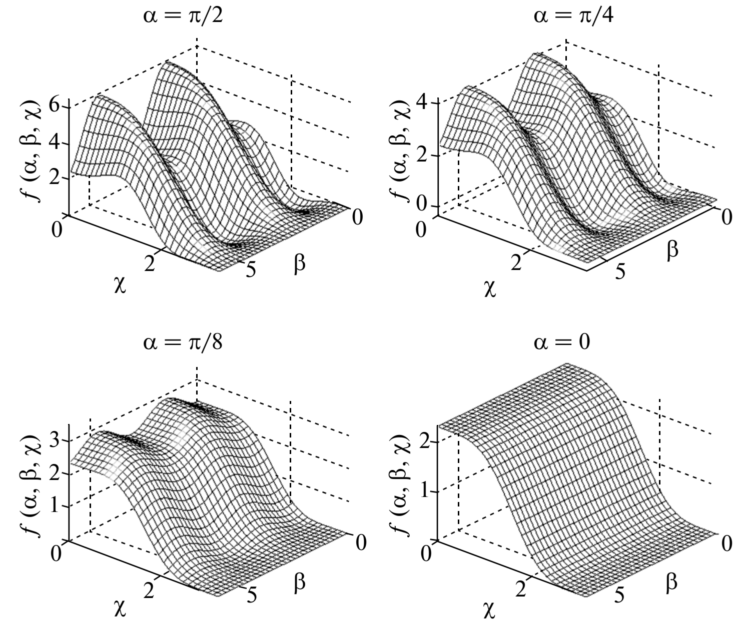
where

$$\begin{aligned} f(\alpha, \beta, \chi) = \frac{1}{512}\{ & [2\sin(6\chi - 2\beta) + 9\sin(6\chi + 2\beta) - 40\sin(4\chi - 2\beta) - 56\sin(4\chi + 2\beta) \\ & - 155\sin(2\chi - 2\beta) + 158\sin(2\chi + 2\beta) + 8\sin(6\chi) - 56\sin(4\chi) - 304\sin(2\beta) + 232\sin(2\chi) \\ & - 312(\pi - \chi)\cos(2\chi - 2\beta) + 288(\pi - \chi)]\sin^2\alpha + 32[8\sin(2\chi) - \sin(4\chi) + 12(\pi - \chi)]\}. \end{aligned} \tag{24}$$

Each pulsar rotates with some frequency Ω_1 about the axis passing through its center of mass, which does not coincide with the vector of dipole moment $\mathbf{m}$; in addition, the rotation axis performs regular precession with the frequency $\Omega_2 > \Omega_1$. However, for the majority of typical neutron stars possessing a strong magnetic field, the periods of these rotations $T_1 = 2\pi/\Omega_1$ and $T_2 = 2\pi/\Omega_2$ prove to be significantly greater than the time of propagation of an electric pulse in the region of a strong magnetic field, $T \approx 2R_s/c$. Therefore, when solving our problem of the nonlinear electrodynamics and gravitational effects of the fields of a neutron star on the propagation of electromagnetic pulses, we might consider the angles α and β independent of the time and, only in the final result (23), take into account that they are slowly varying functions of time. This means that there are pulsars in which, due to the rotation, the angles α and β can vary in the ranges $0 \le \alpha \le \pi$, $0 \le \beta < 2\pi$. The angle χ is determined by the position of the X- and gamma radiation source with respect to the point of the ray of the first normal wave nearest to the center of the pulsar.

Simple analysis of expression (24) shows that, as $\chi \longrightarrow 0$, the source is found at the spatial infinity; in this case, the pulsar is situated between the radiation source and the satellite. This case is possible if the X- and gamma radiation source is the magnetosphere of another neutron star or even a Seyfert galaxy.

At $\chi = \pi/2$, the radiation source is situated at the point of a ray nearest to the center of the pulsar. If $\chi > \pi/2$ but $\chi < \pi$, the X- and gamma radiation source and the satellite are found on the same side of the pulsar. Since, in this case, the electromagnetic pulse travels in the periphery of the magnetic field of a pulsar without hitting the region of stronger magnetic field, it is obvious that the effect of delay for $\chi > \pi/2$ will be strongly suppressed.



Function $f(\alpha, \beta, \chi)$ at some particular values of angle α.

Let us analyze the behavior of the function $f(\alpha, \beta, \chi)$ in the admissible range of the angles α, β, and χ. Expression (24) implies that the $f(\alpha, \beta, \chi)$ attains the maximum at $\alpha = \pi/2$. The numerical analysis has shown that (see the figure) that, at $\alpha = \pi/2$, the maximum value of the function $f(\alpha, \beta, \chi)$ is attained if $\chi = 0$ and $\beta = \pi/2$ or $\beta = 3\pi/2$. If $\alpha = \pi/2$, $\beta = \pi/2$, but $\chi = \pi/2$, the function $f(\alpha, \beta, \chi)$ is smaller by a half.

The condition of applicability of the parametric post-Maxwellian approximation used in the calculation of the delay is the smallness of the parameter $\xi B^2(r)$ at all point of the rays. Then, since $r \le b$, this condition is satisfied if $\xi B^2(b) \sim 0.1$. Then, assuming that some of the pulsars can provide the conditions $\alpha = \pi/2$, $\beta = \pi/2$, and $b = 10R_s$ and taking into account that, for typical pulsars, $R_s = 10$ km, we obtain the estimate $\Delta\tau \sim 10^{-9}$ s.

Thus, the delay $\Delta\tau$ of the pulse carried by the second normal wave relative to the pulse carried by the first normal wave from their common point of origin to the satellite, in the most favorable case, can attain 10^{-9} s. Such time intervals can be detected by modern electronic devices. Therefore, the inclusion of this experiment into programs of future satellite mission will make possible an independent check of whether the vacuum electrodynamics is a nonlinear theory or is fundamentally linear.

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