Adiabatic modulation of a cnoidal wave by a breather with orthogonal circular polarization in an isotropic gyrotropic nonlinear medium

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Abstract: An approximate analytical solution of the non-integrable problem of steady-state adiabatic interaction of a cnoidal wave with a breather is obtained. The solving algorithm is described by the example of one-dimensional problem of steady-state interaction of a plane cnoidal wave with circular polarization (the “information” signal) with orthogonally polarized rational soliton (the “control” signal) in an isotropic nonlinear gyrotropic medium with Kerr nonlinearity and second-order group-velocity dispersion. It is shown that such the interaction results in a strong amplitude and frequency modulation of the information signal and this modulation is localized in the region where intensity of the control signal changes.

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References and links

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1. Introduction

Features of interaction of self-consistent solutions of nonlinear problems (breathers, solitons, and cnoidal waves [1]) have recently attracted an increasing interest [2–11]. This follows from some rather evident reasons. Firstly, the use of optical solitons is a very promising tool in ultra-fast data transmission [12–14]. Therefore, solution of the interaction problem provides the answer to a crucial question about possibility or impossibility of simultaneous transmission of multiple streams of undistorted optical information through the common channel. Secondly, due to such processes we can control characteristics of radiation, passed through a nonlinear medium, by means of optical signals [15–17]. For the media with electronic nonlinearity such control can be extremely fast. And finally, thirdly, this problem is of large fundamental interest. The fact is that this problem is of multi-parametric character and in most cases is described by a non-integrable system of nonlinear equations [18]. Therefore, to solve this system, the numerical methods are commonly used, what does not always make it possible to trace the role of each parameter and make any definite conclusions.

To yield an analytical answer providing an unambiguous interpretation, some approximate methods are often used. Under certain restrictions this is a very promising approach. The most commonly used methods are the perturbation theory [19] and the linearization technique [20]. However, since breathers, solitons, and cnoidal waves are “purely nonlinear objects”, they cannot be properly described in terms of lower-order perturbation theory [19] and require too long series of successive approximations. Results obtained by the linearization technique must always be mentally extrapolated to the nonlinear case [20].

The adiabatic approximation enables one to describe nonlinear interactions between “fast” and “slow” subsystems (see [21] and reference in it) is very popular both in the quantum, classical and semi-classical descriptions of nonlinear dynamics of quite different systems [22]. Combining this approximation with a procedure of separation of variables has recently enabled obtaining the approximate analytical solutions of non-integrable problem of interaction of two cnoidal waves [21]. In this paper, we demonstrate that quite similar approach can be successfully used in situations where this procedure cannot be used. The solving scheme is described by the example of one-dimensional problem of interaction of a plane cnoidal wave with circular polarization (the “information” signal) with orthogonally polarized rational (or Peregrine) soliton [23,24] (the “control” signal) in an isotropic gyrotropic medium with Kerr nonlinearity and second-order group-velocity dispersion. It is shown that in this case interaction with the control signal results in appreciable amplitude and frequency modulation of the information signal and this modulation is localized in a region where the control signal intensity changes.
2. Statement of the problem and the calculation scheme

Let’s write a system of two coupled nonlinear Schrödinger equations (NSEs) for the problem of interaction of two plane light waves with orthogonal circular polarizations (the information and control signals) in a standard [20,21] form

\[
\frac{\partial A}{\partial z} - i \frac{k_z}{2} \frac{\partial^2 A}{\partial t^2} + i \left[ -\rho_0 + \frac{1}{2} (\sigma_i / 2 - \rho_i) |A_i|^2 + \frac{1}{2} (\sigma_i / 2 + \sigma_z) |A_z|^2 \right] A = 0, \quad (1a)
\]

\[
\frac{\partial A}{\partial z} - i \frac{k_z}{2} \frac{\partial^2 A}{\partial t^2} + i \left[ +\rho_0 + \frac{1}{2} (\sigma_i / 2 + \rho_i) |A_i|^2 + \frac{1}{2} (\sigma_i / 2 + \sigma_z) |A_z|^2 \right] A = 0. \quad (1b)
\]

Here \( A_z(z,t) \) are the truncated amplitudes of the field components \( E_z = A_z \exp[i(\omega t - kz)] \) with right and left circular polarizations and the frequency \( \omega \), propagating along the \( z \) axis; \( t \) is the time in own (running) coordinate system; the constant \( k_z = \frac{\partial^2 k}{\partial \omega^2} < 0 \) characterizes the group-velocity dispersion; \( k \) is the wave number. The parameters \( \sigma_0 = 4\pi \omega^2 \chi_{xxyy} / k c^2 \) and \( \sigma_z = 2\pi \omega^2 \chi_{xxyy} / k c^2 \) are determined by two independent components of the local cubic nonlinearity tensor \( \chi^{(3)}(\omega_-, \omega_-, \omega_0) \) while \( \rho_{0,1} = 2\pi \omega^2 \gamma_{0,1} / c^2 \) are defined through the pseudoscalar constants \( \gamma_{0,1} \) of linear and nonlinear gyration.

To solve the system Eq. (1) we firstly separate the variables in Eq. (1b) by substituting \( A_z(z,t) = A(t) \exp(i \kappa z) \),

\[
A(z,t) = r(t) \exp(i \kappa z), \quad (2)
\]

where the constant \( \kappa \) and the function \( r(t) \) are real. This results in an ordinary differential equation

\[
\frac{d^2 r}{dt^2} - i \frac{2k_z}{2} \left[ \Delta \kappa_0 + \left( \frac{\sigma_0}{2} + \rho_0 \right) r^2 + \left( \frac{\sigma_i}{2} + \sigma_z \right) |A|^2 \right] r = 0. \quad (3)
\]

Here \( \Delta \kappa_0 = \kappa_0 + \rho_0 \). Further, taking into account the fact that the separation of variables greatly simplifies the averaging procedure, intrinsic to adiabatic approximation, we consider the field component \( A_z(z,t) \) as the “fast” one. It means the field component \( A_i(z,t) \) is supposed to be the “slow” one and, after substituting Eq. (2) into Eq. (1a), the latter can be averaged over the fast oscillations in time:

\[
\frac{\partial A}{\partial z} - i \frac{k_z}{2} \frac{\partial^2 A}{\partial t^2} + i \left[ -\rho_0 + \frac{1}{2} (\sigma_i / 2 + \sigma_z) \langle r^2 \rangle \right] A_i + i \left( \frac{\sigma_i}{2} + \sigma_z \right) |A|^2 A_i = 0. \quad (4)
\]

Here \( \langle ... \rangle_t \) represents averaging over the time. Following the adiabatic approach [21], we firstly solve Eq. (3) on the assumption that \( |A_i| \equiv \text{const} \). This enables us to write down the solution of Eq. (3) in well-known forms of possible cnoidal waves, for example, in the form \( r(t) = B \cdot \text{cn}(v t, \mu) \) [2,21], where \( \text{cn}(x, \mu) \) is Jacobi elliptic function [25],

\[
B^2 = \frac{4\mu_0 \Delta \kappa_0}{(2\mu_0 - 1)(\sigma_i + 2\sigma_z)} \left[ 1 + (\sigma_i + 2\sigma_z) |A_i|^2 / (2\Delta \kappa_0) \right] > 0,
\]

\[
v^2 = \frac{2\Delta \kappa_0}{k_z (2\mu_0 - 1)} \left[ 1 + (\sigma_i + 2\sigma_z) |A_i|^2 / (2\Delta \kappa_0) \right] > 0. \quad (5)
\]

It means that
Here $K(\mu)$ and $E(\mu)$ are the complete elliptic integrals of the first and second kinds [25]. After substituting the Eq. (6) into Eq. (4), we obtain the following equation

$$\frac{\partial A}{\partial z} - i \frac{k_2}{2} \frac{\partial^2 A}{\partial t^2} + i \left( -\hat{\rho}_0 + \frac{d}{2} |A|^2 \right) A = 0,$$

(7)

where

$$\hat{\rho}_0 = \rho_0 \left[ 1 + \frac{2\Delta \kappa \cdot (\sigma_i + 2\sigma_2)}{\rho_0 (2\mu_i - 1)(\sigma_i + 2\rho_i)} \left[ \mu_i^2 - 1 + E(\mu_i) / K(\mu_i) \right] \right],$$

(8a)

$$d = (\sigma_i - 2\rho_i) \left[ 1 - \frac{2(\sigma_i + 2\sigma_2)^2}{(2\mu_i - 1)(\sigma_i^2 - 4\rho_i^2)} \left[ \mu_i^2 - 1 + E(\mu_i) / K(\mu_i) \right] \right].$$

(8b)

Because Eq. (7) is independent NSE, we can write down its solution in forms of so-called breathers, for example, in the form of a rational soliton [23,24]:

$$A_i(z,t) = \sqrt{P_i} \left[ 1 - \frac{4(1 - izP_1^2 d)}{1 + (zP_1^2 d)^2 + 2izP_1^2 d / |k_2|} \right] \exp \left[ iz(\hat{\rho}_0 - P_1 d / 2) \right], \quad k_2 < 0,$$

(9)

where $P_0$ is the free constant related to the power flux in the coherent continuous background. In Eqs. (7) and (9) the minus sign before the imaginary units is responsible for introducing the concrete representation of the truncated amplitudes in Eq. (1). While the rational soliton is a limiting case of Akhmediev breather and Kuznetsov-Ma soliton, the control signal in the form of Eq. (9) looks much better for the case considered here since the rational soliton describes an isolated peak on the coherent pedestal on the plane $\{z, t\}$. Substituting Eq. (9) into Eq. (5), we obtain the final approximate solution for the fast field component $A(z,t)$ as

$$A_i(z,t) = r_i(z,t) \exp(i\kappa_i z) = B_i(z,t) \exp(i\nu_i(z,t) t, \mu) \exp(i\kappa_i z),$$

(10a)

$$B^2 = B_i^2(z,t) = -\frac{4\mu_i^2 \Delta \kappa [1 + m(z,t)]}{(2\mu_i - 1)(\sigma_i + 2\rho_i)} > 0, \quad \nu_i^2 = \nu_i^2(z,t) = \frac{2\Delta \kappa [1 + m(z,t)]}{k_i(2\mu_i - 1)} > 0.$$

(10b)

Here, the “modulation” function $m(z,t)$ is defined by

$$m(z,t) = \frac{\sigma_i + 2\sigma_2}{2\Delta \kappa} \left| A_i(z,t) \right|^2 = \frac{\sigma_i + 2\sigma_2}{2\Delta \kappa} \left[ 1 + \frac{8(1 + (zP_1 d)^2 - 2izP_1^2 d / |k_2|)}{[1 + (zP_1^2 d)^2 + 2izP_1^2 d / |k_2|]^2} \right].$$

(10c)

Note that in Eq. (10a) the odd function $r_i(z,t) = B_i \exp[i\nu_i(z,t) + K(\mu), \mu]$ can be used along with the even function $r_i(z,t)$. Thus, we have obtained the analytical solution of the interaction problem for a cnoidal wave with a rational soliton in the adiabatic approximation.

3. Applicability of the adiabatic approximation and the obtained solution character

Firstly, to use the adiabatic approximation we have supposed that the component $A_i(z,t)$ varies in time much faster than the component $A_i(z,t)$. It means the period $T = 4K(\mu)\nu_i^{-1}$ of the cnoidal wave must be much less than the time width...
$\Delta t = \left| k^2 \right|^{1/2} \left( 2P_0 |d| \right)^{1/2}$ of the rational soliton peak, that results in the inequality $(P_0 |d|)^{1/2} << \nu_\perp \left| k \right|^{1/2} / 8K(\mu)$.  

Secondly, the solution Eqs. (9) and (10) has been obtained by the use of separation of variables and of averaging the Eq. (6). Therefore, space-time modulation of $r_\perp$ must not violate the Eqs. (2) and (6). It means that $|\delta r_\perp (t, z)/\delta z| << |\kappa_\perp r_\perp (t, z)|$ and the maximal value of the time-dependent part of $\nu_\perp (t, z)$ must be much smaller than the “tailes” (its continuous part). The later requirement is satisfied if

$$|\sigma_1 + 2\sigma_2| P_0 << 0.3 |\Delta \kappa_\perp| . \quad (11)$$

It follows from Eq. (11) that $m(z, t) << 1$. This simplifies the first requirement and enables us to obtain the final inequalities by estimating the maximal derivative values in the second one:

$$(P_0 |d|)^{1/2} << \Delta \kappa_\perp \left| k \right|^{1/2} \left( 4K(\mu) \right)^2 |2\mu^2 - 1|^{1/2} ]$$

$$P_0^{3/2} \left| \sigma_1 + 2\sigma_2 \right| |d|^{1/2} << 2.5 |\kappa_\perp| \left| \Delta \kappa_\perp \left| 2\mu^2 - 1 \right|^{1/2} \right.$$

$$\quad \left(13\right)$$

Inequalities Eqs. (11)–(13) determine applicability conditions for the adiabatic approximation. Equations (12) and (10b) show that $P_0^{3/2} << |B_\perp|_{\max}$, i.e. the control signal must be weak.

Parameters of non-integrable interaction problem must satisfy the Eqs. (11)–(13), (9), and (10b). To illustrate a character of the obtained approximate solution we consider here all the variables and parameters as the dimensionless ones because of the following renormalization: $A, P_0^{1/2} \rightarrow A_\perp, \rho_\perp \rightarrow z_\perp, \rho_\parallel / \sigma_1 \rightarrow \rho_\parallel, \sigma_2 / \sigma_1 \rightarrow \sigma_2, \epsilon (\rho_\parallel \left| k \right|^{1/2} ) \rightarrow t, r_\parallel (\sigma_1 / \rho_\parallel )^{1/2} \rightarrow r_\parallel, \sigma_1 / \rho_\parallel \rightarrow \kappa_\perp, \kappa_\perp / \rho_\parallel \rightarrow \kappa_\perp$. This is equivalent to the choice of the unit values for $P_0, \rho_\parallel, \sigma_1, k_\parallel$. The used values of all other parameters are defined as: $\kappa_\perp = -1.5, \rho_\perp = 0.1, \sigma_2 = -0.6, \mu_\perp = 0.7193$. Dependencies of the rational soliton amplitude $|A_\perp (z, t)|$ and the cnoidal wave “frequency” $\nu_\perp (z, t)$ are shown in Fig. 1. As it was mentioned above,
$v(z,t)$ determines the cnoidal wave period. It is easy to see that the character of slow space-time modulation of $v(z,t)$ [Eq. (10)] exactly follows the shape of the control signal. Dependencies $r(z,t)$ and $r'(z,t)$ for the odd and even cnoidal waves are illustrated in Fig. 2. We can see here a well-defined space-time modulation of the cnoidal waves of both types.

An amplitude-frequency modulation is also intrinsic to solutions of some integrable nonlinear equations [6,7,11].

6. Conclusions

In this paper we have used the adiabatic approximation to obtain an approximate analytical solution of a non-integrable problem of interaction of a breather with a cnoidal wave. The used algorithm have been described by the example of one-dimensional problem of steady-state interaction of a plane cnoidal wave with circular polarization (an “information” signal) with orthogonally polarized rational (or Peregrine) soliton (a “control” signal) in an isotropic nonlinear gyrotrropic medium with Kerr nonlinearity and second-order group-velocity dispersion. We have demonstrated the appreciable space-time modulation of the amplitude and the frequency of the powerful information signal by the week control signal takes place in a region where the control signal significantly changes. Spatial and temporal widths of the rational soliton change too due to interaction with the cnoidal wave.

Note that the obtained solution is only the first iteration of the adiabatic approximation, if it is necessary its accuracy may be improved in the following iterations [21].

In principle, the same calculation scheme can be used to analyze other wave combinations: interaction of Akhmediev breather or Kuznetsov-Ma soliton with other cnoidal waves.

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