Multiple-scattering Henyey-Greenstein phase function
and fast path-integration

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ABSTRACT

It is shown that stability of Henyey-Greenstein phase function gives a possibility to solve quickly a multiple-scattering light propagation problem with the same a priori information about interaction as in the primary problem definition.

Keywords: Multiple-scattering, Henyey-Greenstein phase function, stable probability distribution, fast path integration

1. INTRODUCTION

It is well known that multiple-scattering light propagation problems can not be resolved by direct use of Maxwell or wave equations and are usually solved numerically by so-called transport theory 1,2. The same result can be obtained in the framework of so-called probabilistic description, that is, by the Monte-Carlo simulation 3,4 or by the path-integration technique 5,7. The basis of such approach is ‘a priori’ statistical information about possible ‘photon-medium’ interaction processes, that is, about the indexes µs, a of absorption and scattering of photons as well as about the single-scattering phase function P_s(θ), which describes probability distribution for single-scattering on two-dimensional (2D) angle θ = (θ, ϕ), where θ and ϕ are azimuth and polar angles 1,4. Because, in such an ‘exact’ statement, operating only with a priori statistics, multiple-scattering problems can not be analytically solved too, additional simplifying approximations are usually introduced 2,7-11. Unfortunately, when the photon propagation distance is about 1000 scattering lengths and more, verification of results, obtained with the use of such approximations by means of listed above exact techniques, becomes practically impossible due to enormous increase of the calculation time. It is happened because more or less reliable (with relative error ~ 1%) calculation of the only probability distribution for the photon paths from a source to a photo-detector needs in simulation of more than 10^{13} realizations, which directly follows from the necessity to fill a data array (10^4 photons per cell) corresponding to this distribution.

A cardinal gain in the calculation rate of listed above ‘exact’ calculation techniques can be realized by forced introducing the 2D distributions P_s^{(k)} (θ), that is, by ‘a priori’ description of k-order scattering processes 11-14. To realize this promising idea, the authors of 13,14 supposed that

\[ P_s^{(k)}(\theta) = \frac{1}{4\pi} \frac{1 - g_s^2}{(1 + g_s^2 - 2g_s \cos \theta)^{3/2}}, \quad k = 1, 2, \ldots \]  \hspace{1cm} (1)

is the Henyey-Greenstein phase function 15, whose anisotropy parameter g_s = g_s = \langle \cos \theta \rangle determines the mean cosine of k-order scattering angle θ and vary in the same limits from 0 (the case of isotropic scattering) to 1 (the case of forward scattering). It was assumed that full number k of scattering events on a trajectory part becomes a new effective constant \textit{k}_{\text{eff}} , which value depends on this part length Δz and can be expressed through the mean value \langle k \rangle = \mu_s \Delta z of single-scattering events 12-14. The needed dependence \textit{k}_{\text{eff}} (Δz) was introduced from a simple semi-empirical consideration.

In the paper, we show that, with taking into account stability 16 of P_s^{(k)} (θ) distribution and following similarity of the P_s^{(1)} (θ) and P_s^{(k)} (θ) [see (1)], there is no need to use a semi-empirical consideration and the needed dependence P_s^{(k)} (Δz) can be found in exact analytic form. It means a cardinal gain in calculation rate for multiple-scattering problems can be easily realized by means of any from the listed above ‘exact’ approaches with the use of the same ‘a priori’ statistical information about the interaction processes as in the single-scattering primary problem definition.
2. MULTIPLE-SCATTERING PHASE FUNCTION

Let suppose that, propagating along a part of the trajectory, a photon many times changes its propagation direction by the angle \( \theta_k = (\theta_k, \phi_k) \), where \( k = 0, 1, \ldots \). Choosing \( z \)-axis direction corresponding to \( \theta_0 = 0 \), supposing all the single-scattering events are independent, and interesting in the overall change \( \theta = \sum_{k=0}^{\infty} \theta_k \) of the propagation direction, we can introduce an effective multiple-scattering phase function in rather evident form

\[
P_\infty(\theta, \Delta z) = \sum_{k=0}^{\infty} P_k(\theta) P_k(\Delta z).
\]  

(2)

Here \( P_k(\Delta z) \) describes probability of \( k \)-order scattering without absorption (the case of path-integration technique); \( \Delta z \) is the length of the trajectory part;

\[
P_k(\theta) = \int \int d\theta' P_{k-1}(\theta') P_{k-1}(\theta - \theta'), \quad k = 1, 2, \ldots
\]

(3)

is the \( k \)-order scattering phase function, and \( P_k(0) = \delta(\theta, \phi) \) is the \( \delta \)-function.

By means of characteristic function approach, one can show that, in case of independent single-scattering events with arbitrary single-scattering phase function \( P_{s1}(\theta) \),

\[
\langle \cos^2 \sum_{k=0}^{\infty} \theta_k \rangle = \langle \cos \theta_1 \rangle^2.
\]

(4)

Therefore, if the single-scattering phase function would be defined by (1) and this distribution would be stable \( g_1 = g_1 \), we obtain a very simple and convenient relationship

\[
g_1 = g_1 \quad (1).
\]

Stability of Henyey-Greenstein phase function (1) (as well as of any phase function depending on \( \cos \theta \)) can be easily proved rigorously using analytical approach \( 11,14 \). However, here we illustrate this statement only by a result of numerical integration of (3) for \( P_{s1}(\theta) \) determined by (1) for \( g_1 = 0.95 \) and \( k = 1, 2, \ldots, 30 \). Dependencies \( F^{(k)}(\theta) = \int_0^{2\pi} d\phi P^{(k)}(\theta) \) calculated numerically (points) and by the use of an analytic expression

\[
F^{(k)}(\theta) = \frac{1}{2} \frac{1 - g_1^k}{(1 + g_1^k - 2g_1^k \cos \theta)^{3/2}},
\]

(5)

(solid lines) following from (1) and (4) are shown on the plane \( (\theta, k) \) in Figure 1. It is easy to see that, for \( g_1 = g_1^k \) in the range from 0.99 to 0.07 (almost isotropic scattering), deviation of numerical integration data from (5) is less than \( 10^{-3} \).

![Fig. 1.](http://proceedings.spiedigitallibrary.org/)

\[ F^{(k)}(\theta) \] for different \( k \) values: \( g_1 = 0.95 \); the data are obtained by numerical integration of (3) (points) and by analytical expression (5) following from (1) and (4) (solid lines)
3. MULTIPLE-SCATTERING STATISTICAL MOMENTS

Notice that, because different lengths of photon trajectories with different order of scattering, \( P^{(k)}(\Delta z) \) in (2) cannot be supposed given by a standard Poisson distribution (see \(^{13,14}\)). It is why, to calculate statistical moments of \( P^{(k)}(\Delta z) \) distribution, we will use the following simple consideration.

In case of \( k \)-order scattering, photon trajectory looks like a broken line consisted of \((k+1)\) straight segments \( \Delta l_i \) \((i = 0, 1, \ldots, k)\). The ends of all these segments are the points where a photon changes its propagation direction by the 2D angle \( \theta_i \) (see Figure 2).

![Fig. 2. Photon trajectory in case of 4-order scattering](image)

Let us suppose that the lengths \( \Delta l_i \) of these segments are distributed according to the exponential law with statistical moments of the first \( \langle \Delta l_i \rangle = \mu_i \) and the second \( \langle \Delta l_i^2 \rangle = 2\mu_i^2 \) orders correspondingly. It means the mean length \( \langle \Delta l^{(k)} \rangle = (k+1)\mu_i \) of the considered trajectories depends on the total number \( k \) of single-scattering events and

\[
\langle [\Delta l^{(k)}]^2 \rangle = 2(k+1)\mu_i^2.
\]

Projecting the scattering points to \( z \)-axis, we obtain a set of \((k+1)\) serially placed segments with not equal lengths \( \Delta z_i = \Delta l_i \cos(\sum_{m=0}^{i-1} \theta_m) \), where \( \theta_0 = 0 \). Averaging \( \Delta z_i \) on \( \Delta l_i \) and \( \theta_i \) with assumptions of a statistical independence of free path lengths \( \Delta l_i \) and single-scattering angles \( \theta_i \) (which corresponds to approximation of ‘point’ scattering centers) we obtain

\[
\langle \Delta z^{(k)} \rangle = \sum_{i=0}^{k} \langle \Delta z_i \rangle = \mu_i^\prime \sum_{i=0}^{k} g_i = \mu_i^\prime (1-g_i^{k+1})/(1-g_i),
\]

where \( \langle \Delta z^{(k)} \rangle \) is mean displacement of a photon along \( z \) axis in \( k \)-order scattering case. It follows from (6) that

\[
k = \left\lfloor \ln \left[ 1 - \mu_i^\prime \left( \langle \Delta z^{(k)} \rangle/(1-g_i) \right) \right] / \ln g_i \right\rfloor - 1,
\]

and the overall mean length of the trajectory for such order scattering is equal to

\[
\langle \Delta l^{(k)} \rangle = \mu_i^\prime \ln \left( 1 - \mu_i^\prime \left( \langle \Delta z^{(k)} \rangle/(1-g_i) \right) \right) / \ln g_i.
\]

It is easy to check that, while \( \langle \Delta l^{(k)} \rangle \to \infty \) when \( k \to \infty \), the mean displacement \( \langle \Delta z^{(k)} \rangle \) of a photon along \( z \) axis can not exceed a critical value \( \Delta z^{(\infty)} = (\mu_i^\prime)^{-1} \), where \( \mu_i^\prime = \mu_i (1-g_i) \) is the reduced scattering index. This familiar result [see expression (28) of the paper \(^{17}\)] follows from an evident fact that \( \langle \Delta z^{(k)} \rangle \) grows only thanks to a regular (with nonzero mean value of \( z \)-axis projection) component of photon velocity. And this is the distance \( \Delta z^{(\infty)} \), which determines a possibility to pass to the diffusion approximation, in which a further displacement of a photon along \( z \)-axis must be described through the second-order statistical moment, that is, with taking into account an irregular (with zero mean value of \( z \)-axis projection) component of the photon velocity.

To take this component of the overall displacement into account, let us calculate the value of

\[
\langle [\Delta z^{(k)}]^2 \rangle = \left( \sum_{i=0}^{k} \langle \Delta z_i \rangle^2 \right) = \sum_{i=0}^{k} \langle \Delta z_i^2 \rangle + 2 \sum_{j>i=0}^{k} \langle \Delta z_i \Delta z_j \rangle \]

(9)
It should be noticed here that the main problem of such a calculation lies in statistical dependence of \( \Delta z_i \) and \( \Delta z_j \) in the second term of right part of (9). Really, the photon propagation direction after the \( j \)-th scattering event, described by the overall angle \( \sum_{m=0}^{j} \theta_m \), depends on all the previous scattering events because this angle includes the sum \( \sum_{m=0}^{i} \theta_m \), describing the photon propagation direction after the \( i \)-th scattering event. However, with taking into account a stable character of Henyey-Greenstein phase function (see above) and statistical independence of free-path lengths \( \Delta l_i \) and of single-scattering angles \( \theta_i \), there is an evident possibility to average (9) because

\[
\sum_{i=0}^{k} \langle \Delta z^2 \rangle = \langle \Delta l^2 \rangle \sum_{i=0}^{k} \left( \cos^2 \left( \sum_{m=0}^{i} \theta_m \right) \right) = \langle \Delta l^2 \rangle \sum_{i=0}^{k} \left( 1 + \frac{2 \cos(2 \theta)}{3} \right) = \frac{1}{3} \langle \Delta l^2 \rangle \left( k + 3 + \frac{2}{1 - g_i^2} \right),
\]

(10)

\[
2 \sum_{i>j=0}^{k} \langle \Delta z_i \Delta z_j \rangle = 2 \langle \Delta l^2 \rangle \sum_{i>j=0}^{k} \left( \cos \left( \sum_{m=0}^{i} \theta_m \right) \cdot \cos \left( \sum_{m=0}^{j} \theta_m \right) \right) = 2 \langle \Delta l^2 \rangle \left[ g_i^2 \left( 1 - g_i^2 \right) + \frac{2}{1 - g_i^2} \left( g_i^2 + 1 - g_i^2 \right) \right].
\]

(11)

With taking into account (10), (11) and the relationship \( \langle \Delta l^2 \rangle = 2 \langle \Delta l \rangle \), after a series of rather simple transformations, we easily obtain an exact analytic expression

\[
\left\langle \left[ \Delta l_i \right]^2 \right\rangle = \frac{2}{3} \left\langle \Delta l \right\rangle \left\{ k + 3 \left( 1 - g_i \right) + \left( 2 - g_i \right) \frac{1 - g_i^2}{1 - g_i^2} \right\}.
\]

(12)

It should be noticed here that, in case when \( k = 0 - 3 \), this expression coincides with expressions, obtained earlier by the authors of 17. At the same time, relationship (12) differs from analytic expression (13) of 18 because the last is incorrect in case of small values of \( k \). Really, deriving this analytic expression, the authors of 18 have supposed that the second-order moments of all spatial projections of photon displacement are equivalent. Correct analytic expressions for the second-order moments \( \left\langle \left[ \Delta x^{(k)} \right]^2 \right\rangle = \left\langle \left[ \Delta y^{(k)} \right]^2 \right\rangle \) of a photon displacement along two other Cartesian coordinates \( x \) and \( y \) can be easily found with taking into account an exact analytic relationship

\[
\left\langle \left[ \Delta x^{(k)} \right]^2 \right\rangle + \left\langle \left[ \Delta y^{(k)} \right]^2 \right\rangle + \left\langle \left[ \Delta z^{(k)} \right]^2 \right\rangle = 2 \left\langle \Delta l \right\rangle \left( 1 - g_i \right) \left\{ k - \frac{1 - g_i^2}{1 - g_i^2} \right\},
\]

(13)

which fully corresponds to well-known expression (25) of paper 18.

### 4. FAST PATH-INTEGRATION

With taking into account obtained analytic relationship (12), effective multiple-scattering phase function (2), describing the probability distribution of changing the propagation direction by 2D angle \( \theta = (\theta, \phi) \) on the trajectory segment with length \( \Delta z \), can be written now in a very simple form

\[
P_s (\theta, \Delta z) = \exp \left\{ - \left[ k_{eff} (\Delta z) + 1 \right] \frac{\mu_s}{\mu_s} \right\} \frac{1 - g_i^{2k_{eff}(\Delta z)}}{1 + g_i^{2k_{eff}(\Delta z)} - 2 g_i^{k_{eff}(\Delta z)} \cos \theta} \left[ 1 + g_i^{2k_{eff}(\Delta z)} - 2 g_i^{k_{eff}(\Delta z)} \cos \theta \right]^{1/2},
\]

(14)

where \( k_{eff} (\Delta z) \) is determined as a solution of the transcendental equation (12). A typical (\( g_i = 0.95 \)) relation between \( k_{eff} \) and \( \Delta z \) [the last term is normalized on the reduced scattering length \( (\mu_s')^{-1} \)] is illustrated by Figure 3. Here, in a double logarithmic scale, dependences \( \Delta z (k_{eff} ) \), calculated from analytic expression (12) and relationship (13) of 18, are shown by solid and dashed lines correspondingly. It is easy to see that results of both variants of calculation coincide only in so-called diffusion approximation limit \( \Delta z \gg (\mu_s')^{-1} \) and are different in most interesting case \( \Delta z \leq 2 + 3 (\mu_s')^{-1} \).
Because obtained analytic relationship (12) is exact and expressed through the same ‘a priory’ information about the single-scattering ($\mu_s$ and $g_1$) and absorption ($\mu_a$) processes as in the primary problem definition, a possibility of fast and exact (in described above sense) solution of the small-angle multiple-scattering problem by path-integration technique can be supposed to be proved \cite{5,6,12,13}.

Last statement is illustrated by Figure 4, where central cross-section of probability distributions $P_\Sigma (r)$ are shown (Figure 4a) for the case of light propagating through a model object [weakly absorbing and highly scattering ($\mu_s = 0.01$ and $\mu_a = 14$ mm$^{-1}$, $g_1 = 0.95$) medium inside a cylindrical container (with diameter $2R = 35$ mm) with absolutely absorbing walls]. The logarithmic scale along the ordinate axis is used. The photo-detector is supposed to be located on the container wall at the central angle $90^0$ from the source (Figure 4b). Here $r$ is the distance from the container axis. The distributions $P_\Sigma (r)$ are calculated by the Monte-Carlo (squares) and path-integration techniques \cite{13} for $\Delta z = 8$ mm with the use of Heney-Greenstein phase function when $k_{\text{eff}}$ (\Delta z) is determined through $\langle k \rangle (\Delta z) $ (dashed line) or calculated from the expression (12) (solid line). In Monte-Carlo technique, the source angular aperture and photo-detector receiving aperture are supposed to be equal to $10^0$ and 1 mm$^2$ correspondingly.

5. CONCLUSIONS

Thus, numerical solution of a problem of light propagation through a multiple-scattering object can be substantially accelerated by means of introducing the multiple-scattering phase function (1). In case of independent single-scattering events and stable probability distribution $P_s^{(i)} (\theta)$, to do this, the same ‘a priory’ statistical information about scattering and absorption processes [$\mu_s$ and $P_s^{(i)} (\theta)$] must be used. In case of small-angle ($g_1 = 0.95$) multiple-scattering, varying $\Delta z$ in the range from $\Delta z < \mu_s^{-1}$ to $\Delta z \sim (\mu_s')^{-1}$ enables one to realize the gain in calculation rate which may be as high as $\sim 10^4$ and more. Here $\mu_s' = (1-g_1) \mu_s$ is the reduced scattering index. Of course such an acceleration of numerical calculations is inevitably accompanied by a gradual decrease of the calculation accuracy from the values typical for the Monte-Carlo technique to the errors of so-called diffusion approximation \cite{8}. This enables one to optimize (in relation to the speed and the accuracy) the scheme of numerical solution of multiple-scattering problems and verify fast approximate algorithms, proposed earlier for the diffusion optical tomography of multiple-scattering objects with size about 1000 scattering lengths and more \cite{19}.

It should be noticed here that described above approach can be easily transferred to much more complicated models of
single-scattering phase function where \( P_1^{(1)}(\theta) \) is determined as a linear superposition of two or more standard Henyey-Greenstein phase functions \(^8\),\(^9\).

![Graph](image)

**Fig. 4.** Central cross-sections of \( P_2(r) \) (a) in the geometry (b). Scattering medium \((\mu_s = 0.01 \text{ and } \mu_t = 14 \text{ mm}^{-1}, g_s = 0.95)\) is placed inside a 35-mm-diameter cylindrical container with absolutely absorbing walls; photo-detector is located at central angle 90° from source; \( P_2(r) \) distributions are calculated by Monte-Carlo (squares) and path-integration techniques \(^{13}\) (lines) for \( \langle \Delta z \rangle = 8 \mu_s \), with Henyey-Greenstein phase function; \( k_{\text{eff}}(\Delta z) \) is determined through \( \langle k \rangle(\Delta z) \) (dashed line) and analytic expression (12) (solid line).

**REFERENCES**