OPERATOR PENCILS ARISING IN ELASTICITY AND HYDRODYNAMICS: THE INSTABILITY INDEX FORMULA.

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Dedicated to Professor Peter Lancaster on the occasion of his 65th birthday

The main object of the paper is a quadratic operator pencil of the form

$$A(\lambda) = F\lambda^2 + (D + iG)\lambda + T,$$

with unbounded operator coefficients acting in Hilbert space. It is assumed that $F, T$ are selfadjoint and boundedly invertible and $D \geq 0, G$ are symmetric and $T$-bounded. Pencils of this form arise as abstract models for concrete problems in elasticity and hydrodynamics. We investigate the relations between the classical and generalized spectra and under additional hypotheses prove the formula for the number of eigenvalues of $A(\lambda)$ in the right-half plane. The proof of this formula is based on the preliminary investigation of maximal semidefinite invariant subspaces in the root subspaces corresponding to the pure imaginary eigenvalues of a dissipative operator in Krein or Pontrjagin spaces.

INTRODUCTION

The plan of the present paper is the following. In Section 1 we consider some concrete problems arising in elasticity and hydrodynamics. Further we prefer to work with abstract formulations of physical problems under consideration. For this purpose we provide general classes of operator pencils with unbounded operator coefficients related to problems of origin. The main object of the paper is an operator pencil of the form

$$A(\lambda) = \lambda^2 F + (D + iG)\lambda + T,$$

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where $F$ and $T$ are selfadjoint and boundedly invertible operators, while $D \geq 0$ and $G$ are symmetric and $T$-bounded. The study of the pencil $A(\lambda)$ is realized in Section 3. In particular, we introduce the concepts of the classical and the generalized spectra and investigate the relations between them. We associate the linear pencil

$$
A(\lambda) := T - \lambda W := \begin{pmatrix} D + iG & T \\ -J & 0 \end{pmatrix} - \lambda \begin{pmatrix} F & 0 \\ 0 & J \end{pmatrix}, \quad J = T|T|^{-1},
$$

with the quadratic pencil $A(\lambda)$. It turns out that the operator $T$ is dissipative in the space $H = H \times H_1$, where $H_1$ coincides with the domain of the operator $|T|^{1/2}$ and equipped with the norm $(\cdot, \cdot)_1 = (|T|^{1/2}, |T|^{1/2})$. Generally, the spectrum $\sigma(A)$ of the linearization $A(\lambda)$ coincides with neither the classical nor the generalized spectrum of $A(\lambda)$. However, we prove that $\sigma(A)$ coincides with the generalized spectrum of $A(\lambda)$ in the open right half plane if the operator $W$ generates a Pontrjagin space metric. In this case $\sigma(A)$ in the right half plane consists of finitely many eigenvalues, say $\kappa(A)$, and the number $\kappa(A)$ characterizes the index of instability of the equation

$$
A \left( \frac{du}{dt} \right) = F \frac{d^2u}{dt^2} + (D + iG) \frac{du}{dt} + Tu = 0, \quad u = u(t).
$$

The problem on stability for such kind of equations has a long background and apparently was originated by Kelvin and Tait [KT] (in the end of Section 3 we present a short historical review related to this problem). The main result of the paper is the instability index formula

$$
\kappa(A) = \nu(F) + \nu(T) - \varepsilon^+(A)
$$

where $\nu(F)$ and $\nu(T)$ are the numbers of the negative eigenvalues of the operators $F$ and $T$ respectively, while $\varepsilon^+(A)$ is expressed in terms of the lengths and the sign characteristics of Jordan chains corresponding to the pure imaginary eigenvalues of $A(\lambda)$. In particular, if all the pure imaginary eigenvalues of $A(\lambda)$ are of definite type then $\varepsilon^+(A)$ coincides with the number of the first type eigenvalues of $A(\lambda)$ (see the definitions in Section 3).

The results of Section 2 on root subspaces of linear dissipative pencils seem at the first sight to be isolated from the main subject of the paper. However, these results form a theoretical base to prove the index formula in Section 3. In our opinion, they have also an independent interest.

In Section 4 we return to the physical problems of origin and present the corollaries of our abstract results. Here we also demonstrate how the index formula can be applied to estimate the number of the nonreal eigenvalues of a selfadjoint operator pencil.
1. Classes of unbounded operator pencils.

Small oscillations of an elastic thin beam of unit length with external and internal damping (so called Kelvin-Voigt material) are described by the equation

\[(1.1) \quad \frac{\partial^4 u}{\partial x^4} + \frac{\partial}{\partial t} \frac{\partial^2}{\partial x^2} \left( \alpha(x) \frac{\partial^2 u}{\partial x^2} \right) + \frac{\partial}{\partial x} \left( g(x) \frac{\partial u}{\partial x} \right) + \beta(x) \frac{\partial u}{\partial t} + \rho(x) \frac{\partial^2 u}{\partial t^2} = 0.\]

Here \( x \in [0, 1] \), \( t \in \mathbb{R}^+ \), and \( u(x, t) \) is the transverse displacement at position \( x \) and time \( t \). The function \( \alpha(x) \) determines the internal damping and takes generally small values. The function \( \beta(x) \geq 0 \) determines the distribution of viscous damping, \( \rho(x) > 0 \) defines the mass distribution and \( g(x) \) is responsible for the forces of contraction or tension (see more details in [PI], for example).

As the equation is considered on the finite interval, we have to submit solutions of (1.1) to some boundary conditions. For the sake of definiteness we consider the case when both ends of the beam are clamped, i.e.

\[(1.2) \quad u(0, t) = \frac{\partial u(x, t)}{\partial x} \bigg|_{x=0} = u(1, t) = \frac{\partial u(x, t)}{\partial x} \bigg|_{x=1} = 0.\]

Separating variables \( u(x, t) = y(x) e^{\lambda t} \), we obtain the following spectral problem

\[(1.3) \quad \rho^{-1}(x)[y^{(4)}(x) + (g(x)y'(x))'] + \lambda \rho^{-1}(x)[(\alpha(x)y''(x))'' + \beta(x)y(x)] + \lambda^2 y(x) = 0,\]

\[(1.4) \quad y(0) = y'(0) = y(1) = y'(1) = 0.\]

Suppose that \( \rho(x), \beta(x) \in C[0, 1], g(x) \in C^1[0, 1] \) and \( \alpha(x) \in C^2[0, 1] \). According to the physical sense we have \( \rho(x) > 0, \alpha(x) \geq 0, \beta(x) \geq 0 \) and either \( g(x) \geq 0 \) or \( g(x) \leq 0 \). Then quadratic eigenvalue problem (1.3), (1.4) is represented in the form

\[(1.5) \quad [\lambda^2 I + \lambda(D_\alpha + D_\beta) + A + C] y(x) = 0,\]

where operators \( D_\alpha, D_\beta, A, C \) act in Hilbert space \( H = L_2([0, 1], \rho(x)) \) with the scalar product

\[(y, z) = \int_0^1 \rho(x) y(x) \overline{z(x)} \, dx,\]
and are defined by the equalities

\begin{align}
(Ay)(x) &= \rho^{-1}(x)y^{(4)}(x), \\
(D_\alpha y)(x) &= \rho^{-1}(x)(\alpha(x)y''(x))'', \\
(D_\beta y)(x) &= \rho^{-1}(x)\beta(x)y(x), \\
(Cy)(x) &= \rho^{-1}(x)(g(x)y'(x))'
\end{align}

on the domains

\[ D(A) = D(D_\alpha) = D(C) = \{ y \mid y \in W^4_2[0,1], y(0) = y'(0) = y(1) = y'(1) = 0 \}, \]
\[ D(D_\beta) = L_2([0,1], \rho(x)) = H. \]

We denote by \( I \) the identity operator and by \( W^k_2[0,1] \) (\( k \in \mathbb{N}^+ \)) the Sobolev spaces.

Naturally, it is more fruitful to study an abstract operator pencil of the form (1.5) rather than problem (1.3), (1.4). We have only to extract the most essential properties of the operators (1.6). We observe that these operators satisfy the following conditions (the terminology of unbounded operator theory we borrow from the book [Ka]):

i) \( A = A^* \succ 0 \) (i.e. \( A \) is selfadjoint and uniformly positive), and \( T := A + C \) is selfadjoint and bounded below;

ii) \( D_\alpha \) and \( D_\beta \) are nonnegative symmetric \( A \)-bounded operators.

iii) the identity operator \( I \) and the operator \( C \) are \( A \)-compact (or \( T \)-compact) and hence \( T \) has finitely many negative eigenvalues.

Some results on spectrum of problem (1.3), (1.4) in the case \( \alpha(x) = \text{const} \) were reported by Pivovarchik [P1]. The comprehensive study of abstract pencil (1.5) with \( D_\alpha = \alpha A \), \( C = 0 \) was carried out by Lancaster and Shkalikov [LS]. Additional results in the case \( D_\alpha = \alpha A \), \( C \neq 0 \) were obtained in a recent paper by Shkalikov and Griniv [SG]. New problems appear in the case \( \alpha \neq \text{const} \), as pencil (1.5) in this situation has nontrivial essential spectrum. However, we leave an interesting problem on the spectrum localization of pencil (1.5) with \( D_\alpha \neq \alpha A \) for another occasion. We will deal with pencil (1.5) (and more general ones) mainly in view of the application of our index formula.

A more interesting example for the application of the index formula comes from hydrodynamics. Namely, small transverse oscillations of ideal incompressible fluid in a pipe of finite length are described by the equation which is obtained from (1.1) if we add in the left hand side of (1.1) the "gyroscopic" term

\[ 2sv\delta^2 u/\delta x \delta t. \]
Here \( v \) is the velocity of the fluid and \( s \) depends on the mass of the pipe and the fluid (see [ZKM], for example). The physical meanings of the functions in (1.1) are subject to change in this situation. In particular, \( g(x) = v^2 \). Assuming \( sv = \text{const} \) and repeating the previous arguments we come to the following quadratic spectral problem

\[
[\lambda^2 I + \lambda(D_\alpha + D_\beta + iG) + A + C]y = 0,
\]

where

\[
Gy = -2sviy', \quad D(G) = D(A),
\]

and \( D_\alpha, D_\beta, A, C \) are defined as in (1.6). The last operators retain the properties i)-iii). The most essential properties of the operator \( G \) are the following:

iv) \( G \) is a symmetric \( T \)-bounded operator;

v) \( G \) is a \( T \)-compact operator.

It is also of interest to consider equation (1.1) on the semiaxis \( x \in \mathbb{R}^+ \) (see the papers of Pivovarchik [P2] and Griniv [Gr]). Assuming that the left end of a beam is clamped we define the operator coefficients in (1.5) by equalities (1.6) on the domains

\[
\mathcal{D}(A) = \mathcal{D}(D_\alpha) = \mathcal{D}(D_\beta) = \mathcal{D}(C) = \{y \mid y \in W_2^4[0, \infty], y(0) = y'(0) = 0\}.
\]

Obviously, this definition is correct if we assume in addition that all the functions \( \rho(x) \), \( \rho^{-1}(x) \), \( \alpha(x) \), \( \beta(x) \), \( g(x) \) are bounded on \( \mathbb{R}^+ \). In this case, the properties i)-ii) are retained, however, the property iii) is not true any more. This makes the problem much more complicated. Nevertheless, under some additional assumptions on the behaviour of the function \( g(x) \) at \( \infty \) (see [Gr]) the important property

vi) \( T = A + C \) has finitely many negative eigenvalues

remains valid.

Analogously, equation (1.1) with the additional "gyroscopic" term can be considered on the semiaxis \( \mathbb{R}^+ \) with respect to the variable \( x \). In this case we obtain a pencil of the form (1.7) whose coefficients satisfy the properties i)-ii), iv) and also the property vi) under additional assumptions on the behaviour of the function \( g(x) \) found in the paper [Gr].
2. Root subspaces of linear dissipative pencils and their properties

In this section we deal with a linear dissipative operator pencil

\[ A(\lambda) = T - \lambda W, \]

where \( W \) is a bounded selfadjoint operator, while \( T \) is a closed dissipative operator in Hilbert space \( H \). This means that \( T \) is closed and

\[ \text{Im}(Tx, x) \geq 0 \quad \text{for all} \quad x \in D(T), \]

and \( D(T) \) is the domain of \( T \). Throughout all the section we also assume that there exists at least one point \( \mu_0 \) belonging to the open upper half plane \( \mathbb{C}^+ \) such that \( A(\mu_0) \) has a bounded inverse, i.e. \( \mu_0 \in \rho(A) \).

If the operator \( W \) has a bounded inverse then the spectrum \( \sigma(A) \) and the root subspaces \( \mathcal{L}_\mu(A) \) of the pencil \( A(\lambda) \) coincide with those of the operator \( A = W^{-1}T \). Hence, in this case spectral problems for the pencil \( A(\lambda) \) are equivalent to those for dissipative operators in Krein or Pontrjagin spaces (see [AI], ch.II, §2). In the sequel we prefer to deal with the linear pencil \( A(\lambda) \). The motivation for this becomes clear when considering the corresponding operator differential equations. Moreover, at least formally, we obtain more general results, as we do not always assume that \( W \) generates a regular indefinite metric.

The basic goal of this section is to prove formula (2.17). This formula is based on the well-known fundamental result on the existence of a maximal \( W \)-nonnegative \( A \)-invariant subspace in Pontrjagin space and on the explicit construction of maximal \( W \)-nonnegative subspaces corresponding to real normal eigenvalues of the pencil \( A(\lambda) \) or of the operator \( A = W^{-1}T \). In the paper [S1] the author considered dissipative operator pencils of an arbitrary order \( n \geq 1 \) and constructed for such pencils regular canonical systems corresponding to real normal eigenvalues. This construction allows us to define the sign characteristics for Jordan chains and to realize the construction of a maximal \( W \)-nonnegative subspace \( \mathcal{L}_\mu^+ \) in the root subspace \( \mathcal{L}_\mu \) corresponding to a real eigenvalue \( \mu \). The additional details for linear pencils were given in the unpublished manuscript [S2]. We note also the papers of Kostyuchenko and Orazov [KO] (devoted to the case of a selfadjoint operator \( T \)) and Gomilko [G] related to this topic. However, our construction is new and, perhaps simpler, even for selfadjoint pencils. In addition we obtain the information on the connection of the middle elements of mutually adjoint canonical systems. This information is essentially used when considering half range completeness and minimality problems (see [S1]). Recently Ran and Temme [RT] investigated an analogous problem from another point of view. Here
we present some results of [S2] concerning this subject.

Let $\mu$ be an eigenvalue of the pencil $A(\lambda) = T - \lambda W$ and

$$y_j^0, y_j^1, \ldots, y_j^{p_j}, \quad j = 1, \ldots, N,$$

be a canonical system of eigen and associated elements (or Jordan chains) corresponding to $\mu$ (see [Ke]). The linear span of all elements (2.1) is denoted $\mathcal{L}_\mu(A)$ or simply $\mathcal{L}_\mu$ and is called the root subspace corresponding to the eigenvalue $\mu$. An eigenvalue $\mu$ is said to be normal if $A(\lambda)$ is invertible in a punctured neighbourhood of $\mu$ and the number $N = \ker(T - \mu W)$ as well as the lengths $p_j + 1$ of Jordan chains (2.1) are finite. It is known [Ke] that the principal part of the Laurent expansion of the function $A^{-1}(\lambda)$ at the pole $\mu$ has the representation

$$\sum_{j=1}^{N} \sum_{s=0}^{p_j} \frac{(\cdot, x_j^s) y_j^0 + \ldots + (\cdot, x_j^0) y_j^s}{(\lambda - \mu)^{p_j + 1 - s}},$$

where the adjoint system

$$x_j^0, x_j^1, \ldots, x_j^{p_j}, \quad j = 1, \ldots, N,$$

is uniquely determined by the choice of system (2.1). It turns out that the adjoint system (2.3) is a canonical system of Jordan chains corresponding to the eigenvalue $\bar{\mu}$ of the pencil $A^*(\lambda) = T^* - \lambda W$.

Further the upper index is always used for numeration of associated elements while the the lower one numerates eigenvalues and canonical chains simultaneously, i.e. each eigenvalue is counted as many times as its geometric multiplicity. The set of all eigenvalues of the pencil $A(\lambda)$ is denoted $\sigma_p(A)$. For the subset in $\sigma_p(A)$ consisting of the normal eigenvalues we reserve the notation $\sigma_d(A)$ (the discrete spectrum). Notice that canonical system of Jordan chains (2.1) is well defined for any $\mu \in \sigma_p(A)$ (possibly, consisting of infinitely many elements), however, adjoint system (2.3) is well defined only for $\mu \in \sigma_d(A)$. As usually the indefinite scalar product $(Wx, x)$ is denoted $[x, x]$.

Although some of the subsequent propositions are essentially known, we present their proofs here for the reader's convenience. New constructions are started from Proposition 2.6.

**Proposition 2.1.** Let $\mathcal{L}_+^0$ be the minimal subspace containing the root subspaces corresponding to all $\mu \in \mathbb{C}^+ \cap \sigma_p(A)$. Then $\mathcal{L}_+^0$ is a $W$-nonnegative subspace.
Proof. (Cf. [AI], Ch.2, Corollary 2.22). We present here another, shorter proof. Suppose eigenvalues are numerated as many times as their geometric multiplicity. Let us consider the functions

$$u^h_j(t) = e^{i\mu_j t} \left( y_j^h + \frac{it}{1!} y_j^{h-1} + \ldots + \frac{(it)^h}{h!} y_j^0 \right), \quad h = 0, 1, \ldots, p_j,$$

where $y_j^0, \ldots, y_j^{p_j}$ are Jordan chains corresponding to the eigenvalues $\mu_j \in \mathbb{C}^+$. It is easily seen that the functions $u^h_j(t)$ satisfy the equation

$$i W u'(t) + T u(t) = 0.$$

Any linear combination $u(t) = \sum c_{j,h} u^h_j(t)$ also satisfies this equation, therefore

$$[u(\xi), u(\xi)]' = (W u'(\xi), u(\xi)) + (u(\xi), W u'(\xi))$$

$$= (i T u(\xi), u(\xi)) + (u(\xi), i T u(\xi)) = -2 \text{Im} (T u(\xi), u(\xi)).$$

As all the functions $u^h_j(t)$ vanish at $\infty$, so does $u(t)$. Integrating the last equality from $t$ to $\infty$ we obtain

$$[u(t), u(t)] = 2 \int_t^\infty \text{Im} (T u(\xi), u(\xi)) \, d\xi \geq 0.$$

In particular $[u(0), u(0)] \geq 0$ for all $u(0) = \sum c_{j,h} y_j^h$. By the definition the set of these elements is dense in $\mathcal{L}_+^0$, hence, $\mathcal{L}_+^0$ is a $W$-nonnegative subspace. \qed

**Proposition 2.2.** Let (2.1) be a canonical system corresponding to a real eigenvalue $\mu$. If $[\gamma]$ is the integer part of a number $\gamma$ then the elements

$$(2.4) \quad y_k^0, y_k^1, \ldots, y_k^{\alpha_k}, \quad k = 1, \ldots, N, \quad \alpha_k = \left[ \frac{p_k}{2} \right],$$

belong to $\mathcal{D}(T^*)$ and $T^* y_k^h = T y_k^h$ for all $1 \leq k \leq N, \ 0 \leq h \leq \alpha_k$.

**Proof.** First we notice that $T^*$ is well defined, as the operator $T$ is closed by assumption (see [Ka], Ch.3, §5.5). Now, let us prove the following: If $x \in \mathcal{D}(T)$ and $\text{Im} (T x, x) = 0$ then $x \in \mathcal{D}(T^*)$ and $T^* x = T x$. (Cf. [AI], Ch.2, Theorem 2.15). To prove this fact, we introduce an indefinite product in the space $\mathbf{H} = H \times H$ as follows

$$\langle \{x_1, x_2\}, \{y_1, y_2\} \rangle = i(x_1, y_2) - i(x_2, y_1).$$

As $T$ is dissipative, we have

$$\langle x, x \rangle = 2 \text{Im}(T x, x) \geq 0 \quad \text{for all} \quad x = \{x, T x\} \in \Gamma(T),$$
where $\Gamma(T)$ is the graph of $T$. If $x \in \mathcal{D}(T)$ and $\text{Im}(Tx, x) = 0$ then by virtue of Cauchy-Schwarz-Bunyakovskii inequality we obtain
\[
|\langle x, Tz \rangle - \langle Tx, z \rangle| = |\langle x, z \rangle| \leq \langle x, x \rangle^{1/2} \langle z, z \rangle^{1/2} = 0 \quad \text{for all } z = \{z, Tz\} \in \Gamma(T).
\]
Hence, $(Tz, x) = (z, Tx)$ for all $z \in \mathcal{D}(T)$. From the definition of the adjoint operator we obtain $x \in \mathcal{D}(T^*)$ and $T^*x = Tx$.

Now let us prove the assertion of Proposition 2.2. As the elements of system (2.1) are Jordan chains, we have
\[
(T - \mu W)y_k^h = Wy_k^{h-1}, \quad 0 \leq h \leq p_k \quad (y_k^{-1} := 0).
\]
In particular,
\[
\text{Im} \left((T - \mu W)y_k^0, y_k^0\right) = \text{Im} \left(Ty_k^0, y_k^0\right) = 0.
\]
Therefore, $y_k^0 \in D(T^*)$ and $Ty_k^0 = T^*y_k^0$. Now we can end the proof by induction. Suppose that for some $h \leq \alpha_k$ we have proved that
\[
y_k^s \in D(T^*) \quad \text{and} \quad Ty_k^s = T^*y_k^s \quad \text{for } s = 0, 1, \ldots, h - 1.
\]
As $2h \leq p_k$, we find
\[
(Wy_k^{h-1}, y_k^h) = (y_k^{h-1}, (T - \mu W)y_k^{h+1}) = (Wy_k^{h-2}, y_k^{h+1}) = \ldots = (y_k^0, (T - \mu W)y_k^{2h}) = 0.
\]
Hence, $\text{Im} \left(Wy_k^{h-1} + \mu Wy_k^h, y_k^h\right) = 0$ and
\[
\text{Im} \left(Ty_k^h, y_k^h\right) = \text{Im} \left((T - \mu W)y_k^h - Wy_k^{h-1}, y_k^h\right) = 0.
\]
As before we deduce that $y_k^h \in D(T^*)$ and $Ty_k^h = T^*y_k^h$. \qed

Proposition 2.3. Let (2.1) be a canonical system corresponding to a real eigenvalue $\mu$. Then
\[
[y_k^h, y_j^s] = 0, \quad j = 1, \ldots, N, \quad h \leq \lfloor(p_k - 1)/2\rfloor, \quad s \leq \lfloor(p_j - 1)/2\rfloor.
\]
If $p_j \neq p_k$ then (2.6) hold for all $s \leq \lfloor p_j/2\rfloor, h \leq \lfloor p_k/2\rfloor$.

Proof. Suppose $p_j \leq p_k$. Then it follows from our assumptions that $h + s + 1 \leq p_k$. Taking into account (2.5) and the equalities $Ty_j^s = T^*y_j^s$ (Proposition 2.2) we find
\[
(y_k^h, Wy_j^s) = ((T - \mu W)y_k^{h+1}, y_j^s) = (y_k^{h+1}, Wy_j^{s-1}) = \ldots = (y_k^{h+s+1}, (T^* - \mu W)y_j^0) = 0.
\]
and the equalities (2.6) follow. \qed
Proposition 2.4. Let $\nu$ and $\mu$ be eigenvalues of the pencils $A(\lambda)$ and $A^*(\lambda)$ respectively. If $\nu \neq \bar{\mu}$ then the root subspaces $\mathcal{L}_\nu(A)$ and $\mathcal{L}_\mu(A^*)$ are $W$-orthogonal. In particular, truncated Jordan chains (2.4) corresponding to a real eigenvalue $\mu$ of the pencil $A(\lambda)$ are $W$-orthogonal to any root subspace $\mathcal{L}_\nu(A)$ if $\nu \neq \mu$.

Proof. Let $y^0, \ldots, y^p \in \mathcal{L}_\nu(A)$, $x^0, \ldots, x^q \in \mathcal{L}_\mu(A^*)$ be Jordan chains and $\nu \neq \bar{\mu}$. Using (2.5) we obtain

$$(Ty^s, x^l) = \nu[y^s, x^l] + [y^{s-1}, x^l]$$

$$(y^s, T^*x^l) = \bar{\mu}[y^s, x^l] + [y^s, x^{l-1}], \quad (y^{-1} := x^{-1} := 0).$$

In particular, from these equalities we have $[y^0, x^0] = 0$. Now, the proof of the first assertion is ended by induction with respect to the index $s + l$. The second assertion follows from Proposition 2.2. □

Proposition 2.5. Let (2.1) and (2.3) be mutually adjoint canonical systems corresponding to normal eigenvalues $\mu_j$ which are enumerated according to their geometric multiplicity. Then the following biorthogonality relations hold:

$$(2.7) \quad [y^h_k, x^s_j] = -\delta_{k,j}\delta_{h,p_j-s},$$

where $\delta_{m,n}$ is the Kronecker symbol.

Proof. (Cf.[Ke]). We have

$$A(\lambda)y^h_k = [A(\mu_k) - (\lambda - \mu_k)W]y^h_k = W^{-1}y^h_k - (\lambda - \mu_k)Wy^h_k, \quad 0 \leq h \leq p_k,$$

where as before it is assumed that $y^{-1} := 0$. Using the representation (2.2) we obtain

$$(2.8) \quad y^h_k = A^{-1}(\lambda)A(\lambda)y^h_k$$

$$= \sum_{j=N_1}^{N_2} \sum_{s=0}^{p_j} \left[ (\cdot, x^s_j)y^0_j + \ldots + (\cdot, x^0_j)y^s_j \right] + R(\lambda) \right] \left[ -(\lambda - \mu_k)Wy^h_k + W^{-1}y^h_k \right],$$

where $R(\lambda)$ is a holomorphic operator function at the point $\mu = \mu_j$ and $N_2 - N_1 + 1$ is the geometric multiplicity of the eigenvalue $\mu$. We may assume that $N_1 = 1$, $N_2 = N$, $p_1 \geq p_2 \geq \ldots \geq p_N$. 

Suppose that $\mu_k \neq \mu_j$. If we take $h=0$ and compare the coefficients of the powers $(\lambda - \mu_j)^{-p_j-1+s}$, $0 \leq s \leq p_j$, we find

\begin{align}
(2.9) & \quad - \sum_{p_j=p_1} [y_k^0, x_j^0]y_j^0 = 0, \\
(2.10) & \quad - \sum_{p_j=p_1} [y_k^0, x_j^1]y_j^0 - \sum_{p_j=p_1} [y_k^0, x_j^1]y_j^1 - \sum_{p_j=p_1-1} [y_k^0, x_j^0]y_j^0 = 0.
\end{align}

We do not write out the other coefficients corresponding to the indices $s \geq 2$. We also notice that the third term in (2.10) should be omitted if there are no Jordan chains of length $(p_1 + 1) - 1$. It follows from the definition of a canonical system that the elements \{$y_j^{0,1}\}_1^N$ are linearly independent. Hence, from (2.9) we have

\begin{equation}
(2.11) \quad [y_k^0, x_j^0] = 0, \quad \text{for all indices } j \text{ such that } p_j = p_1.
\end{equation}

Now, it follows from (2.10) and (2.11) that

\begin{equation}
[y_k^0, x_j^0] = 0 \quad \text{if } p_j = p_1 - 1; \quad [y_k^0, x_j^1] = 0 \quad \text{if } p_j = p_1.
\end{equation}

Repeating the argument we find $[y_k^{0,s}, x_j^s] = 0$ for all indices $0 \leq s \leq p_j$. Using the last equalities and taking $h = 1, 2, \ldots, p_k$, we find subsequently

\begin{equation}
[y_k^1, x_j^1] = 0, \ldots, [y_k^{p_k}, x_j^{p_k}] = 0 \quad \text{for all } 0 \leq s \leq p_j.
\end{equation}

The same arguments can be applied in the case $\mu_k = \mu_j$. Comparing the coefficients of the powers $(\lambda - \mu_j)^{v}$ in (2.10) it is found that, for $h = 0, 1, \ldots, p_k$,

\begin{equation}
- [y_j^h, x_j^s] = \delta_{h,p_j - s},
\end{equation}

and relations (2.7) follow. □

Let a canonical system (2.1) correspond to a real normal eigenvalue $\mu$. Denote by $S_\mu^0$ the span of elements

\begin{equation}
(2.12) \quad y_k^0, y_k^1, \ldots, y_k^{\beta_k}, \quad k = 1, \ldots, N, \quad \beta_k = [(p_k - 1)/2]
\end{equation}

(if $p_k = 0$, we assume that $\beta_k = -1$ and the element $y_k^0$ does not belong to $S_\mu^0$). Let us fix an index $k$, $1 \leq k \leq N$. If the number $p_k + 1$ is even we set $S_{\mu_k} := S_\mu^0$. If $p_k + 1$ is odd we denote by $S_{\mu_k}$ the span of elements (2.12) combined with the elements $y_j^{\alpha_j}, \alpha_j = [p_j/2]$, where index $j$ runs through all the values such that $p_j = p_k$. Similarily, by replacing chains (2.1) with adjoint chains (2.3) we construct subspaces $(S_\mu^0)^*$ and $S_{\mu_k}^*$. We emphasize that, according to our agreement about the enumeration of eigenvalues, the subspaces $S_{\mu_k}$ are generally different although $\mu_k = \mu$. 
Proposition 2.6. For all nonzero real normal eigenvalues $\mu$ the following equalities hold

$$S_\mu^0 = (S_\mu^0)^*, \quad S_{\mu_k} = S_{\mu_k}^* \quad \text{for all} \ 1 \leq k \leq N.$$  

Proof. Suppose that $y_k^h \in S_{\mu_k}$ and $x_k^h \notin S_{\mu_k}$. It follows from Proposition 2.2 that

$$x_k^0, x_k^1, \ldots, x_k^{\alpha_k}, \quad k = 1, \ldots, N, \quad \alpha_k = [p_k/2],$$

are chains of EAE of the pencil $A(\lambda)$ as well as of $A^*(\lambda)$. Since (2.1) is a canonical system, we have the representation

$$x_k^h = \sum_{j=1}^N \sum_{s=0}^{h} c_{j,s} y_j^s, \quad \text{if} \ 0 \leq h \leq \alpha_j = [p_j/2].$$  

We have assumed that $x_k^h \notin S_{\mu_k}$, therefore, at least one of the numbers $c_{j,s}$ in (2.12) is not equal to zero for $s > \beta_j = [(p_j - 1)/2], \quad p_j < p_k$. In this case, however, $x_j^{p_j-s} \in S_{\mu_j}^*$ i.e. $p_j - s \leq [p_j/2]$. Applying Proposition 2.3 with respect to the pencil $A^*(\lambda)$ we find

$$[x_k^h, x_j^{p_j-s}] = 0.$$  

On the other hand it follows from Proposition 2.5 and representation (2.13) that

$$[x_k^h, x_j^{p_j-s}] = -c_{j,s}.$$  

Hence, the assumption $x_k^h \notin S_{\mu_k}$ is not valid. The equality $S_\mu^0 = (S_\mu^0)^*$ is proved in a similar way.  

□

Proposition 2.7. A canonical system (2.1) corresponding to a real normal eigenvalue $\mu$ of the pencil $A(\lambda)$ can be chosen in such a way that

$$[y_j^{\alpha_j}, y_l^{\alpha_l}] = \varepsilon_j \delta_{j,l}, \quad \alpha_j = [p_j/2], \quad \varepsilon_j = \begin{cases} 0 & \text{if } p_j + 1 \text{ is even} \\ \pm 1 & \text{if } p_j + 1 \text{ is odd} \end{cases}$$

for all indices $1 \leq j, l \leq N$.

Proof. Fix an index $k$ such that $p_k + 1$ is odd. Assume that there are $q$ chains of the length $p_k + 1$, i.e. $p_j = p_k$ for $j = k, k + 1, \ldots, k + q - 1$. According to the definition of $S_{\mu_k}$ we have dim $S_{\mu_k} \oplus S_\mu^0 = q$. Let $P_k$ be the orthoprojector onto the subspace $S_{\mu_k}$. It follows from the biorthogonality relations (2.7) that the selfadjoint operator $P_k W P_k$ has exactly $q$ nonzero eigenvalues which correspond to an orthogonal basis $\{\varphi_s\}^q_1$. We can replace, if
necessary, chains (2.1) corresponding to indices \( l = k, k + 1, \ldots, k + q - 1 \), by their linear combinations and obtain a new canonical system such that the system \( \{ \varphi_l \}_{i=1}^{q} \) coincides with \( \{ y_s^{\alpha_k} \}_{k}^{k+q-1} \). Then, after a proper norming, the relations (2.14) hold for all indices \( l, j = k, k + 1, \ldots, k + q - 1 \). We can repeat the same arguments for any other index \( r \) such that \( S_{\mu, \neq S_{\mu, k}} \). Taking into account that the subspaces \( S_{\mu, r} \) and \( S_{\mu, k} \) are \( W \)-orthogonal (Proposition 2.3), we obtain relations (2.14) for all indices such that \( 1 \leq j, l \leq N \). □

**Proposition 2.8.** Let a canonical system (2.1) correspond to a real normal eigenvalue \( \mu \) and satisfy relations (2.14). Then for all indices \( j \) such that \( p_j = 2\alpha_j \) the elements \( x_j^{\alpha_j} \) of the adjoint system (2.3) have the representation

\[
(2.15) \quad x_j^{\alpha_j} = -\varepsilon_j y_j^{\alpha_j} + y, \quad \varepsilon_j = \pm 1, \quad \text{where } y \in S_{\mu}^0.
\]

In other words: there exists a canonical system (2.1) such that for Jordan chains of odd length the middle elements \( x_j^{\alpha_j} \) of its adjoint system have representation (2.15).

**Proof.** As \( x_j^{\alpha_j} \in S_{\mu, j}^* = S_{\mu, j} \), we have

\[
x_j^{\alpha_j} = \sum_{l=1}^{p_j} c_l y_l^{\alpha_k} + y, \quad \text{where } y \in S_{\mu}^0.
\]

Now, if canonical system (2.1) satisfies relations (2.14) then \( c_l = -\varepsilon_j \delta_{j, l} \), and relation (2.15) follow. □

A canonical system (2.1) which satisfies relations (2.14) or (2.15) is said to be **regular**. The numbers \( \varepsilon_j \) in (2.15) are said to be **sign characteristics**. We note that for linear selfadjoint pencils the sign characteristics are determined in a different way, namely, \( \varepsilon_j = \pm 1 \) for Jordan chains of any length (see [GLR], Ch.3, and [KS], Lemma 2). Simple examples show that for dissipative pencils the definite sign characteristics can not be well defined for Jordan chains of even length. In this situation it is convenient to assume that the sign characteristics \( \varepsilon_j = 0 \) for all chains of even length \( p_j + 1 \). It is supposed that this agreement holds through the rest of the paper.

Let (2.1) be a regular canonical system corresponding to a normal real eigenvalue \( \mu \). Denote by \( \mathcal{L}_\mu^+ (\mathcal{L}_\mu^-) \) the span of elements (2.12) combined with \( y_j^{\alpha_j} \) satisfying relations (2.14) with \( \varepsilon_j = +1 (\varepsilon_j = -1) \). Then according to the definition of the sign charactereristics we have

\[
(2.16) \quad \dim \mathcal{L}_\mu^+ = \sum_{k=1}^{N} (\varepsilon_k^+ + [(p_k - 1)/2]), \quad \text{where } \varepsilon_k^+ = \max(0, \varepsilon_k).
\]
Proposition 2.9. Let $\mu$ be a real normal eigenvalue of the pencil $A(\lambda) = T - \lambda W$. Then $\mathcal{L}_\mu^+$ is a maximal $W$-nonnegative subspace in the root subspace $\mathcal{L}_\mu$.

Proof. It follows from Propositions 2.2 and 2.7 that $\mathcal{L}_\mu^+$ is a $W$-nonnegative subspace. Assume that $\mathcal{L}_\mu^+ \subset \mathcal{L}' \subset \mathcal{L}_\mu$, where $\mathcal{L}'$ is also $W$-nonnegative subspace, and there exists an element $y \in \mathcal{L}'$ such that $y \notin \mathcal{L}_\mu^+$. Obviously, $y \notin \mathcal{L}_\mu^-$, as the assumptions $y \in \mathcal{L}_\mu^-, y \notin \mathcal{L}_\mu^+$ imply $[y, y] < 0$. Therefore, $y \notin \mathcal{L}_\mu^+ \cup \mathcal{L}_\mu^-$. Now, using (2.7) we can find an element $y_k^h \in S^0_\mu$ such that $[y_k^h, y] = \gamma \neq 0$. Denote $z = ay_k^h + \gamma y$. Then $[z, z] = |\gamma|^2(a + [y, y]) \rightarrow -\infty$ if $a \rightarrow -\infty$. On the other hand $[z, z] \geq 0$, as $z \in \mathcal{L}'$ and $\mathcal{L}'$ is by assumption $W$-nonnegative. This contradiction ends the proof. □

Denote by $\mathcal{L}$ the minimal subspace containing the root subspaces $\mathcal{L}_\mu(A)$ corresponding to all the eigenvalues $\mu \in \mathbb{C}^+$ and all the root subspaces $\mathcal{L}_\mu(A)$ corresponding to normal real eigenvalues. Analogously, let $\mathcal{L}^+$ be the minimal subspace containing $\mathcal{L}_\mu$ for all $\mu \in \mathbb{C}^+ \cap \sigma_p(A)$ and all the subspaces $\mathcal{L}_\mu^+$ corresponding to the normal real eigenvalues. For a selfadjoint operator $C$ we introduce the (well-known) notations

$$\pi(C) = \text{rank } C^+, \quad \text{where } C^+ = (|C| + C)/2, \quad \nu(C) = \pi(-C).$$

Further, we use the following fundamental result.

**Theorem on a maximal nonnegative invariant subspace.** Suppose $W$ generates a Pontrjagin space, i.e. $W$ is boundedly invertible and $\nu(W) < \infty$. If $A = W^{-1}T$ and $\rho(A) \cap \mathbb{C}^+ \neq \emptyset$ then there exists a maximal $A$-invariant $W$-nonnegative subspace $H^+ \subset H$, $\dim H^+ = \nu(W)$, such that the spectrum of the restriction $A/H^+$ lie in $\overline{\mathbb{C}}^+$, and in $\mathbb{C}^+$ coincides with the spectrum of $A$.

Proof. In the case $T = T^*$ this is a well-known Pontrjagin theorem [P]. For a maximal $W$-dissipative operator $A$ in Pontrjagin space the theorem was proved by Krein and Langer [KL], and by Azizov [A] (see [AI] and references therein). □

**Theorem 2.10.** The subspace $\mathcal{L}^+$ defined above is a maximal $W$-nonnegative subspace in $\mathcal{L}$. If $W$ generates a Pontrjagin space and all the real eigenvalues of the pencil $A(\lambda)$ are normal then $\mathcal{L}^+$ is a maximal $W$-nonnegative subspace in the whole space $H$.

Proof. It follows from Proposition 2.4 and the definition that $\mathcal{L}^+$ is a $W$-nonnegative subspace. As $\mathcal{L}_\mu^+$ is a maximal $W$-nonnegative subspace in $\mathcal{L}_\mu$ for any $\mu \in \sigma_d(A) \cap \mathbb{R}$ (Proposition 2.9), we have that $\mathcal{L}^+$ possesses the same property in $\mathcal{L}$.

Now, let $W$ generate a Pontrjagin space and all the real eigenvalues of the pencil $A(\lambda)$
are normal. According to the generalized Pontrjagin theorem there exists a maximal $W$-nonnegative subspace $H^+$ in $H$, $\dim H^+ = \nu(W)$, such that $L_+^0 \subset H^+ \subset L$, where $L_+^0$ is defined in Proposition 2.1. As the subspace $H^+ \cap L_\mu$ is $W$-nonnegative in $L_\mu$ and $L_+^\mu$ is a maximal nonnegative subspace in $L_\mu$ (Proposition 2.9), we have: $\dim(H^+ \cap L_\mu) \leq \dim L_+^\mu$ (see, for example, [AI], Ch.I, §4). Then it follows that

$$
dim H^+ = \dim L_+^0 + \sum_{\mu \in \mathbb{R} \cap \sigma_d} \dim(H^+ \cap L_\mu) \leq \dim L^+, \quad \sigma_d := \sigma_d(A).$$

On the other hand, it is known ([AI], Ch.I, §4) that $\dim L^+ \leq \nu(W) = \dim H^+$. Hence, $\dim L^+ = \dim H^+$ and from this it follows that $L^+$ is a maximal $W$-nonnegative subspace in the whole $H$. □

**Corollary 2.11.** Let $W$ be boundedly invertible, $\nu(W) < \infty$, and all the real eigenvalues of $A(\lambda)$ be normal. Then the following formula is valid

$$
(2.17) \quad \kappa(A) + \sum_{\mu_k \in \mathbb{R} \cap \sigma_d} (\epsilon_k^+ + [(p_k - 1)/2]) = \nu(W), \quad \epsilon^+_k = \max(0, \epsilon_k).
$$

Here $\kappa(A)$ is the total algebraic multiplicity of all eigenvalues in $\mathbb{C}^+$ and $\epsilon_k (p_k + 1)$ are the sign characteristics (the lengths) of Jordan chains of regular canonical systems corresponding to real normal eigenvalues $\mu_k$.

**Proof.** It follows from formula (2.16) and Theorem 2.10. □

**Remark 2.12.** Formula (2.17) is not applicable if the pencil $A(\lambda)$ has real eigenvalues which are embedded into the essential spectrum. In this case we do not know how to determine the sign characteristics and how to realize the explicit construction of a maximal $W$-nonnegative subspace in the the root subspace $L_\mu(A)$. However, the following inequality is always valid (cf. [AI], Ch.2, Theorem 2.26)

$$
(2.18) \quad \kappa(A) + \sum_{\mu_k \in \mathbb{R} \cap \sigma_p} [(p_k - 1)/2] \leq \nu(W), \quad \sigma_p := \sigma_p(A).
$$

This inequality is much more simple and follows directly from Propositions 2.1, 2.3 and 2.4. It expresses the fact that the linear span of all root subspaces $L_\mu$ corresponding to $\mu \in \sigma_p(A) \cap \mathbb{R}$ and all the truncated root subspaces $S^0_\mu$ corresponding to $\mu \in \sigma_p(A) \cap \mathbb{R}$ forms a $W$-nonnegative subspace (not necessarily a maximal one). Indeed, using (2.17) we can improve (2.18) and write the following inequality

$$
(2.19) \quad \kappa(A) + \sum_{\mu_k \in \mathbb{R} \cap \sigma_p} (\epsilon_k^+ + [(p_k - 1)/2]) \leq \nu(W), \quad \sigma_p := \sigma_p(A),
$$

where $\epsilon_k^+ = \max(0, \epsilon_k)$ if $\mu_k \in \sigma_d$ and $\epsilon_k^+ = 0$ if $\mu_k \in \sigma_p \setminus \sigma_d$. 
3. Quadratic dissipative pencils and the instability index formula.

In this section we study a quadratic operator pencil of the form

\[(3.1) \quad A(\lambda) = \lambda^2 F + (D + iG)\lambda + T.\]

Further it is always assumed that the coefficients in (3.1) are operators in Hilbert space $H$ satisfying the following conditions:

\begin{enumerate}
  \item $F$ is a selfadjoint bounded and boundedly invertible operator;
  \item $T$ is defined on the domain $\mathcal{D}(T)$, $T = T^*$ and $T$ is boundedly invertible;
  \item $D$ and $G$ are symmetric $T$-bounded operators (i.e. $D$ and $G$ are symmetric, $\mathcal{D}(D) \subseteq \mathcal{D}(T)$ and $\mathcal{D}(G) \subseteq \mathcal{D}(T)$). Moreover, $D \geq 0$.
\end{enumerate}

These assumptions imply that $A(\lambda)$ is a quadratic dissipative pencil with respect to the imaginary axis in the following sense (see [S1])

\[\text{Im}(\zeta A(i\zeta)x, x) = \zeta^2(Dx, x) \geq 0 \quad \text{for all } x \in \mathcal{D}(T) \text{ and } \zeta \in \mathbb{R}.\]

One may expect that the quadratic dissipative pencil (3.1) can be transformed into a linear dissipative pencil. Indeed, such a linearization will be realized below. However, working with unbounded pencils we come to some new problems which do not arise when considering pencils with bounded coefficients. In particular, the spectrum of a linearization may not coincide with the spectrum of the original pencil.

According to our assumptions $A(\lambda)$ is well defined for each $\lambda \in \mathbb{C}$ on the domain $\mathcal{D}(T)$. Hence, the first natural definition of the resolvent set $\rho(A)$ is the following: $\zeta \in \rho(A)$ if $A(\zeta)$ with the domain $\mathcal{D}(T)$ has a bounded inverse. To give another definition, we consider the scale of Hilbert spaces $H_\theta$, $\theta \in \mathbb{R}$ ($H_0 = H$) generated by the selfadjoint operator

\[S^2 := |T| := (T^2)^{1/2}.\] Namely, if $\theta > 0$ we set $H_\theta = \{x | x \in \mathcal{D}(S^\theta)\}$ with the norm $\|x\|_\theta = \|S^\theta x\|$. If $\theta < 0$, the space $H_\theta$ is defined as the closure of $H$ with respect to the norm $\|x\|_\theta = \|S^\theta x\|$.

Let us associate the pencil

\[\hat{A}(\lambda) = \lambda^2 \hat{F} + \lambda(\hat{D} + i\hat{G}) + J\]

with the pencil $A(\lambda)$. Here

\[\hat{F} = S^{-1}FS^{-1}, \hat{D} = S^{-1}DS^{-1}, \hat{G} = S^{-1}GS^{-1}, J = T^{-1}|T|.\]

Obviously $\hat{F}$ and $J$ are bounded. From the next Proposition it follows that $\hat{D}$ and $\hat{G}$ are also bounded in $H$. 
Proposition 3.1. Let $S$ be an uniformly positive selfadjoint operator and $B$ be a symmetric operator such that $D(B) \supset D(S^2)$. Then the operator $S^{\theta-2}BS^{-\theta}$ defined on the domain $D(S^{\theta-2})$ is bounded in $H$ for all $0 \leq \theta \leq 2$. Equivalently, $B$ is bounded as an operator acting from $H_\theta$ into $H_{\theta-2}$.

Proof. As $B$ is closable, the assumption $D(B) \supset D(S^2)$ implies that $B : H_2 \to H$ is a bounded operator (this follows immediately from the closed graph theorem). Hence, the adjoint operator $B^* : H \to H_{-2}$ is also bounded. As $B^* \supset B$, we have that $B : H \to H_{-2}$ is bounded. Now, applying the interpolation theorem (see [LM], Ch.1, for example) we find that $B : H_\theta \to H_{\theta-2}$ is bounded for all $0 \leq \theta \leq 2$. □

Let $\sigma(\hat{A})$ be the spectrum of the pencil $\hat{A}(\lambda)$ with bounded operator coefficients in the space $H$. It is easily seen that $\sigma(\hat{A})$ coincides with the spectrum of $A(\lambda)$ considered as the operator function in the space $H_{-1}$ on the domain $D(A) = H_1$. Both our definitions of the spectra are better understood (especially for the specialists working with partial differential operators) if we say the following: $\sigma(A)$ is the spectrum of the pencil $A(\lambda)$ considered in the "classical" space $H$ while $\sigma(\hat{A})$ is its spectrum in the generalised space $H_{-1}$.

Generally, $\sigma(A) \neq \sigma(\hat{A})$. What is the connection between the classical and the generalized spectra? Some light is cast on this problem by the next propositions. It will be convenient to define in the complex plane the open set $\rho_m(A) := \rho(A) \cup \sigma_d(A)$. The set $\rho_m(\hat{A})$ is defined analogously. In the other words $\rho_m(A)$ and $\rho_m(\hat{A})$ are the domains where the operator functions $A^{-1}(\lambda)$ is finite meromorphic in the spaces $H$ and $H_{-1}$, respectively.

Proposition 3.2. In the domain $\rho_m(A) \cap \rho_m(\hat{A})$ all the eigenvalues and Jordan chains of $A(\lambda)$ in the spaces $H$ and $H_{-1}$ coincide.

Proof. Let us consider the pencils

$$\begin{align*}
\hat{A}(\lambda) &:= A(\lambda)S^{-2} = \lambda^2 F S^{-2} + \lambda (DS^{-2} + iGS^{-2}) + J, \\
\hat{A}^*(\lambda) &:= \lambda^2 S^{-2} F + \lambda (S^{-2} D - iS^{-2} G) + J.
\end{align*}$$

We have already noticed (Proposition 3.1) that all the operator coefficients of these pencils are bounded operators in $H$. Moreover, $\hat{A}(\lambda)$ and $\hat{A}^*(\lambda)$ are mutually adjoint in $H$. Let $\mu$ be a normal (classical) eigenvalue of $A(\lambda)$ with a corresponding canonical system of Jordan chains

$$y_j^0, \ldots, y_j^{p_j}, \quad j = 1, \ldots, N.$$
Then $\mu$ is a normal eigenvalue of the pencil $\tilde{A}(\lambda)$ and
\begin{equation}
S^2 y_j^0, \ldots, S^2 y_j^{p_j}, \quad j = 1, \ldots, N,
\end{equation}
is a canonical system of Jordan chains of the pencil $\tilde{A}(\lambda)$ corresponding to $\mu$. In this case, obviously, $S y_j^0, \ldots, S y_j^{p_j}$, are eigen and associated elements of the pencil $\tilde{A}(\lambda)$. Therefore, $\mu \in \sigma_p(\tilde{A})$ and we can define a canonical system of Jordan chains
\begin{equation}
z_j^0, \ldots, z_j^{q_j}, \quad j = 1, \ldots, K,
\end{equation}
of the pencil $\tilde{A}(\lambda)$ corresponding to the eigenvalue $\mu$. It follows from the definition of a canonical system that $K \geq N$ and $q_j \geq p_j$ for $j = 1, \ldots, N$. If in addition $\mu \in \rho_m(\tilde{A})$ (as claimed by assumption) then the adjoint system
\begin{equation}
x_j^0, \ldots, x_j^{q_j}, \quad j = 1, \ldots, K,
\end{equation}
with respect to (3.4) (in the sense of the Laurent expansion for $\tilde{A}^{-1}(\lambda)$ at the point $\mu$) is well defined. Now, we observe that $S^{-1} x_j^0, \ldots, S^{-1} x_j^{q_j}$ are the Jordan chains of the pencil $\tilde{A}^*(\lambda)$ corresponding to the eigenvalue $\tilde{\mu}$. Hence any canonical system of $\tilde{A}^*(\lambda)$ corresponding to $\tilde{\mu}$ consists of $l \geq q_1 + \ldots + q_K$ elements. On the other hand, the system which is adjoint to (3.3) is a canonical system of $\tilde{A}^*(\lambda)$ corresponding to $\tilde{\mu}$ and consists of $p_1 + \ldots + p_N$ elements. Therefore, $N = K$ and $p_j = q_j$ for $j = 1, \ldots, N$.

The same arguments can be applied to show that if $\mu \in \rho(A) \cap \rho_m(\tilde{A})$ then $\mu \in \rho(\tilde{A})$.

\begin{proposition}
Let
\begin{equation}
A_\pm(\lambda) = \lambda^2 F + (D \pm iG)\lambda + T \quad (A_+(\lambda) = A(\lambda)).
\end{equation}
Let $\Omega$ be a domain in $\mathbb{C}$ which is symmetric with respect to the real axis and such that $\Omega \subset \rho_m(A_+) \cap \rho_m(A_-)$. Then $\Omega \subset \rho_m(\tilde{A}_+) \cap \rho_m(\tilde{A}_-)$ and all the eigenvalues of the pencils $A_\pm(\lambda)$ and $\tilde{A}_\pm(\lambda)$ in $\Omega$ as well as the structures of the corresponding canonical systems coincide.

\begin{proof}
Denote
\begin{equation}
\tilde{A}_\pm(\lambda) = A_\pm(\lambda)S^{-2}.
\end{equation}
Obviously, $\zeta \in \rho_m(A_+) \cap \rho_m(A_-)$ if and only if $\zeta \in \rho_m(\tilde{A}_+) \cap \rho_m(\tilde{A}_-)$ (see also [Ma], Lemma 20.1). Hence the functions
\begin{equation}
\tilde{A}_\pm^1(\lambda) = S^2 A_\pm^{-1}(\lambda), \quad [\tilde{A}_\pm^{-1}(\lambda)]^* = A^-_\pm^{-1}(\lambda)S^2
\end{equation}
are finitely meromorphic in the domain $\Omega$. From this we find that the functions $SA_\pm^{-1}(\lambda)S$ are finitely meromorphic in $\Omega$, i.e. $\Omega \in \rho_m(\tilde{A}_+) \cap \rho_m(\tilde{A}_-)$. \qed
\end{proof}

Corollary 3.4. Suppose that the operator $G$ is $T$-compact and the set $\rho_m(A)$ is connected, i.e. $\rho_m(A)$ is a domain in $\mathbb{C}$. Then $\rho_m(\hat{A}) \supset \rho_m(A)$ and the eigenvalues of the pencils $A(\lambda)$ and $\hat{A}(\lambda)$ in $\rho_m(A)$ as well as the structures of the corresponding canonical systems coincide.

Proof. Let us consider the pencils $A_{\pm}(\lambda)$ defined by (3.5). Since $0 \in \rho(A_{\pm})$ (as $T$ is boundedly invertible), $G$ is a $T$-compact operator and $\rho_m(A_+)$ is a domain in $\mathbb{C}$, we have

$$\rho_m(A_+) = \rho_m(A_-) = \rho_m(A_0), \quad \text{where} \quad A_0(\lambda) = \lambda^2 F + \lambda D + T.$$

(see [Ka], Ch.4, Theorems 5.26 and 5.31). It is easy to check that the set $\rho_m(A_0)$ is symmetric with respect to the real axis. Now apply proposition 3.3. $\square$

We shall associate a linear pencil with the quadratic pencil $A(\lambda)$. Let us consider the following linear pencil

$$-\begin{pmatrix} D + iG & T \\ -J & 0 \end{pmatrix} - \lambda \begin{pmatrix} F & 0 \\ 0 & J \end{pmatrix} =: T - \lambda W =: A(\lambda). \tag{3.6}$$

We can consider $T$ as an operator acting in the space $H \times H$ with the domain $\mathcal{D}(T) = H_2 \times H_2$ or as an operator acting in $H_{-1} \times H_{-1}$ with the domain $\mathcal{D}(T) = H_1 \times H_1$. However, in both these spaces $T$ is not a dissipative operator. The situation is changed if we define $T$ as an operator acting in the space $H = H \times H_1$ with the domain

$$\mathcal{D}(T) = \left\{ x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \ x_1, x_2 \in H_1, \ (D + iG)x_1 + Tx_2 \in H \right\}. \tag{3.7}$$

We observe that $H_2 \times H_2 \subset \mathcal{D}(T)$, therefore, $T$ is densely defined in $H = H \times H_1$.

Proposition 3.5. The operator $-iT$ with domain (3.7) is dissipative in the space $H \times H_1$.

Proof. Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{D}(T)$. Then $y_j := Sx_j \in H$, for $j = 1, 2$, and

$$i(Tx, x)_H = -i((D + iG)x_1 + Tx_2, x_1) + i(SJx_1, Sx_2)$$

$$= -i(\hat{D}y_1, y_1) + (\hat{G}y_1, y_1) - i(Jy_2, y_1) + iy_1, Jy_2).$$

Therefore, $Im (iTx, x)_H = -(Dx_1, x_1) \leq 0$ for all $x \in \mathcal{D}(T)$. $\square$
Proposition 3.6. Let $\rho(A)$ be the resolvent set of the linear pencil $A(\lambda)$ defined by (3.6). Then $\rho(A) \supset \rho(\hat{A})$. Moreover, $\sigma_p(A) = \sigma_p(\hat{A})$ and the Jordan structures of the root subspaces corresponding to each eigenvalue $\mu$ of $A(\lambda)$ and $\hat{A}(\lambda)$ coincide.

Proof. Let us solve the equation

$$A(\lambda)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \quad \text{where} \quad \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H, \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(T).$$

After simple calculations we obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -\begin{pmatrix} \lambda A^{-1}(\lambda) & \lambda - J - A^{-1}(\lambda)T \\ A^{-1}(\lambda) & \lambda - A^{-1}(\lambda)T \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$  

If $\lambda \in \rho(\hat{A})$ then $A^{-1}(\lambda) : H_{-1} \to H_1$ is a bijection, and so is $A^{-1}(\lambda)T : H_1 \to H_1$. Therefore $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in H_1 \times H_1$ if $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in H = H \times H_1$. Moreover, it follows from (3.8) that

$$(D + iG)x_1 + T x_2 = -f_1 - \lambda F x_1 \in H, \quad \text{i.e.} \quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in D(T).$$

Hence, the inclusion $\rho(A) \supset \rho(\hat{A})$ is proved.

Suppose that system (3.2) forms a canonical system of Jordan chains corresponding to an eigenvalue $\mu$ of $A(\lambda)$ acting in the space $H_{-1}$. Then it is easily seen that the elements

$$\begin{pmatrix} \mu y_j^0 \\ y_j^0 \end{pmatrix}, \begin{pmatrix} \mu y_j^0 + y_j^0 \end{pmatrix}, \ldots, \begin{pmatrix} \mu y_j^{p_j} + y_j^{p_j-1} \end{pmatrix}$$

belong to $D(T)$ and form Jordan chains of $A(\lambda)$ corresponding to $\mu \in \sigma_p(A)$. The converse assertion can also be easily verified, namely: all Jordan chains of $A(\lambda)$ have representation (3.9), and if (3.9) is a canonical system of $A(\lambda)$ then (3.2) is a canonical system of $A(\lambda)$ in $H_{-1}$. \quad \Box

Let $\mu$ be a normal pure imaginary eigenvalue of the pencil $A(\lambda)$ in the space $H_{-1}$. Then according to Proposition 3.6 $\zeta = i\mu$ is a normal real eigenvalue of the linear dissipative pencil $iA(-i\zeta)$. Using the results of section 2 we can choose a regular canonical system (3.9) of the pencil $iA(-i\zeta)$ corresponding to the eigenvalue $\zeta = i\mu$. A canonical system (3.2) is said to be regular if the corresponding system (3.9) is regular. Further we define the sign characteristics $\varepsilon_j$ of regular system (3.2) to be equal to those of the corresponding system (3.9). As in section 2 we define the numbers $\varepsilon_j^+ = \max(0, \varepsilon_j)$ for all $\mu_j \in \sigma_d(A)$ and assume $\varepsilon_j^+ = 0$ for all $\mu_j \in \sigma_p(A) \setminus \sigma_d(A)$. Recall also that $\nu(F) = \text{rank } F_-$ and $\nu(T) = \text{rank } T_-$ are equal to the numbers of negative eigenvalues counting with multiplicities of the operators $F$ and $T$, respectively.
**Theorem 3.7.** Let the numbers $\nu(F)$ and $\nu(T)$ be finite. Then the generalized spectrum of $A(\lambda)$ in the open right-half plane $\mathbb{C}_r$ consists of only normal eigenvalues, and hence coincides with the spectrum of the linear pencil $A(\lambda)$. If $\kappa(\hat{A})$ is the total algebraic multiplicity of all the eigenvalues lying in $\mathbb{C}_r$ then

$$
\kappa(\hat{A}) + \sum_{\mu_k \in \mathbb{R} \cap \sigma_p} (\varepsilon_k^+ + [(p_k - 1)/2]) \leq \nu(T) + \nu(F).
$$

Here the numbers $\varepsilon_k$ are defined as above and $p_k + 1$ are the lengths of Jordan chains of regular canonical systems corresponding to the pure imaginary eigenvalues $\mu_k$. If $A(\lambda)$ (and hence $A(\lambda)$) has only normal eigenvalues on the imaginary axis then equality holds in (3.10).

**Proof.** It was already noticed that the spectrum of $A(\lambda)$ in the space $H_{-1}$ coincides with the spectrum of the pencil

$$
\hat{A}(\lambda) = \lambda^2 \hat{F} + \lambda(\hat{D} + i\hat{G}) + J
$$

in the space $H$, where $\hat{F}, \hat{D}, \hat{G}$ are bounded operators in $H$, while $J = I - 2J_-$ is a finite rank perturbation of the identity operator. Let

$$
Q(\lambda) = \lambda|\hat{F}| + (\hat{D} + i\hat{G}) + \lambda^{-1}I,
$$

$$
\lambda = \eta + i\tau.
$$

Then

$$
Re\left(Q(\lambda)x, x\right) = \eta(|\hat{F}|x, x) + (\hat{D}x, x) + \frac{\eta}{\eta^2 + \tau^2}(x, x) > \frac{\eta}{\eta^2 + \tau^2}(x, x)
$$

for all $\eta = Re\lambda > 0$. Therefore, $Q(\lambda)$ is boundedly invertible in the open right-half plane (see, for example, [Ma], Theorem 26.2). It follows from our assumptions that $\lambda^{-1}\hat{A}(\lambda)$ is a finite rank perturbation of $Q(\lambda)$. Hence, by virtue of the theorem on holomorphic operator function (see [GS], for example) the spectrum of $\hat{A}(\lambda)$ in the open right-half plane $\mathbb{C}_r$ consists only of normal eigenvalues. According to Proposition 3.6 the linear pencil $A(\lambda) = T - \lambda W$ has the same spectrum in $\mathbb{C}_r$. Now, notice that $\nu(W) = \nu(T) + \nu(F)$ and apply formula (2.19) with respect to the pencil $iT - \zeta W$, $\zeta = i\lambda$. □

**Corollary 3.8.** Let the numbers $\nu(F)$ and $\nu(T)$ be finite and let the operators $F$ and $G$ be $T$-compact. Then the generalized spectrum of $A(\lambda)$ consists of normal eigenvalues with the possible exception of a closed subset lying on the negative semiaxis and the following formula is valid

$$
\kappa(\hat{A}) + \sum_{\mu_k \in \mathbb{R} \cap \sigma_d} (\varepsilon_k^+ + [(p_k - 1)/2]) = \nu(T) + \nu(F).
$$
If in addition the classical spectrum of the linear pencil $\lambda \hat{D} + T$ in the closed right-half plane consists of only normal eigenvalues then $\kappa(\hat{A})$ can be replaced by $\kappa(A)$.

**Proof.** Let us consider the linear pencil $\lambda \hat{D} + I$ in the space $H$. Obviously, its nondiscrete spectrum is a closed subset, say $\Delta$, belonging to $\mathbb{R}_-$ ($\Delta = \emptyset$ if $\hat{D}$ is a compact operator). It follows from the interpolation theorem (see Proposition 3.1) that $\hat{F}$ and $\hat{G}$ are compact operators in $H$ if $F$ and $G$ are $T$-compact. Hence the pencil $\hat{A}(\lambda)$ is a compact perturbation of the pencil $\lambda \hat{D} + I$. Now apply the theorem on holomorphic operator functions and Theorem 3.7.

As the operators $F$ and $G$ are $T$-compact, the complement to the nondiscrete spectrum of the linear pencil $\lambda D + T$ in $H$ coincides with the set $\rho_m(A)$ (this follows as before from the theorem on holomorphic operator functions). Hence we may apply Proposition 3.2 and replace $\kappa(\hat{A})$ by $\kappa(A)$.

Suppose that $A(\lambda)$ has only semisimple eigenvalues on the imaginary axis. Then the sign characteristics $\varepsilon_k$ are defined as follows $(\zeta_k = i\mu_k)$

$$
\varepsilon_k = \left( \begin{array}{cc}
\zeta_k y_k \\
y_k
\end{array} \right), \left( \begin{array}{cc}
\zeta_k y_k \\
y_k
\end{array} \right) \right) = \\
= -\mu_k^2 (Fy_k, y_k) + (Ty_k, y_k) = (A(\mu_k)y_k, y_k) \\
- \mu_k (A'(i\mu_k)y_k, y_k) = -\mu_k (A'(i\mu_k)y_k, y_k).
$$

An eigenvalue $i\mu \in i\mathbb{R}$ is said to be of the first (the second) type if $\varepsilon = -\mu(A'(i\mu)y, y) > 0(< 0)$ for all $y \in \text{Ker} A(\mu)$. It is well-known fact that all eigenvalues of definite type are semisimple.

**Corollary 3.9.** If assumptions of Theorem 3.7 are fulfilled and all the pure imaginary eigenvalues of $A(\lambda)$ are of definite type then

$$
(3.12) \quad \kappa(\hat{A}) = \nu(T) + \nu(F) - \varepsilon^+(\hat{A}).
$$

where $\varepsilon^+(A)$ is the number of the first type eigenvalues counting with multiplicities belonging to the imaginary axis. In particular, if $D > 0$ then

$$
(3.13) \quad \kappa(\hat{A}) = \nu(T) + \nu(F).
$$

**Proof.** We have only to notice that $A(\lambda)$ has no pure imaginary eigenvalues if the condition $D > 0$ is fulfilled. □
Let us consider the operator differential equation

\[(3.14) \quad A \left( \frac{du}{dt} \right) = F \frac{d^2 u}{dt^2} + (D + iG) \frac{du}{dt} + Tu = 0, \quad u = u(t). \]

If (3.2) are Jordan chains of \(A(\lambda)\) corresponding to an eigenvalue \(\mu_k\) then the functions

\[u^h_k(t) = e^{\mu_k t} \left( y^h_k + \frac{t}{1!} y^{h-1}_k + \ldots + \frac{t^h}{h!} y^0_k \right), \quad h = 0, \ldots, p_k,\]

are called elementary solution of (3.14). Under the assumptions of Corollary 3.9 the number \(\kappa(\dot{A})\) coincides with the number of linearly independent elementary solutions of equation (3.14) which are not bounded when \(t \to \infty\). Hence the number \(\kappa(\dot{A})\) characterizes the index of instability of equation (3.14). Strictly speaking, the index of instability \(\kappa'(\dot{A})\) has to be defined as the number of linearly independent generalized solutions of (3.14) which are not bounded when \(t \to \infty\). Generally, \(\kappa'(\dot{A}) \geq \kappa(\dot{A})\). We know abstract examples when \(\kappa'(\dot{A}) > \kappa(\dot{A})\) even if there are no pure imaginary eigenvalues (see [M1], for example).

We can show that \(\kappa'(\dot{A}) = \kappa(\dot{A})\) if in addition to the assumptions of Theorem 3.7 the whole spectrum of \(\dot{A}(\lambda)\) is discrete. However, the rigorous proof of this fact (and more general ones) requires additional preparations and is left for a future occasion. With these reservations we may consider (3.12) as the instability index formula. The relation (3.11) may be considered as the generalized instability index formula.

In the end of this section we would like to make some historical remarks concerning formula (3.11). Apparently, the first investigation of the pencil (3.1) with matrix coefficients was carried out by Kelvin and Tait [KT]. They considered the case \(F = I\) and made the following interesting observations.

1. \textit{If matrix } \(T\) \textit{is positive then the problem is stable for all gyroscopic matrices } \(G\) \textit{and all } \(D \geq 0\).

2. \textit{The condition } \(T > 0\) \textit{is not necessary for stability. Even if } \(T < 0\) \textit{the motion can be stabilized by gyroscopic forces (an example was given). However, if rank } \(T_- = \kappa\) \textit{is odd then the problem can not be stabilized by the action of gyroscopic forces.}

3. \textit{If } \(\text{rank } T_- = \kappa > 0\) \textit{and } \(D > 0\) \textit{(complete dissipative forces) then the problem can not be stable for any gyroscopic forces } \(G\).

All these observations were rigorously proved by Chetaev [Ch] by introducing the Lyapunov function. The next step was made by Zajac [Z]. He considered the matrix pencil
(3.1) with $F = I$, $D > 0$ and proved the formula $\kappa(A) = \nu(T)$. Wimmer [W] and later Lancaster and Tismenetsky [LT] studied matrix pencil (3.1) and admitted an indefinite leading coefficient $F$. In particular, they proved the relations

\begin{align*}
\kappa(A) &\leq \nu(F) + \nu(T) \quad \text{if } D \geq 0, \\
\kappa(A) &= \nu(F) + \nu(T) \quad \text{if } D > 0.
\end{align*}

(3.15)

The second relation in (3.15) for the case $D = 0$ has been more closely investigated recently by Barkwell, Lancaster, and Markus [BLM]. The pencils of the form (3.1) with unbounded operator coefficients were studied by Miloslavsky, Pivovarchik et al. in the papers [M1], [M2],[ZKM], [P3], [P4]. The main aim of these papers was to obtain relations (3.15) under the assumptions $F = I$ and various hypotheses on the operator coefficients D, G, T, which were essentially stronger than our assumptions i) -iv). As far as we know, formulas (3.11), (3.12) presented in this paper are new even for matrix pencils.

4. Applications

In this section we shall apply the obtained abstract results to concrete problems considered in Section 1.

**Theorem 4.1.** Formula (3.11) or its simplifications (3.12) or (3.13) are valid for operator pencil (1.7) associated with the problem of small oscillations of ideal incompressible fluid in a pipe of finite length if the condition $\ker T = \{0\}$ is fulfilled ($T := A + C$). For a pipe of infinite length the assertion of Theorem 3.7 is valid if $g(x)$ is such a function that $\ker T = \{0\}$ and $\nu(T) < \infty$.

**Proof.** The conditions i)-iv) of Section 1 imply conditions i)-iii) of Section 3 if it is assumed in addition that $\ker T = \{0\}$. Moreover, for a pipe of finite length the conditions iii) and iv) of Section 1 hold. Hence, for a pipe of finite length the assumptions of Corollary 3.8 are fulfilled. For a pipe of infinite length the operators $G$ and $I$ are not $T$-compact and we must use Theorem 3.7. In the last case we can not guarantee the absence of pure imaginary eigenvalues belonging to the nondiscrete spectrum. □

If $\ker T \neq \{0\}$ then $\lambda = 0$ is an eigenvalue of pencil (3.1). In this case the analogue of formula (3.11) can also be obtained. For this purpose one has to modify the results of Section 2 for the case $\ker W \neq \{0\}$. Technically this is not a trivial work. However, the estimates for the number $\kappa(\hat{A})$ can be obtained easily if $\ker T \neq \{0\}$. 
**Theorem 4.2.** Suppose that a pencil $A(\lambda)$ is defined by (3.1) and its operator coefficients satisfy the assumptions i)-iii) of Section 3 with the possible exception that the operators $F$ and $T$ are not necessarily boundedly invertible. Suppose that there exists a point $\mu$, $Re \mu > 0$ such that $A(\mu)$ is boundedly invertible. Then

(4.1) \[ \kappa(\hat{A}) \leq \nu(F) + \nu(T). \]

**Proof.** Let us consider the pencil

\[ A_r(\lambda) = \lambda^2 (F + \tau I) + (D + iG)\lambda + T + \tau I, \quad \tau > 0. \]

Obviously, $\nu(T + \tau I) = \nu(T)$, $\nu(F + \tau I) = \nu(F)$, if $\tau \in (0, \tau_0$ and $\tau_0$ is sufficiently small. By virtue of Theorem 3.7 we have

(4.2) \[ \kappa(\hat{A}_r) \leq \nu(F) + \nu(T) \quad \text{for all} \quad 0 < \tau < \tau_0. \]

Repeating the arguments from the proof of theorem 3.7 and taking into account that $\mu \in \rho(\hat{A})$ for some $\mu$ with $Re \mu > 0$ we obtain that the spectrum of $\hat{A}(\lambda) := \hat{A}_0(\lambda)$ in the open right half plane consists only of normal eigenvalues. These eigenvalues continuously depend on $\tau$ (see [Ka], Ch. 7). Then (4.2) implies (4.1). \(\square\)

The results of Sections 2 and 3 can also be applied to selfadjoint pencils. Lancaster and Shkalikov [LS] considered an operator pencil $L(\lambda)$ defined by (1.5) with $C = 0, D_\alpha = \alpha A$ and obtained the following estimate

(4.3) \[ \eta/2 \leq \min_{k \in \mathbb{R}} \pi(L(k)), \quad \pi(L) := \nu(-L), \]

where $\eta$ is the number of nonreal eigenvalues of the pencil $L(\lambda)$ counting with algebraic multiplicities. Using an analytic approach Shkalikov and Griniv proved a sharper estimate for the case $C = 0$ and reproved (4.3) for $C \neq 0$ (if $C$ is an $A$-compact operator). Here we refine the corresponding results from [LS] and [SG].

**Theorem 4.3.** Let

\[ L(\lambda) = \lambda^2 F + \lambda D + T, \]

where $T = T^*$ and $F, D$ are symmetric and $T$-bounded operators. Let $S^2 = |T| + I$ and the scale of Hilbert spaces $H_\theta$ be generated by the operator $S \gg 0$. Suppose that there exist real points $a$ and $b$ belonging to $\rho(\hat{L})$ such that

\[ \pi(L(a)) < \infty, \quad \nu(L(b)) < \infty. \]
Then the nonreal spectrum of $L(\lambda)$ in the space $H_{-1}$ consists of finitely many, say $\eta$, nonreal eigenvalues, and the following estimate is valid

\[(4.4) \quad \eta/2 \leq \pi(L(a)) + \nu(L(b)) - \delta^+(L),\]

where $\delta^+(L)$ is the number of real eigenvalues $\mu_k$ of $L(\lambda)$ counting with multiplicities such that

\[(b - a) \left( \frac{\mu_k - a}{b - \mu_k} \right) (L'(\mu_k)y, y) > 0 \quad \text{for all } y \in \text{Ker}L(\mu_k).\]

**Proof.** We use the same idea as in [LS] where estimate (4.4) was obtained in a slightly different situation not taking into account the number $\delta^+(L)$. It was shown in Section 3 that the spectrum of $L(\lambda)$ in the space $H_{-1}$ coincides with the spectrum of $\hat{L}(\lambda) = S^{-1}L(\lambda)S^{-1}$ in the space $H$. The pencil $\hat{L}(\lambda)$ has the bounded operator coefficients $\hat{F}, \hat{D}, \hat{T}$. After the substitution $\lambda = (b\xi + a)(\xi + 1)^{-1}$ we obtain the quadratic pencil

$$\hat{L}(\xi) := (\xi + 1)^2\hat{L}(\lambda(\xi)) = \xi^2 \hat{F} + \xi \hat{D} + \hat{T}, \quad \hat{F} = \hat{L}(b), \quad \hat{T} = \hat{L}(a).$$

Let us consider the linearization of $\hat{L}(\xi)$

$$L(\xi) = - \begin{pmatrix} \hat{D} & \hat{T} \\ \hat{T} & 0 \end{pmatrix} - \xi \begin{pmatrix} \hat{F} & 0 \\ 0 & -\hat{T} \end{pmatrix}. $$

Suppose that $\xi_k$ is a simple (or semisimple) real eigenvalue of $\hat{L}(\xi)$ with a corresponding eigenvector $y_k$. Then the sign characteristic $\varepsilon_k$ (see section 2) is defined as follows

$$\varepsilon_k = \begin{pmatrix} \hat{F} & 0 \\ 0 & -\hat{T} \end{pmatrix} \begin{pmatrix} \xi_k y_k \\ y_k \end{pmatrix}, \quad \begin{pmatrix} \xi_k y_k \\ y_k \end{pmatrix}_{H \times H} = \xi(\hat{L}'(\xi_k)y_k, y_k)$$

$$= \xi_k \lambda'(\xi_k)(\hat{L}'(\lambda_k)y_k, y_k)) = (b - a)(\mu_k - a)(b - \mu_k)^{-1}(\hat{L}'(\mu_k)y_k, y_k).$$

Now apply Corollary 2.11. □

We note that the estimate (4.4) is also new for matrix pencils.

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