ON MINIMALITY AND COMPLETENESS OF SYSTEMS
CONSTRUCTED FROM SOME OF THE EIGEN- AND ASSOCIATED
ELEMENTS OF QUADRATIC OPERATOR PENCILS

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We consider the quadratic operator pencil

\[ L(\lambda) = A + \lambda B + \lambda^2 C; \]

here \( A \) and \( C \) are bounded selfadjoint operators acting in a Hilbert space \( \mathcal{H} \), and \( B \) is a bounded dissipative operator acting in \( \mathcal{H} \), i.e., either \( B_f = i(B^* - B) \geq 0 \) or \( B_f \leq 0 \). It is assumed that the spectrum of the pencil \( L(\lambda) \) is discrete. In this case the operator-valued function \( L^{-1}(\lambda) \) is meromorphic, and the principal part of its expansion at a pole \( \lambda = \lambda_k \) has the form

\[ \sum_{k=N_1}^{N_2} \sum_{h=0}^{p_k} \frac{(\cdot, z^0_k)y_k^h + (\cdot, z^1_k)y_k^{h-1} + \cdots + (\cdot, z^h_k)y_k^0}{(\lambda - \lambda_k)^{p_k+1-h}}, \]

where

\[ y_k^{0}, \ldots, y_k^{p_k}, \quad k = N_1, \ldots, N_2, \]

is some canonical (in the Keldysh sense) system of eigen- and associated elements (EAE’s) of the pencil \( L(\lambda) \) corresponding to the eigenvalue (EV) \( \lambda_k \) (which is counted as many as there are chains of EAE’s corresponding to it), and

\[ z_k^{0}, \ldots, z_k^{p_k}, \quad k = N_1, \ldots, N_2, \]

is the conjugate canonical system of EAE’s of the pencil \( L^*(\lambda) \) corresponding to the EV \( \bar{\lambda}_k \).

Denote the spectrum of \( L(\lambda) \) by \( \sigma(L) \). It will be assumed that \( 0 \notin \sigma(L) \), i.e., \( A \) is invertible. The case \( 0 \in \sigma(L) \) requires additional considerations, and the corresponding changes will be indicated in a remark.

**Lemma 1.** If \( \lambda_k \in \mathbb{R} \cap \sigma(L) \) and \( \lambda_k \neq 0 \), then the canonical system (3) can be chosen so that

\[ y_k^h = \varepsilon_k z_k^h, \quad h = 0, 1, \ldots, [p_k/2], \quad k = N_1, \ldots, N_2, \]

where \([p]\) is the integer part of a number \( p \), and \( \varepsilon_k = \pm 1 \).

**Proof.** It was proved in [2] that canonical systems satisfying (5) for \( 0 \leq h \leq p_k \) can be chosen in the case where \( B = B^* \). But \( y_k^h \in \text{Ker}(B^* - B) \) for \( h \leq [p_k/2] \) by Lemma 4 in [3], i.e., the chains (3) of the pencil \( L(\lambda) \) corresponding to an EV \( \lambda_k \in \mathbb{R} \) coincide for \( h \leq [p_k/2] \) with the corresponding chains of the pencil \( L_R(\lambda) = A + \lambda B_R + \lambda^2 C \), where \( B_R = (B^* + B)/2 \). Lemma 12 follows from this.

Canonical systems corresponding to an EV \( \lambda_k \in \mathbb{R} \) and having the properties indicated in Lemma 1 are said to be normal. It is assumed everywhere below that normal canonical systems correspond to real EV’s.
Denote by $\Lambda$ the set of pairs $(k, h)$, $0 \leq h \leq p_k$, of numbers used to label the elements $y^k_h$ of all chains (3) of the pencil $L(\lambda)$. The collection of pairs $(k, h) \in \Lambda$ of indices such that $\text{Im } \lambda_k > 0$ $(< 0)$ is denoted by $\Lambda^+$ $(\Lambda^-)$. We also introduce the subsets $\Lambda^+_R$, $\Lambda^-_R \subset \Lambda$ of pairs of indices which label the part of the EAE’s corresponding to EV’s $\lambda_k \in \mathbb{R}$. It will be assumed that $(k, h) \in \Lambda^+_R$ $(\Lambda^-_R)$ if $\text{Im } \lambda_k = 0$ and $0 \leq h \leq \nu^+_k (\nu^-_k)$, where

$$
\nu^+_k (\nu^-_k) = \begin{cases} 
  l_k & \text{if } p_k = 2l_k + 1 \text{ or } p_k = 2l_k \text{ and } \lambda_k \varepsilon_k > 0 < (0), \\
  l_k - 1 & \text{if } p_k = 2l_k \text{ and } \lambda_k \varepsilon_k < 0 > (0).
\end{cases}
$$

In the Hilbert space $\mathcal{H}^2 = \mathcal{H} \oplus \mathcal{H}$ we consider the elements $y^h_k = (y^h_k, \lambda_k y^h_k + y^{h-1}_k)$, the derived chains in the Keldysh sense, constructed from the chains of EAE’s (3) of $L(\lambda)$ (it is assumed that $y^{h-1}_k = 0$ if $h = 0$; the elements of the space $\mathcal{H}^2$ are denoted by boldface letters in contrast to the elements of $\mathcal{H}$). We represent the operators $A$ and $C$ in the form $A = A_+ - A_-$ and $C = C_+ - C_-$, where $A_+$, $A_-$ and $C_+$, $C_-$ are nonnegative operators such that $A_+ A_- = 0$ and $C_+ C_- = 0$. We consider the operators

$$
\mathcal{T}_\pm \left( \begin{array}{cc} A^{1/2}_\pm & 0 \\
0 & C^{1/2}_\pm \end{array} \right)
$$

acting in $\mathcal{H}^2$, and we denote by $\mathcal{H}^2_\pm$ the closures of their ranges in $\mathcal{H}^2$. Obviously, $\mathcal{H}^2_\pm = \text{Im } A^\mp \oplus \text{Im } C^\mp$.

Finally, we define the systems which make up the object of investigation in this note:

$\mathcal{Y}^\pm = \{ \mathcal{T}_\pm y^h_k \}$, where $(k, h) \in \Lambda^+ \cup \Lambda^+_R$;

$\mathcal{L}^\pm = \{ \mathcal{T}_\pm y^h_k \}$, where $(k, h) \in \Lambda^- \cup \Lambda^-_R$.

**Theorem 1.** Suppose that the pencil $L(\lambda)$ is given by (1) and that $B_J \geq 0$ $(\leq 0)$.

Then the systems $\mathcal{Y}^-$ and $\mathcal{Z}^+$ ($\mathcal{Z}^-$ and $\mathcal{Y}^+$) are minimal in the respective spaces $\mathcal{H}^2_-$ and $\mathcal{H}^2_+$.

The proof can be obtained from a result in [3] which is a generalization of minimality theorems obtained previously in [2], where Theorem 1 was proved for $A > 0$ and $B = B^*$. In turn, [2] was preceded by [4]–[6]. We give another proof of Theorem 1 which is much shorter and, on the other hand, clears up the essence of the matter. For definiteness we consider a system $\mathcal{Y}^-$ and assume that $B_J \geq 0$.

**Step 1.** Consider the operators

$$
\mathcal{G} = \begin{pmatrix} A & 0 \\
0 & -C \end{pmatrix}, \quad \mathcal{H} = \begin{pmatrix} B & C \\
C & 0 \end{pmatrix}, \quad \mathcal{L} = \begin{pmatrix} A^{-1}B & A^{-1}C \\
-I & 0 \end{pmatrix}
$$

acting in $\mathcal{H}^2$, where $I$ is the identity operator on $\mathcal{H}$. Obviously, the derived chains in the Keldysh sense $y^h_k$, $\ldots, y^p_k$ constructed from the chains (3) are EAE’s of the operator $\mathcal{L}$ corresponding to the EV $\mu_k = -\lambda^{-1}_k$ (these chains are also EAE’s of the linear pencil $\mathcal{G} + \lambda \mathcal{H}$ corresponding to the EV $\lambda_k$). It is also obvious that the operator $\mathcal{L}$ is $\mathcal{G}$-dissipative, i.e., $\text{Im } (\mathcal{G} \mathcal{L} y, y) \geq 0$ for all $y \in \mathcal{H}^2$. But then (see Proposition 10° in [7]) we have that

$$
(\mathcal{G} y, y) \geq 0 \quad \text{if } y = \sum \epsilon^h_k y^h_k, \quad (k, h) \in \Lambda^+;
$$

here the sum is finite and the summation is over the indices $(k, h) \in \Lambda^+$.

**Step 2.** For EAE’s of the linear pencil $I + \lambda \mathcal{L}$ (or for EAE’s of the pencil $\mathcal{G} + \lambda \mathcal{H}$ in the space $\mathcal{H} \oplus \text{Im } C$) the biorthogonality relations (1) take the form

$$
y^{h}_k, \lambda^{*} z^{s}_{m} = \delta_{k,m} \delta_{h,p_m-s},
$$

from which, using the equalities $(\mathcal{G} + \lambda_m \mathcal{H}^*) z^s_m + \lambda^{*} z^{s-1}_m = 0$, we get

$$
(\mathcal{G} y^h_k, z^s_m - \lambda^{-1}_m z^{s-1}_m + \ldots + (-1)^s \lambda^{-s}_m z^0_m) = -\lambda^{-1}_m \delta_{k,m} \delta_{h,p_m-s}.
$$
Here $z_k^0, \ldots, z_k^{p_k}$ are the chains of EAE’s of the pencil $I + \lambda L^*$ that are conjugate to the chains $y_k^0, \ldots, y_k^{p_k}$. Here it turns out that the chains of elements $\lambda_k^2 z_k^0, \ldots, \lambda_k^2 z_k^{p_k}$ are the derived chains in the Keldysh sense constructed from the chains (4). This can be obtained along the same lines as in [8] and [10]. Suppose that $(m, s) \in \Lambda^*_R$, and let $(k, h) \in \Lambda^+ \cup \Lambda^*_R$. By Lemma 1 and (9),

$$
(G y_k^h, y_m^s) = \begin{cases} 
0 & \text{if } k \neq 0 \text{ or } k = m, \text{ but } h + s < p_k, \\
-\varepsilon_k \lambda_k & \text{if } k = m \text{ and } h = s = [p_k/2].
\end{cases}
$$

It follows from (7), (10), and the definition of the set $\Lambda^*_R$ of index pairs that

$$
(G y, y) \geq 0 \quad \text{if } y = \sum c_k^h y_k^h, \quad (k, h) \in \Lambda^+ \cup \Lambda^*_R.
$$

Step 3. Since $(G y, y) = ||\mathcal{F}_- y||^2 - ||\mathcal{F}_+ y||^2$, (11) implies the estimate $||\mathcal{F}_- y||^2 \geq ||\mathcal{F}_+ y||^2$. Let $\mathcal{F} = \mathcal{F}_+ - \mathcal{F}_-$; then

$$
\sqrt{2} ||\mathcal{F}_- y|| \geq ||\mathcal{F} y|| \geq ||\mathcal{F}||^{-1} ||\mathcal{F}^2 y|| = ||\mathcal{F}|| y.
$$

If $\{e_k\}$ is a minimal system in the Hilbert space $\mathcal{L}$, and the system $\{f_k\} \in \mathcal{L}$ is such that for any finite sequence $\{c_k\}$ of numbers we have the estimate $||\sum c_k f_k||_\mathcal{L} \geq \delta ||\sum c_k e_k||_\mathcal{L}$, $\delta > 0$, then the system $\{f_k\}$ is obviously also minimal in $\mathcal{L}$. By (9), the system $\{G y_k^h\}$ is minimal in $\mathcal{F}^2$; therefore, (12) implies that the system $\mathcal{Y}^- = \{\mathcal{F}_- y_k^h\}$ is minimal in the space $\mathcal{F}^2 \subset \mathcal{F}^2$. Theorem 1 is proved.

We now give two theorems on completeness of the systems $\mathcal{Y}^\pm$ and $\mathcal{Z}^\pm$. The theorems differ both in the methods of proof and in the conditions on the resolvent. The ideas for proving them represent a development of ideas in [2] and [8].

**Theorem 2.** Suppose that the pencil $L(\lambda)$ is given by (1), $B_J \geq 0 \leq 0$, and the following conditions hold ($v_0, v_1, w_0, w_1$ and $w_1$ are arbitrary fixed elements of $\mathcal{F}$):

a) If the function $L^{-1}(\lambda)[A^{1/2}_- v_0 + \lambda C^{1/2}_+ v_1], w_0 + \lambda w_1$ is holomorphic outside some disk, then the principal part of its Laurent expansion at $\infty$ is a polynomial of order at most $N$ (N is arbitrary, but is independent of $v_0$ and $v_1$).

b) There exists a sequence of points $\lambda_n \to \infty$ such that

$$
(L^{-1}(\lambda)[A_- w_0 + \lambda C^{1/2}_+ v_1], A_+ v_0 + \lambda C^{1/2}_- v_1) \to 0 \quad \text{as } \lambda = \lambda_n \to \infty.
$$

Then the system $\mathcal{Y}^- (\mathcal{Z}^-)$ is complete in $\mathcal{F}^2$. If conditions a) and b) are satisfied when the signs “$+$” and “$-$” are interchanged, then the system $\mathcal{Y}^+ (\mathcal{Z}^+)$ is complete in $\mathcal{F}^2$.

**Proof.** Assume that $B_J \geq 0$, and consider the system $\mathcal{Y}^-$. By Theorem 1, there is a system $w^s_m = (w^s_{0,m}, w^s_{1,m})$ such that

$$
(A^{1/2}_- z_k^h, w^s_{0,m}) + (C^{1/2}_+ [\lambda_k z_k^h + z_k^{h-1}], w^s_{1,m}) = (\mathcal{F}_+ z_k^h, w^s_m) = \delta_{k,m} \delta_{s,h}, \quad (k, h) \in \Lambda^- \cup \Lambda^*_R
$$

(here we use the fact that the system $\{\mathcal{F}_+ z_k^h\}, (k, h) \in \Lambda^- \cup \Lambda^*_R$, coincides with the system $\mathcal{Y}^+$ for the pencil $L^*(\lambda)$).

Assume that the element $v = (v_0, v_1) \in \mathcal{F}^2$ is orthogonal to the system $\mathcal{Y}^-$, i.e., for all $(k, h) \in \Lambda^+ \cup \Lambda^*_R$ we have

$$
(v_0, A^{1/2}_- y_k^h + (v_1, C^{1/2}_- [\lambda_k y_k^h] + y_k^{h-1}) = (v, \mathcal{F}_- y_k^h) = 0.
$$

We consider the functions

$$
\Phi^s_m(\lambda) = \lambda^{-1}(L^{-1}(\lambda))^{*}[A^{1/2}_+ v_0 + \lambda C^{1/2}_- v_1], A^{1/2}_- w^s_{0,m} + C^{1/2}_+ w^s_{1,m}).
$$

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Then $\lambda = 0$ is regular for this function, because $A^{1/2}AA^{1/2} = 0$. It follows from (2) that the principal part of the expansion of $\Phi_m^s(\lambda)$ in a neighborhood of the pole $\lambda = \bar{\lambda}_k$ has the form

$$
\lambda_k^{-1} \sum_{k=N_1}^{N_2} \sum_{h=0}^{p_k} \frac{(v, \mathcal{F}_+ y^k_h)(\mathcal{F}_+ z^k_h, w^s_m) + \cdots + (v, \mathcal{F}_- y^h_k)(\mathcal{F}_- z^h_k, w^s_m)}{(\lambda - \bar{\lambda}_k)^{p_k+1-h}}.
$$

The equalities (14) and (15), the form of the principal part (16) of $\Phi_m^s(\lambda)$ at the poles, and the definition of the sets $\Lambda^\pm_R$ and $\Lambda^\pm$ (it is also taken into account that the canonical systems (3) corresponding to the EV $\lambda_k \in R$ are chosen to be normal) give us that $\Phi_m^s(\lambda)$ has a unique pole $\lambda = \bar{\lambda}_m$. Im $\lambda_m \geq 0$. Suppose that Im $\lambda_m > 0$. Letting $s = p_m$ and using (14), we find that $\Phi_m^p(\lambda)$ has a simple pole at the point $\lambda = \bar{\lambda}_m$, and the residue at this pole is $a_m = \bar{\lambda}_m^{-1}(v, \mathcal{F}_- y^0_m)$. We then have that $\Phi_m^p(\lambda) = P_N(\lambda) + o(\lambda - \bar{\lambda}_m)^{-1}$ from condition a) of the theorem, where $P_N(\lambda)$ is a polynomial, and we get that $\Phi_m^p(\lambda) \equiv 0$ from the condition b). Consequently, $(v, \mathcal{F}_- y^0_m) = 0$. If we now set $s = p_m - 1$, then the last equality gives us that $(v, \mathcal{F}_- y^1_m) = 0$. Repeating the same procedure step by step, we get that

$$
(v, \mathcal{F}_- y^h_m) = 0, \quad h = 0, 1, \ldots, p_m.
$$

If Im $\lambda_m = 0$, then, by (15), (17) is valid for $h \leq \nu_m$. Considering the functions $\Phi_m^s(\lambda)$ successively for $s = \nu_m^+, \nu_m^+ - 1, \ldots, 0$, just as before, we get (17) for $h = \nu_m^- + 1, \ldots, p_m$. But then the vector-valued function

$$
\Phi(\lambda) = [L^{-1}(\bar{\lambda})]^*(A^{1/2}_+ \nu_0 + \lambda C^{1/2}_+ v_1)
$$

is entire, and it follows from a) that $\Phi(\lambda)$ is a polynomial. Applying the operator $L^*(\lambda)$ to $\Phi(\lambda)$ and comparing both sides of the equality, we get that $v_0 = v_1 = 0$. This proves Theorem 2.

**Theorem 3.** Suppose that the pencil $L(\lambda)$ is given by (1), $B_J \geq 0$ ($\leq 0$), and there exists a $q > 0$ such that the following conditions hold $(v = (v_0, v_1) \in \mathfrak{F}^2)$.

c) For any $\varepsilon > 0$ there is a sequence of semicircles $C_n = \{\lambda: |\lambda| = r_n, \text{Im} \lambda \geq 0 \leq 0\}$, $r_n \to \infty$, on which the function

$$
\Psi(\lambda) = [(L^{-1}(\lambda))^*[A^{1/2}_- v_0 + \lambda C^{1/2}_- v_1], A^{1/2}_- v_0 + \lambda C^{1/2}_+ v_1]
$$

admits the estimate $|\Psi(\lambda)| \leq M(\varepsilon) \exp \varepsilon|\lambda|^q$.

d) If the function $\Psi(\lambda)$ is holomorphic in the upper (lower) half-plane, then $\Psi(\lambda) - ||v_1||^2 \to 0$ as $\lambda \to \infty$, $\lambda \in \Omega_q (\Omega_q)$, where $\Omega_q = \{\lambda: \pi/2q' \leq \arg \lambda \leq \pi(1 - 1/2q')\}$, $q' = \max(1, q)$. Moreover, if $q \geq 1$, then there is a number $p > q$ such that $|\Psi(\lambda)| \leq M(\varepsilon) \exp \varepsilon|\lambda|^q$ for $\lambda \in \Omega_p$ and for any $\varepsilon > 0$.

Then the system $\mathcal{Y}^-(Z^-)$ is complete in the space $\mathfrak{F}^2_\Omega$. The theorem remains true if the signs "-" and "+" are interchanged.

**Proof.** Suppose that $B_J \geq 0$ and the element $v = (v_0, v_1) \in \mathfrak{F}^2_\Omega$ is orthogonal to the system $\mathcal{Y}^-$. Then the function $\chi(\lambda) = \lambda^{-1}(\Psi(\lambda) - ||v_1||^2)$ is holomorphic in the upper half-plane, and can have only simple poles with residues $a_k \leq 0$ at the points $\lambda_k \in R$; furthermore, the residue at zero is $a_0 = -(||A^{1/2}_- v_0||^2 + ||v_1||^2) < 0$ (this follows from Lemma 1 and arguments in [2]). Since $\Psi(\lambda) = (L(\lambda)g(\lambda), g(\lambda))$, where $g(\lambda) = L^{-1}(\lambda)[A^{1/2}_- v_0 + \lambda C^{1/2}_+ v_1]$, it follows that

$$
\text{Im} \chi(\lambda) \geq 0 \quad \text{for} \quad \lambda \in R, \quad \lambda \neq \lambda_k.
$$

For $t > 0$ we consider the function

$$
u_{in}(t) = \int_{t_k} e^{-\lambda t_k} \chi(\lambda) d\lambda + \int_{t_k} e^{-(\lambda) t_k} \chi(\lambda) d\lambda,
$$

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where $l_n^+$ and $l_n^-$ are parts of the contour (see Figure 1) consisting of segments of the positive and negative rays and segments of circles of small radii $\delta_m$ about the points $\lambda_m$. Let the functions $u_{C_n}(t)$ be defined just as in (19), but let $l_n^+$ and $l_n^-$ be replaced by segments of the semicircles $C_n^+$ and $C_n^-$. Similarly, define functions $u_{\omega_n}(t)$ and $u_\delta(t)$ by replacing $l_n^+$ and $l_n^-$ in (19) by segments of the rays $\omega_q^+$ and $\omega_q^-$ directed along the sides of the angle $\Omega_q$ and, respectively, by segments of arcs of the semicircle $\gamma_{\delta_0} = \{ \lambda : |\lambda| = \delta_0, \text{Im} \lambda \geq 0 \}$. For an appropriate choice of the direction of integration we have

$$ u_{l_n}(t) + u_{C_n}(t) + u_{\omega_n}(t) + u_\delta(t) = 0. \quad (20) $$

Using arguments in Lemma 1.3 of [2], we get that $u_{\omega_n}(t) + u_\delta(t) = -i\pi a_0 + o(1)$ as $t \to +0$ and $\delta_0 \to 0$ independently of $n$.\(^{(1)}\) It follows from condition d) that $\chi(\lambda) \exp -\lambda^2 t \to 0$ as $\lambda \to \infty$ on the rays $\omega_q^+$ and $\omega_q^-$. We then conclude from the Phragmèn-Lindelöf principle (see [9], Chapter I) that $\chi(\lambda) \exp -\lambda^2 t \to 0$ uniformly in the sector bounded by these rays. The Jordan lemma and condition c) give us that $u_{C_n}(t) \to 0$ as $n \to \infty$. Finally, letting $\delta_m$ go to 0 and considering the imaginary part of (20), we get from (18) that $a_0 + \sum a_m + \rho_n = o(1)$ ($\rho_n \leq 0$), which implies that $a_0 = 0$, i.e., $v_0 = v_1 = 0$. Theorem 3 is proved.

**Remark 1.** If $0 \in \sigma(L)$, i.e., Ker $A \neq 0$, then all three theorems remain true, but the definitions of the sets $\Lambda_R^+$ and $\Lambda_R^-$ must be changed by analogy with [3]. Namely, the index pairs $(k, h)$ such that $\lambda_k = 0$ go into $\Lambda_R^+ \setminus (\Lambda_R^-)$ if $0 \leq h \leq \kappa_k^+$ ($\kappa_k^-$), where

$$ \kappa_k^+ (\kappa_k^-) = \begin{cases} l_k & \text{if } p_k = 2l_k \text{ or } p_k = 2l_k - 1 \text{ and } \varepsilon_k > 0 \, (0) , \\ l_k - 1 & \text{if } p_k = 2l_k - 1 \text{ and } \varepsilon_k < 0 \, (0) . \end{cases} $$

The selection of index pairs $(k, h)$ such that $\lambda_k \neq 0$ remains as before. The changes in the proofs of the completeness theorems are very obvious. To prove Theorem 1 it is necessary to observe two facts in addition: 1) for $\lambda_m = 0$ the relations (8) take the form $(G y_k^h, z_{m+1}^{s+1}) = -\delta_{k,m} \delta_{h,p_m-s}$, and 2) the inequalities (7) remain in force. Indeed, if $\mathcal{P}$ is the orthogonal projection of $\mathcal{H}_2$ onto the linear span of the elements $\{ y_k^h \}$, $(k, h) \in \Lambda^+$, then the domains of the operators $G \mathcal{P}$ and $\mathcal{H} \mathcal{P}$ coincide. Therefore, the operator $(G \mathcal{P})^{-1} \mathcal{H} \mathcal{P}$, which is $\mathcal{P} G \mathcal{P}$-dissipative, is defined in $\mathcal{H}_2$, and (7) follows from Proposition 10\(^2\) in [7].

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\(^{(1)}\) Under our weaker assumptions this estimate must be carried out more subtly than in [2].


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