MATHEMATICS OF FINANCE AND INVESTMENT

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G.I.Falin. Mathematics of finance and investment.

This text is written for students of Moscow State University, studying actuarial science. It is based on syllabus of The Institute and Faculty of Actuaries for the subject CT1 (Financial Mathematics) of the Actuarial Profession. The present (first draft) version of the text covers units 1-8, 11,12, 14 (of the total 14 units). A distinguished feature of our text is that many theoretical concepts are introduced through detailed solutions of problems. Besides we show how standard financial functions of Microsoft Excel can be used to solve these problems. The problems are taken from the past exams of The Institute and Faculty of Actuaries and The Society of Actuaries (USA).

Contents

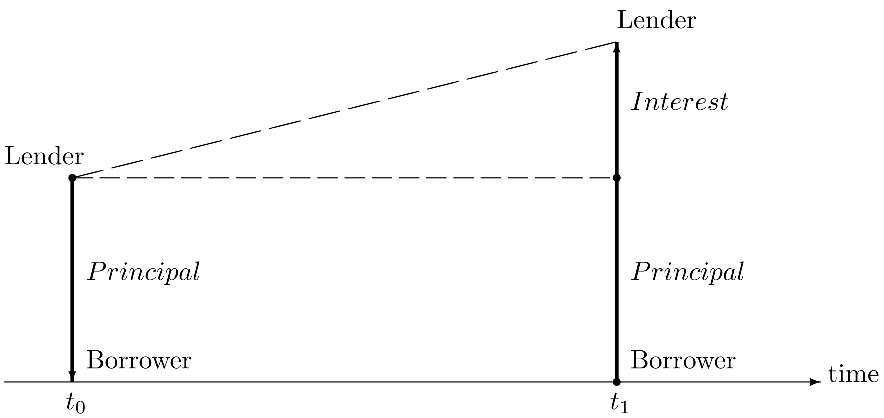
Chapter 1

Interest

1.1 The effective rate of interest

The notion of interest arises in the following simple situation (see figure 1.1).

Figure 1.1:



Assume that at time t_0 a person, organization (*the lender*) gives some amount of money P (we will refer to this as a *principal*) to other person, organization (*the borrower*) and both parties agree that the loan must be paid off in some period of time h, i.e. at time $t_1 = t_0 + h$. In other applications we may say that a customer deposits some money into his savings account, or an investor invests his capital/buys some securities, etc.

It is generally accepted that the lender should receive an amount A (*accumulation*) which is greater than P. The additional amount $I = A - P$ (*the interest*) is a reward to the lender for the use of his capital. The addition of interest to the principal is called *compounding*.

The interest is usually expressed in relative terms as a ratio $i = \frac{I}{P}$. This ratio is said to be *the rate of interest* or *the effective rate of interest* to distinguish from *the nominal rates of interest* which will be introduced later. Sometimes the rate of interest is called *the rate of return*.

In finance rates of interest are usually expressed as a percentage rather than a common or decimal fraction. However, to perform calculations it is necessary to transform a percentage in a fraction.

Since the accumulation A can be any positive number, greater than the principle P, the rate of interest can take any value $i > 0$. It is possible that the accumulation equals the principle. In this case the interest $I = A - P$ equals 0. Correspondingly, the rate of interest is $i = \frac{I}{P} = 0$. Moreover, in many cases financial losses are possible. It means, that "the accumulation" is less than the principle. Correspondingly the rate of interest $i = \frac{A-P}{P}$ is negative. However, since $A \geq 0$ this number is always greater than (or equal to) $\frac{-P}{P} = -1$. The case $i = -1$ means that the accumulation is 0, i.e. the principle is lost. If we assume that "the accumulation" can be negative (which means that finally, as the result of the transaction, the lender owes money to the borrower), then the rate of interest can take the value less than -1. However, sensible theory can be developed only if will consider the rates of interest i greater than -1.

The definition of the interest rate can be rewritten in the following form:

$$I = iP,$$

so that the total amount to be paid to the lender is

$$A = P + I = P + iP = P \cdot (1 + i). \tag{1.1}$$

The coefficient $k = 1 + i$ is called *accumulation factor*.

To draw the importance of the interval $[t_0, t_1]$, the rate of interest is said to be the effective rate of interest for this interval.

Usually 1 year is considered as the basic unit of time and correspondingly usually annual interest rates are used to describe profitability of financial operations.

Amounts of money are measured by integer numbers of pounds and pence (or dollars and cents, etc.) But even if the amount P is expressed by an integer number and the rate of interest i by an integer number or by a decimal fraction, the accumulation A calculated with the help of equation (1.1) is not necessary an integer number. Say, if $P = £147$, $i = 3.5\%$, then $A = £152.145$, i.e. 152 pounds, 14 pence and a half of a penny (which does not exist). In more complex financial calculation the amounts can be irrational real numbers. In such cases it is generally accepted in commercial practice that the amount is rounded down to integer number of pence. In theoretical considerations it is convenient to assume that amounts of money can take any real value.

The rates of interest which are used in majority of actuarial calculations in insurance, are determined based on conservative assumptions about profitability of future investments of the insurance company. These rates are considerably lower real interest rates offered by financial market. The function of the interest rates in actuarial calculations is to take into account the time value of money which are paid as a price for insurance cover. To draw this fact, in the actuarial mathematics the rate of interest used in actuarial calculations is often said to be *technical* or *actuarial* rate of interest. As a matter of fact insurance companies earn much higher interest; moreover it is one of the most important sources of income for them.

1.2 The real rate of interest

If inflation in economy should be taken into account, then the interest rate is said to be the money (or, sometimes, nominal) interest rate and inflation adjusted interest rate is said to be the real interest rate.

Let (real or projected) rate of inflation over the year is f. Then amount of money $A(1+f)$ at the end of the year has the same purchase power as amount A in the beginning of the year, or, to put this in another words, the amount $A' = A(1+f)$ at the end of the year and amount $A = \frac{A'}{1+f}$ in the beginning of the year are equivalent.

If i is the effective rate of interest for one year deposit of P, then the nominal (i.e. money) accumulation is $A = P(1+i)$. But this accumulation is due at the end of the year. If it is measured in money at the beginning of the year, then this amount is equivalent to $\frac{A}{1+f}$, i.e. the inflation adjusted interest is $\frac{A}{1+f} - P = \frac{P(i-f)}{1+f}$. Correspondingly, the inflation adjusted rate of interest is $\frac{i-f}{1+f}$. Since f usually is relatively small, $\frac{i-f}{1+f} \approx i - f$, i.e. to calculate the real rate of interest one should subtract the rate of inflation from the effective rate of interest.

1.3 The simple and compound interest

Assume that the principal P can be invested into two successive intervals: $[t_0, t_1]$ and $[t_1, t_2]$; let i_1 and i_2 be the effective rates of interest for these intervals and $k_1 = 1 + i_1$, $k_2 = 1 + i_2$ the corresponding accumulation factors.

There are two principles to calculate the total interest (or equivalently, the accumulation) over the joint interval $[t_0, t_2]$.

The principle of simple interest

According to this principle, only the principal earns the interest. Thus, the total interest is $I_1 + I_2 = Pi_1 + Pi_2$ and the accumulation is $A = P + Pi_1 + Pi_2$.

Correspondingly, the total rate of interest is the sum i_1+i_2 and the accumulation factor is $k = k_1 + k_2 - 1$.

If the principal P is invested **under simple interest** into n successive intervals:

$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$$

and $i_1, i_2, \dots, i_n$ be the effective rates of interest for these intervals, then the amount to be paid off at time t_n is

$$A = P + Pi_1 + Pi_2 + \cdots + Pi_n. \tag{1.2}$$

This amount is the accumulation at time t_n of an investment of the principal P at time t_0 under the simple interest.

If all intervals

$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$$

have the same length (without loss of generality we can assume that the common length is 1) and the effective rates of interests over these intervals are identical ($i_1 = i_2 = \cdots = i_n \equiv i$), then equation (1.2) becomes:

$$A = P + Pin. \tag{1.3}$$

Equation (1.5) is used even if n is not integer.

The principle of compound interest

According to this principle, the interest earned during the first period of time is added to the principal and thus can earn further interest during the second period of time. Thus, by the end of the second interval the accumulation is $A = P(1+i_1)(1+i_2)$. Correspondingly, the interest is $A - P = P(i_1 + i_2 + i_1 i_2)$, so that the effective rate of interest for the period $[t_0, t_2]$ is $i = i_1 + i_2 + i_1 i_2$ and the accumulation factor is $k = k_1 k_2$.

If the principal P is invested **under compound interest** into n successive intervals:

$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$$

and $i_1, i_2, \dots, i_n$ be the effective rates of interest for these intervals, then the amount to be paid off at time t_n is

$$A = P(1 + i_1)(1 + i_2)\dots(1 + i_n). \tag{1.4}$$

This amount is the accumulation at time t_n of an investment of the principal P at time t_0.

If $i_k > 0$ then during the interval $[t_{k-1}, t_k]$ the accumulation grows, if $i_k = 0$ then during the interval $[t_{k-1}, t_k]$ the accumulation does not change and if $-1 < i_k < 0$ then during the interval $[t_{k-1}, t_k]$ the accumulation decreases.

If all intervals

$$[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$$

have the same length (without loss of generality we can assume that the common length is 1) and the effective rates of interests over these intervals are identical ($i_1 = i_2 = \cdots = i_n \equiv i$), then equation (1.4) becomes:

$$A = P(1+i)^n \equiv Pk^n, \tag{1.5}$$

where $k = 1 + i$ is the accumulation factor.

It can be shown that equation (1.5) must be used even if n is not integer.

In actuarial calculations it is always assumed that the time value of money is estimated with the help of equation (1.5) (which expresses the principle of compound interest) even if n is not integer. So we will do later on unless otherwise stated.

1.4 The force of interest

Consider again equation (1.4). It yields that to calculate the accumulation over the period $[t_0, t_n]$ it is necessary to perform a multiplication. It is not a problem at all even with the help a simple calculator. Modern software, say Microsoft Excel, further simplifies many financial calculations. To calculate the accumulation given by equation (1.4) with the help of Microsoft Excel one can use the function **=FVSCHEDULE**. For example, if the principal $P = 180$ is invested for 3 years and the rates of interest are 3%, 5%, 2% correspondingly, then to find the accumulation $A = P(1+i_1)(1+i_2)(1+i_3) = 180 \cdot 1.03 \cdot 1.05 \cdot 1.02$ it is necessary to enter in a cell, say A1, the formula **=FVSCHEDULE(180,{0.03,0.05,0.02})** and then press **Enter**. Immediately after this in the cell A1 we will see the result: 198.5634.

As the equation (1.4) is very simple, the same result can be obtained with the help of the formula **=180*1.03*1.05*1.02**.

However without calculator calculation of the value

$$A = 180 \cdot 1.03 \cdot 1.05 \cdot 1.02 = 198.5634$$

a relatively difficult task. Now we cannot even imagine how to perform calculations without a calculator, but a few decades ago scientists, engineers and all the more students did not have these electronic instruments.

To simplify calculations which involves multiplications (as well as divisions, powers, roots) mathematicians developed (as early as in the beginning of 17th century) the method of logarithms, tables of logarithms and shortly after that the slide rule, which as a matter of fact is a mechanical analog computer. The tables of logarithms and slide rules were widely used till the mid of 1970's. Even modern textbooks on actuarial and financial mathematics contain tables for values of various widely used expressions, such as $\frac{1-(1+i)^{-n}}{i}$.

To calculate the accumulation, given by equation (1.4), with the help of the slide rule (or the table of logarithms) it is necessary to rewrite it as follows:

$$\ln A = \ln P + \ln(1+i_1) + \ln(1+i_2) + \cdots + \ln(1+i_n), \tag{1.6}$$

then calculate all logarithms in the right-hand side and sum them. This will give us a value of $\ln A$: $\ln A = V$. And finally, again with the help of either the slide rule or the table of logarithms it is necessary to find A.

To simplify (at least partially) this process it is more convenient to use quantities $\delta_k = \ln(1 + i_k)$ rather than the interest rates i_k, $k = 1, 2, \ldots, n$.

Then (1.4) becomes:

$$A = Pe^{\delta_1+\delta_2+\cdots+\delta_n}. \tag{1.7}$$

Since the effective rate of interest can be expressed through the parameter δ as $i = e^{\delta} - 1$, the parameter δ is just another (convenient) way to describe the rate of interest. However, its real function reveals in continuous models of financial mathematics.

Continuous models of financial mathematics

Let the effective rate of interest is constant. Consider the accumulation over period $[0, t]$, $A = P(1+i)^t$, as a function of time, *assuming that time is continuous*, i.e. can take *any* real value. Then the rate of the growth of the accumulation is the derivative $A'(t) = P(1 + i)^t \ln(1 + i)$ and the relative rate of the growth of the accumulation is

$$\frac{A'(t)}{A(t)} = \frac{P(1 + i)^t \ln(1 + i)}{P(1 + i)^t} = \ln(1 + i) = \delta.$$

Taking this into account the quantity $\delta = \ln(1+i)$ is called *the force of interest.*

Now assume that in some investment project with the only initial investment of the principle P at time $t_0 = 0$ we know the dependence of the accumulation on time, i.e. we know the function $A(t)$ (which need not have the form $A(t) = (1 + i)^t$). Then, *by definition* the instantaneous force of interest at time t, $\delta(t)$, is $\frac{A'(t)}{A(t)}$:

$$\delta(t) = \frac{A'(t)}{A(t)}. \tag{1.8}$$

In continuous models of financial mathematics it is usually assumed that the force of interest $\delta(t)$ is given. In this case the definition (1.8) can be thought as a differential equation for an unknown function $A(t)$. This equation can be easily solved; since $\frac{A'(t)}{A(t)} = (\ln A(t))'$, equation (1.8) can be rewritten as

$$\delta(t) = (\ln A(t))',$$

so that

$$\ln A(t) = \ln A(0) + \int_0^t \delta(u)du \Leftrightarrow A(t) = A(0) \exp\left(\int_0^t \delta(u)du\right).$$

Since $A(0) = P$, we finally have:

$$A(t) = P \exp\left(\int_0^t \delta(u)du\right). \tag{1.9}$$

1.5 Nominal rates of interest

As we noted, usually in financial calculations 1 year is considered as the basic unit of time. Typical value of the annual interest rate say for a 1-year fixed term bank deposit could be a few percent.

However quite often investors deal with investment projects shorter than 1 year, say a lender can lend some amount to a borrower for 1 month only. In this case the effective interest rate i_* is relatively small, say 0.2%, which seems not too attractive.

To present the effective interest rate i_* for a short period of time of length h in a more attractive way and simplify comparison of different investment projects, in financial mathematics the profitability of an investment for a short period of time of length h is usually described with the help of the scaled effective rate of interest

$$i_{(h)} = \frac{i_*}{h}. \tag{1.10}$$

This rate is said to be *the nominal rate of interest*. The adjective "nominal" means that this rate is just a "name" and does not exists in reality (as opposite to the effective rate of interest i_* which represents real income from investment).

If the basic unit of time is one year then the following values of h are of special interest: $h = \frac{1}{2}$ – half year, $h = \frac{1}{4}$ – one quarter, $h = \frac{1}{12}$ – one month, $h = \frac{1}{52}$ – one week, $h = \frac{1}{365}$ – one day. For $h = \frac{1}{p}$ the nominal rate of interest is denoted as $i^{(p)}$ and is said to be *the nominal rate of interest payable (convertible) pthly*:

$$i^{(p)} = pi_*. \tag{1.11}$$

The effective rate of interest for this period is $i_*^{(p)} = \frac{i^{(p)}}{p}$.

It should be noted that in the modern financial mathematics the profitability of an investment project is described with the help of so called internal rate of return (IRR). To define IRR, consider an auxiliary savings account with effective annual rate of return i and assume that the deposit grows according to the formula of compound interest, i.e. an investment of P for a period of length t gives the accumulation $A = P(1+i)^t$. Then IRR is defined as such annual effective rate of interest for this auxiliary savings account that the accumulation at the account at the end of the period under consideration is identical to the income from the investment project.

If i_* is the effective interest rate for a short period of time of length h, then $i = IRR$ is calculated with the help of equation

$$(1+i)^h = 1 + i_*, \tag{1.12}$$

i.e.

$$i = (1+i_*)^{\frac{1}{h}} - 1. \tag{1.13}$$

In particular, if $h = \frac{1}{p}$, then $i = IRR$ is given by the formula:

$$i = \left(1 + i_*^{(p)}\right)^p - 1.$$

Like the nominal rate of interest, $i = IRR$ is an "equivalent" annual rate of interest, but equivalent in the sense of compound interest rather than simple interest (as is the case for the nominal rate of interest).

Equation (1.12) can be rewritten in an equivalent form as

$$\ln(1+i) = \frac{\ln(1+i_*)}{h},$$

or introducing the corresponding forces of interest $\delta = \ln(1+i)$, $\delta = \ln(1+i_*)$, as

$$\delta = \frac{\delta_*}{h}.$$

Thus the use of IRR as an "equivalent" annual rate of interest means normalizing of the effective force of interest, whereas the use of the conventional nominal rate of interest $i = \frac{i_*}{h}$ means normalizing of the effective rate of interest.

Eliminating i_* from equations (1.10) and (1.13)we get a relation connecting the nominal rate of interest and $i = IRR$:

$$i = \left(1 + hi_{(h)}\right)^{\frac{1}{h}} - 1 \Leftrightarrow i_{(h)} = \frac{(1+i)^h - 1}{h}.$$

In particular, if $h = \frac{1}{p}$ we have:

$$i = \left(1 + \frac{i^{(p)}}{p}\right)^p - 1 \Leftrightarrow i^{(p)} = p\left((1+i)^{\frac{1}{p}} - 1\right). \tag{1.14}$$

Thus, if we are given the nominal rate of interest (and the number of periods p) then we can calculate the corresponding IRR, and vice versa.

Microsoft Excel has two functions: `NOMINAL` and `EFFECT`, which allow to make the calculations easier. Say, if we know monthly nominal rate of interest $i^{(12)} = 5\%$ then IRR can be calculated if we enter in a cell the following formula: **=EFFECT(0.05,12)**, and if the annual effective rate of interest $IRR = 6\%$ then quarterly nominal rate of interest $i^{(4)}$ can be calculated if we enter in a cell the following formula: **=NOMINAL(0.06,4)**.

Now consider behavior of the nominal rate of interest $i^{(p)}$ as a function of p with fixed value of the equivalent annual rate of interest i (which is given by (1.14)).

Lets start with the following simple numerical example. Assume that the effective annual rate of interest i is 20%, so that the force of interest is $\delta = 18.23\%$. With the help of (1.14) we can easily construct the table 1.1 for the values of $i^{(p)}$ for $p = 2,\ 4,\ 12,\ 52,\ 365$ (these values of p correspond to a half-year, a quarter, a month, a week, a day):

It is easy to see that as the parameter p grows, the nominal rate of interest $i^{(p)}$ tends to the force of interest $\delta = 18.23\%$.

This result can be obtained in a general case if we calculate the limit

$$\begin{aligned}\lim_{p\to\infty} i^{(p)} &= \lim_{p\to\infty} p\cdot(e^{\delta/p} - 1) = \lim_{p\to\infty} p\cdot\left(1 + \frac{\delta}{p} + o\left(\frac{1}{p}\right) - 1\right) \\ &= \lim_{p\to\infty}(\delta + o(1)) = \delta, \end{aligned} \tag{1.15}$$

Table 1.1:

p	$i^{(p)}$
2	19.09%
4	18.65%
12	18.37%
52	18.26%
365	18.24%

Thus, if p is sufficiently large we can use an approximation

$$i^{(p)} \approx \delta. \tag{1.16}$$

The table 1.1 shows that the accuracy of this approximation is sufficiently good. Say, even for $p = 12$ the relative error is less than 1%.

This result shows that the continuous model discussed in section 1.4 can be used as an approximate way to describe the situation when the nominal rates of interest change sufficiently often, say daily.

The approximation (1.16) can be improved if we take more terms in expansion of the exponential function in the right hand side of (1.15). Say if we add the term $\frac{1}{p^2}$, we get:

$$i^{(p)} = p \cdot \left(1 + \frac{\delta}{p} + \frac{\delta^2}{2p^2} + o\left(\frac{1}{p^2}\right) - 1\right) = \delta + \frac{\delta^2}{2p} + o\left(\frac{1}{p}\right). \tag{1.17}$$

Thus we can use the following approximation:

$$i^{(p)} \approx \delta + \frac{\delta^2}{2p}. \tag{1.18}$$

Table 1.2 contains exact values of the nominal rate of interest $i^{(p)}$ and the approximate values calculated with the help of (1.18) in the case $i = 20\%$. It is easy to see that the accuracy is very high. The approximate values are

Table 1.2:

p	$i^{(p)}$ Exact	$i^{(p)}$ Approximate
1	20%	19.89%
2	19.09%	19.06%
4	18.65%	18.65%
12	18.37%	18.37%

practically identical to the exact.

Chapter 2

Time value of money

2.1 Present values

Opportunity to invest money to get some return means that the value of money changes in the course of time. For example, if the effective annual rate of return is $i = 25\%$, then amount $C = 500$ now (at time $t_0 = 0$) becomes $500(1+i) = 625$ in one year (at time $t_1 = 1$). On the other hand amount $C = 500$ now can be obtained investing $500(1+i)^{-1} = 400$ a year ago (at time $t_2 = -1$). To put this in other words, if, for example, someone must pay us at time $t_0 = 0$ amount 500, then we can agree to get amount 400 at time $t = -1$ (taking the trouble to invest this amount we will get at time $t_0 = 0$ amount $400 \cdot 1.25 = 500$). However, at time $t = 1$ we have to require amount 625 (if at time $t_0 = 0$ we got 500, then investing this amount, at time $t = 1$ we would have 625).

Thus, amounts

- 400 at time $t = -1$
- 500 at time $t = 0$
- 625 at time $t = +1$

as a matter of fact are equivalent (under given rate of return $i = 25\%$). This result means that the value of money changes in the course of time.

Similar considerations show in a general case, for any $t > 0$, the value at time $t_0 = 0$ of amount C due at time $t > 0$ is $C' = C \cdot (1+i)^{-t}$: if we invest this amount at time t_0 then at time $t > 0$ we will get amount $C' \cdot (1+i)^t = C \cdot (1+i)^{-t} \cdot (1+i)^t = C$.

If $t < 0$, then value at time $t_0 = 0$ of amount C due at time t is accumulation over the period of length $-t$, i.e. $C \cdot (1+i)^{-t}$.

Thus, for any value of the sign of t the value (at time $t_0 = 0$) of amount C due at time t is

$$P(t) = C \cdot (1+i)^{-t}. \tag{2.1}$$

The amount $P(t)$ is said to be *the present value* (or *the discounted value*) of amount C due at time t.

The present value of the unit sum, i.e. $C = 1$, is denoted as $v(t)$ and is said to be *the discount function*:

$$v(t) = (1 + i)^{-t}. \tag{2.2}$$

The variable $v = (1 + i)^{-1}$ is said to be *the discount factor*. With its help we can rewrite (2.1) in the form

$$P(t) = Cv^t. \tag{2.3}$$

The discount factor can be expressed in terms of the force of interest $\delta = \ln(1+i)$ as

$$v = e^{-\delta},$$

so that the discount function is

$$v(t) = e^{-\delta t}$$

and the present value $P(t)$ is

$$P(t) = Ce^{-\delta t}.$$

Since the origin of time can be take arbitrary, the value C_1 at time t_1 of amount C_2 due at time t_2 is given by formula: $C_1 = C_2 v^{t_2 - t_1}$. This yields that $C_1 v^{t_1} = C_2 v^{t_2}$ – this equation expresses equal value of both sums at time $t_0 = 0$.

If the rate of interest used to evaluate the present value is assumed to be i_1 for the first period of time $(0, 1)$, i_2 for the second period of time $(1, 2)$, and so on, then the above arguments yield that the present value (at time $t_0 = 0$) of amount C due at time $t = n$ is

$$P(t) = C(1 + i_1)^{-1}(1 + i_2)^{-1} \dots (1 + i_n)^{-1}. \tag{2.4}$$

Correspondingly, the present value of the unit sum, i.e. *the discount function* is

$$v(n) = (1 + i_1)^{-1}(1 + i_2)^{-1} \dots (1 + i_n)^{-1}. \tag{2.5}$$

Thus *the discount factor* varies and for the period $(k - 1, k)$ is $v_k = (1 + i_k)^{-1}$. With its help we can rewrite (2.4) in the form

$$P(n) = Cv_1 v_2 \dots v_n. \tag{2.6}$$

The discount factor v_k can be expressed in terms of the force of interest $\delta_k = \ln(1 + i_k)$ as

$$v_k = e^{-\delta_k},$$

so that the discount function is

$$v(n) = e^{-(\delta_1 + \delta_2 + \dots + \delta_n)}$$

and the present value $P(n)$ is

$$P(n) = Ce^{-(\delta_1+\delta_2+\cdots+\delta_n)}.$$

If time value of money is described with the help of the continuous model discussed in section 1.4 (with the force of interest $\delta(t)$ as the main characteristics), then the above arguments and equation (1.9) yield that the present value (at time $t_0 = 0$) of amount C due at time t is

$$P(t) = C\exp\left(-\int_0^t \delta(u)du\right). \tag{2.7}$$

Correspondingly, the present value of the unit sum, i.e. *the discount function* is

$$v(t) = \exp\left(-\int_0^t \delta(u)du\right). \tag{2.8}$$

2.2 The rate of discount

Assume that at time $t_0 = 0$ a lender lends amount C for one year. Then at time $t = 1$ a borrower have to return amount $C(1 + i) = C + Ci$, which consists of the principal C and the interest $I = Ci$.

The interest Ci (due at time $t = 1$), at time $t_0 = 0$ has the discounted value $Ci(1 + i)^{-1}$. Since amounts $Ci(1 + i)^{-1}$ (at time $t = 0$) and Ci (at time $t = 1$) are equivalent, both parties could agree that the interest is to be paid off in advance, i.e. at time $t_0 = 0$ of the deal. This discounted interest is a quota dC, where $d = \frac{i}{1+i}$, of the loan amount C. The value d is said to be *the effective rate of discount*.

The effective rate of discount d can be expressed in terms of both the force of interest $\delta = \ln(1 + i)$ and the discount factor $v = \frac{1}{1+i}$:

$$d = 1 - v = 1 - e^{-\delta}. \tag{2.9}$$

2.3 The nominal rate of discount

Now assume that a loan of amount C is made at time $t = 0$ at the annual rate of interest i for the period of length $\frac{1}{p}$ with the interest payable in advance. As we saw in section 1.5, the effective rate of interest is $i_*^{(p)} = \frac{i^{(p)}}{p} = (1 + i)^{\frac{1}{p}} - 1$. Thus the interest due at time $t = \frac{1}{p}$ is $Ci_*^{(p)}$. According to (2.1), the present value of this amount at time $t_0 = 0$ is $Ci_*^{(p)} \cdot (1 + i)^{-\frac{1}{p}} = C\left(1 - (1 + i)^{-\frac{1}{p}}\right)$. Thus the discounted interest is a quota $d_*^{(p)}C$, where $d_*^{(p)} = 1 - (1 + i)^{-\frac{1}{p}}$, of the loan amount C. The value $d_*^{(p)}$ is said to be *the effective rate of discount* per period $\frac{1}{p}$.

Since $i = \frac{d}{1-d}$, the effective rate of discount per period $\frac{1}{p}$ can be expressed as

$$d_*^{(p)} = 1 - (1-d)^{\frac{1}{p}}. \tag{2.10}$$

Typically $d_*^{(p)}$ is very low quantity. Taking this into account and simplify comparison of different investment projects, we introduce the scaled effective discount rate

$$d^{(p)} = p \cdot d_*^{(p)}. \tag{2.11}$$

This rate is said to be *the nominal rate of discount convertible pthly.*

From (2.10) we can express the nominal rate of discount in terms of the effective rate of discount d:

$$d^{(p)} = p\left(1 - (1-d)^{\frac{1}{p}}\right). \tag{2.12}$$

Similarly, we can express the nominal rate of discount in terms of the discount factor v, the force of interest δ and the effective rate of interest i:

$$d^{(p)} = p\left(1 - v^{\frac{1}{p}}\right) = p\left(1 - e^{-\frac{\delta}{p}}\right) = p\left(1 - (1+i)^{-\frac{1}{p}}\right). \tag{2.13}$$

It should be noted that if (on the analogy with the basic definition of the discount rate) we define the effective rate of discount for the period $\frac{1}{p}$ as $\frac{i_*^{(p)}}{1+i_*^{(p)}}$, then

$$\begin{aligned} & \frac{i_*^{(p)}}{1+i_*^{(p)}} = \frac{i^{(p)}}{p+i^{(p)}} = \frac{p((1+i)^{1/p}-1)}{p+p((1+i)^{1/p}-1)} \\ = & \frac{(1+i)^{1/p}-1}{(1+i)^{1/p}} = 1-(1+i)^{-1/p} = \frac{d^{(p)}}{p} = d_*^{(p)}. \end{aligned}$$

Now consider behavior of the nominal rate of discount $d^{(p)}$ as a function of p with fixed value of the equivalent annual rate of interest i (which is given by (2.13)).

Lets start with the following simple numerical example. Assume that the effective annual rate of interest i is 20%, so that the force of interest is $\delta = 18.23\%$. With the help of (2.13) we can easily construct the table 2.1 for the values of $d^{(p)}$ for $p = 2,\ 4,\ 12,\ 52,\ 365$ (these values of p correspond to a half-year, a quarter, a month, a week, a day).

It is easy to see that as the parameter p grows, the nominal rate of discount $d^{(p)}$ tends to the force of interest $\delta = 18.23\%$.

This result can be obtained in a general case if we calculate the limit

$$\begin{aligned} \lim_{p\to\infty} d^{(p)} &= \lim_{p\to\infty} p\cdot(1-e^{-\delta/p}) = \lim_{p\to\infty} p\cdot\left(1-1+\frac{\delta}{p}+o\left(\frac{1}{p}\right)\right) \\ &= \lim_{p\to\infty}(\delta+o(1)) = \delta, \end{aligned} \tag{2.14}$$

Table 2.1:

p	$d^{(p)}$
2	17.43%
4	17.82%
12	18.09%
52	18.20%
365	18.23%

Thus, if p is sufficiently large we can use an approximation

$$d^{(p)} \approx \delta. \tag{2.15}$$

The table 2.1 shows that the accuracy of this approximation is sufficiently good. Say, even for $p = 12$ the relative error is less than 1%.

The approximation (2.15) can be improved if we take more terms in expansion of the exponential function in the right hand side of (2.14). Say if we add the term $\frac{1}{p^2}$, we get:

$$d^{(p)} = p \cdot \left(1 - 1 + \frac{\delta}{p} - \frac{\delta^2}{2p^2} + o\left(\frac{1}{p^2}\right)\right) = \delta - \frac{\delta^2}{2p} + o\left(\frac{1}{p}\right). \tag{2.16}$$

Thus we can use the following approximation:

$$d^{(p)} \approx \delta - \frac{\delta^2}{2p}. \tag{2.17}$$

Table 2.2 contains exact values of the nominal rate of discount $d^{(p)}$ and the approximate values calculated with the help of (2.17) in the case $i = 20\%$. It is easy to see that the accuracy is very high. The approximate values are

Table 2.2:

p	$d^{(p)}$ Exact	$d^{(p)}$ Approximate
1	16.67%	16.57%
2	17.43%	17.40%
4	17.82%	17.82%
12	18.09%	18.09%

practically identical to the exact.

2.4 Valuing cash flows

Assume that a borrower must pay to a lender amount $p_1 = 400$ at time $t_1 = 1$ and amount $p_2 = 600$ at time $t_2 = 2$. Assume further that the borrower would like to repay his debts now (at time $t_0 = 0$) and the lender agrees with this. How much should the borrower pay?

If we calculate this amount as a simple algebraic sum $p = p_1 + p_2 = 1000$, then instead of to pay the debts immediately the borrower could deposit this amount to a bank account which pays interest at some annual rate of interest i, for simplicity of calculations assume that $i = 25\% = \frac{1}{4}$ and thus the accumulation factor $1 + i = \frac{5}{4}$. At time $t_1 = 1$ this initial investment accumulates to $1000 \cdot \frac{5}{4} = 1250$. Then (according to the original agreement) at time t_1 the borrower pays to the lender amount 400 and the rest, i.e. $1250 - 400 = 850$ invests for another year. At time $t_2 = 1$ this amount accumulates to $850 \cdot \frac{5}{4} = 1062.50$. Then (according to the original agreement) at time $t_2 = 2$ the borrower pays to the lender amount 600 and the rest, i.e. $1062.50 - 600 = 462.50$ is his pure risk-free income. Thus returning at time $t_0 = 0$ a simple algebraic sum $p = p_1 + p_2 = 1000$ of his debts the borrower overpays to the lender.

Taking these considerations into account, lets try to calculate the "fair" amount p the borrower should pay now.

As in the above example assume that sum p is invested at the annual rate of interest i.

At time $t_1 = 1$ this initial investment accumulates to $p(1+i)^{t_1}$. Then (according to the original agreement) at time t_1 the lender receives amount p_1 and the rest, i.e. $p(1+i)^{t_1} - p_1$ is invested for another year. At time t_2 this amount accumulates to $(p(1+i)^{t_1} - p_1)(1+i)^{t_2 - t_1}$. Then (according to the original agreement) at time t_2 the lender receives amount p_2, so that the rest is $b = (p(1+i)^{t_1} - p_1)(1+i)^{t_2-t_1} - p_2$.

If $b > 0$, then either the borrower or the lender (depending on which of them will invest amount p) gets a risk-free income from the described deal.

If $b < 0$, i.e. $p(1+i)^{t_2} < p_1(1+i)^{t_2-t_1} + p_2$, the lender should prefer to get amounts p_1 and p_2 according to the original schedule, i.e. at times t_1 and t_2 accordingly. In this case at time t_2 he would have accumulation $p_1(1+i)^{t_2-t_1} + p_2$, which is greater than the total accumulation $p(1+i)^{t_2}$ by time t_2 from investment of p at time $t_0 = 0$. Thus in the case $b < 0$ the requirement to pay amount p at time $t_0 = 0$ is not fair with respect to the lender.

Therefore, the fair solution to the problem is given by equation $(p(1+i)^{t_1} - p_1)(1+i)^{t_2-t_1} - p_2 = 0$, which means that neither party can gain. From this we have:

$$p = p_1(1+i)^{-t_1} + p_2(1+i)^{-t_2}.$$

Thus the fair *present value* of the loan due at some times in future is algebraic sum of the amounts p_1 and p_2 due, but with coefficients: $(1+i)^{-t_1} = v^{t_1}$ and $(1+i)^{-t_2} = v^{t_2}$ accordingly. These coefficients are *discount factors*. Amount $p_1 v^{t_1}$ is *the present value* (at time $t_0 = 0$) of amount p_1 due at time t_1 and amount $p_2 v^{t_2}$ is *the present value* (at time $t_0 = 0$) of amount p_2 due at time t_2.

Returning to the specific example we started this section with, we have the following scheme to calculate the fair amount to be paid at time $t_0 = 0$.

- calculate the discount factor $v = (1+i)^{-1} = \frac{4}{5}$;

- calculate the present values of amounts $p_1 = 400$ (due at time $t_1 = 1$) – it is $400 \cdot \frac{4}{5} = 320$, and $p_2 = 600$ (due at time $t_1 = 2$) – it is $400 \cdot \left(\frac{4}{5}\right)^2 = 384$;
- calculate the present value of the debts as the sum of the present values: $320 + 384 = 704$ – this is the amount to be returned at time t_0.

This simple example shows that *we can add (subtract, compare and perform any other operations) sums of money only if all these amounts are considered at the same epoch of time.*

Microsoft Excel has a function `NPV` which allows to calculate the present value p of a series of future payments of amounts $p_1, p_2, \ldots, p_n$ at times $t_1 = 1, t_2 = 2, \ldots, t_n = n$.

The present value p of a series of n future payments of amounts $p_1, p_2, \ldots, p_n$ at times $t_1 = 1, t_2 = 2, \ldots, t_n = n$ can be calculated with the help of a standard function `NPV` of Microsoft Excel. This is done by the formula `=NPV(`i`,` $p_1, p_2, \ldots, p_n$`)`. Say in the above example we must use the formula: `=NPV(0.25, 400, 600)`. Alternatively we may enter the amounts $p_1 = 400$, $p_2 = 600$ in cells, say A1, A2 and use the formula: `=NPV(0.25, A1:A2)`.

2.5 The principle of equivalence

For applications to life insurance and pension schemes one of the most important is the following problem. Assume that a person wishes to buy now (at time $t_0 = 0$) a pension, which pays sums $b_1, b_2, \ldots, b_n$ at times $t_1, t_2, \ldots, t_n$ accordingly. How much is the price a of this cover now, if the pension fund invests the contribution a and guarantees return i per annum? Above-stated arguments shows that a must be equal to the present value of the cash flow of benefits:

$$a = b_1 v^{t_1} + b_2 v^{t_2} + \cdots + b_n v^{t_n}. \tag{2.18}$$

Indeed, if a is given by (2.18), then at time t_1 the pension fund will have amount

$$a(1+i)^{t_1} = av^{-t_1} = b_1 + b_2 v^{t_2 - t_1} + \cdots + b_n v^{t_n - t_1}.$$

This will allow at time t_1 pay the first benefit b_1.

The rest $a_1 = b_2 v^{t_2 - t_1} + \cdots + b_n v^{t_n - t_1}$ at time t_2, i.e. after time $t_2 - t_1$, accumulates to

$$a_1(1+i)^{t_2 - t_1} = a_1 v^{-t_2 + t_1} = b_2 + b_3 v^{t_3 - t_2} + \cdots + b_n v^{t_n - t_2}.$$

This will allow at time t_2 pay the second benefit b_2, an so on.

After payment of $(n-1)$th benefit at time t_{n-1} the fund will have amount $b_n v^{t_n - t_{n-1}}$. At time t_n, i.e. after time $t_n - t_{n-1}$, this amount will grow to b_n, which will allow to pay the final benefit b_n. This means that the formula (2.18) is fair with respect to pension fund (as it is able to make all payments only from the contribution a and subsequent accumulations). On the other hand after the final payment the rest of the money paid by the member of the pension scheme is 0, which means that the formula (2.18) is fair with respect to the person which bought the pension (as he did not overpay for the benefits).

The the right-hand side of (2.18) is the present value of all benefits and left-hand side of (2.18) can be thought of as the present value of contributions to the pension fund. Thus equation (2.18) expresses *the principle of equivalence of obligations of both parties.*

In the problem we have just considered, it is assumed that the member of the pension scheme pays the price a for the future benefits as a lump sum at the time when he buys the cover. However usually contributions to the pension scheme form a sequence of payments $c_1, \cdot, c_k$ at some agreed times $\tau_1, \cdot, \tau_k$.

The present value of all contributions to the pension fund is

$$a_C = c_1 v^{\tau_1} + ... + c_k v^{\tau_k},$$

and the present value of all benefits is

$$a_B = b_1 v^{t_1} + ... + b_n v^{t_n}.$$

Thus according to the principle of equivalence, fair relation between contributions c_i and benefits b_i is given by the formula:

$$c_1 v^{\tau_1} + ... + c_k v^{\tau_k} = b_1 v^{t_1} + ... + b_n v^{t_n}. \tag{2.19}$$

To proof (2.19) consider the sequence of times

$$T_1, T_2, \ldots, T_{n+k},$$

when either the member pays his contribution or the pension fund pays benefits. To put this in other words, join sequences

$$t_1, t_2, \ldots, t_n$$

and

$$\tau_1, \tau_2, \ldots, \tau_k. \tag{2.20}$$

Let u_i be the amount of payment made by the pension fund at time T_i. This means, that if $T_i = t_j$ when a benefit b_j is paid (for some j) then $u_i = b_j$. But if if $T_i = \tau_j$ when a contribution c_j is paid (for some j) then $u_i = -c_j$.

Then equation (2.21) can be rewritten as

$$u_1 v^{T_1} + ... + u_{n+k} v^{T_{n+k}} = 0. \tag{2.21}$$

This relation has the same structure as (2.18). It is easy to see that arguments used to prove relation (2.18) can be applied in the case when some of the values b_i are negative. In this case we consider them as contributions so that when the assets of the fund at time t_i changes from $b_i + b_{i+1} v^{t_{i+1} - t_i} + ...$ to $b_{i+1} v^{t_{i+1} - t_i} + ...$ it mean that the assets increases. Thus relation (2.21) states that the final balance is 0.

At some times T_i a negative balance is possible, which means that the member owes pension fund. During corresponding intervals the debt grows according to the formula of compound interest and later on this debt is paid by the the future contributions. It should be noted that if the benefits are due after all contributions have been paid this situation is not possible.

Chapter 3

Deterministic annuities

3.1 Level annuities

3.1.1 Definitions

Consider a sequence of n consecutive unit intervals

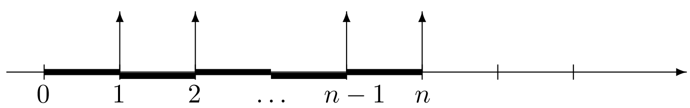
$$(0, 1), \dots, (n-1, n).$$

Usually, time $t_0 = 0$ means now and the unit of time is one year, but equally as the unit of time one quarter, months, etc. can be considered.

A series of n level payments, each of amount 1, made at the **end** of these intervals, i.e. at times $1, 2, \dots, n$, is said to be *immediate annuity* or *annuity payable in arrear.*

This cash flow is shown on figure 3.1.

Figure 3.1:



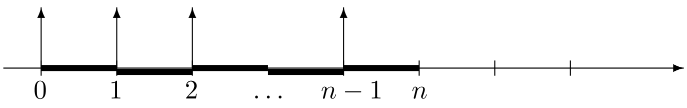
A series of n level payments, each of amount 1, made at the **beginning** of these intervals, i.e. at times $0, 1, \dots, n-1$, is said to be *annuity-due* or *annuity payable in advance.*

This cash flow is shown on figure 3.2.

3.1.2 The present values

The present value of the immediate annuity at time $t_0 = 0$ is denoted by a symbol $a_{\overline{n}|}$ and the present value of the annuity-due at time $t_0 = 0$ is denoted by a symbol $\ddot{a}_{\overline{n}|}$).

Figure 3.2:



To calculate these values it is necessary to reduce each of n payments to the present time $t_0 = 0$ and then sum the present values:

$$a_{\overline{n}|} = v + v^2 + ... + v^n, \tag{3.1}$$

$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1}. \tag{3.2}$$

Applying the formula for n terms of geometrical progression we have (if $v \neq 1$, i.e. $i \neq 0$):

$$a_{\overline{n}|} = \frac{v - v^{n+1}}{1-v} = \frac{v(1-v^n)}{1-v} = \frac{1-v^n}{1/v - 1} = \frac{1-v^n}{i}, \tag{3.3}$$

$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{1-v} = \frac{1-v^n}{d}. \tag{3.4}$$

In the trivial case $i = 0$, when money do not change the value in a course of time, obviously we have: $a_{\overline{n}|} = \ddot{a}_{\overline{n}|} = n$.

Besides it is convenient to define $a_{\overline{0}|}$ and $\ddot{a}_{\overline{0}|}$ as 0. These definitions coordinate with a common agreement that a sum which does not contain summands equals 0.

It is clear from figures 3.1 and 3.2 that the difference between the immediate annuity and annuity-due is connected with the choice of the origin of time: if time $t = 1$ is taken as the origin then immediate annuity can be thought of as an annuity-due. This observation can be expressed in algebraic form as follows:

$$a_{\overline{n}|} = v(1 + v + \cdots + v^{n-1}) = v\ddot{a}_{\overline{n}|}.$$

Relations (3.3) and (3.4) can be rewritten as follows:

$$ia_{\overline{n}|} + v^n = 1, \tag{3.5}$$

$$d\ddot{a}_{\overline{n}|} + v^n = 1. \tag{3.6}$$

Note that although (3.3) and (3.4) hold only for $i \neq 0$, relations (3.6) and (3.6) hold for $i = 0$ as well.

In the form (3.6) and (3.6) formulas for the present values of annuities can be proved with the help of the following "financial" arguments.

At time $t_0 = 0$ deposit amount $P = 1$ to a saving account which pays interest at the rate i per annum.

At time $t_1 = 1$ we will have the accumulation $P(1+i) = 1+i$, which can be divided into interest i and the principal. Withdraw the interest i and reinvest the principal $P = 1$.

At time $t_2 = 2$ we will have the accumulation $P(1+i) = 1+i$, which can be divided into interest i and the principal. Withdraw the interest i and reinvest the principal $P = 1$, and so on, till time t_n when we stop the process of reinvestment.

Thus investment of $P = 1$ at time $t_0 = 0$ produces:

- a series of n payments, each of amount i, at the end of every unit interval, i.e. immediate annuity; its value at time $t_0 = 0$ is $ia_{\overline{n}|}$.
- amount $P = 1$ at time $t_n = n$; its value at time $t_0 = 0$ is $1 \cdot v^n = v^n$.

By the principle of equivalence the value of the total at time $t_0 = 0$ of the income from the investment equals the amount invested. This equality is exactly relation (3.5).

To prove (3.6), at time $t_0 = 0$ deposit amount $P = 1$ to a saving account which pays interest at the rate i per annum.

As opposite to the previous case, withdraw the interest in advance, i.e. withdraw amount $iv = d$ at time $t_0 = 0$.

At time $t_1 = 1$ we will have the same principal $P = 1$. Reinvest it and withdraw the interest in advance, i.e. withdraw amount $iv = d$ at time $t_1 = 1$.

At time $t_2 = 2$ we will have the same principal $P = 1$. Reinvest it and withdraw the interest in advance, i.e. withdraw amount $iv = d$ at time $t_2 = 2$, and so on, till time t_n when we stop the process.

Thus investment of $P = 1$ at time $t_0 = 0$ produces:

- a series of n payments, each of amount d, at the beginning of every unit interval, i.e. annuity-due; its value at time $t_0 = 0$ is $d\ddot{a}_{\overline{n}|}$.
- amount $P = 1$ at time $t_n = n$; its value at time $t_0 = 0$ is $1 \cdot v^n = v^n$.

By the principle of equivalence the value of the total at time $t_0 = 0$ of the income from the investment equals the amount invested. This equality is exactly relation (3.6).

Microsoft Excel has a function `PV` which allows to calculate the present value of both the annuity-due and the immediate annuity. The present value of the immediate annuity with n level payments of p evaluated at the rate of interest i is calculated by the formula: `=-PV(`i`, `n`, `p`, 0)` and the present value of the annuity-due with n level payments of p evaluated at the rate of interest i is calculated by the formula: `=-PV(`i`, `n`, `p`, 1)` (note that the result of the use of the function PV is negative because it represents money that you would pay to get the annuity payments).

3.1.3 Recursive relations

Directly from (3.1) and (3.2) we have:

$$a_{\overline{n}|} = v + v(v + v^2 + \cdots + v^{n-1}) = v + va_{\overline{n-1}|}, \tag{3.7}$$
$$\ddot{a}_{\overline{n}|} = 1 + v + v^2 + ... + v^{n-1} = 1 + v\ddot{a}_{\overline{n-1}|}. \tag{3.8}$$

Starting with $a_{\overline{0}|} = 0$ and $\ddot{a}_{\overline{0}|} = 0$ these formulas allow to calculate recursively the present values $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$.

3.1.4 Accumulations

Sometimes the value of an annuity at the end $t = n$ of the final interval $(n-1, n)$ is of interest. This value can be thought of as the total amount accumulated on a bank account after a series of a regular deposits, each of amount 1. It is denoted similar to the present value, but the letter a is replaced by letter s.

Thus $s_{\overline{n}|}$ is the value of the immediate annuity at time $t_n = n$ when the final payment is made, and $\ddot{s}_{\overline{n}|}$ is the value of the immediate annuity at the same time $t_n = n$ (i.e. one unit of time after the final payment is made).

Formulas for the accumulations $s_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$ can be obtained if we calculate the value of each of n payments at time $t_n = n$ and then sum all these values. Applying the formula for n terms of geometrical progression we have (if $v \neq 1$, i.e. $i \neq 0$):

$$s_{\overline{n}|} = (1+i)^{n-1} + ... + 1 = \frac{(1+i)^n - 1}{i}, \tag{3.9}$$
$$\begin{aligned} \ddot{s}_{\overline{n}|} &= (1+i)^n + ... + (1+i) = \frac{(1+i)^{n+1} - (1+i)}{i} \\ &= \frac{(1+i)^n - 1}{i/(1+i)} = \frac{(1+i)^n - 1}{d}. \end{aligned} \tag{3.10}$$

In the trivial case $i = 0$, when money do not change the value in a course of time, obviously we have: $s_{\overline{n}|} = \ddot{s}_{\overline{n}|} = n$.

Formulas (3.9) and (3.11) can be obtained if we first calculate the present value of the corresponding annuity at time $t_0 = 0$ and then find the accumulation of the obtained value at time $t_n = n$:

$$s_{\overline{n}|} = a_{\overline{n}|}(1+i)^n = \frac{1-(1+i)^{-n}}{i}(1+i)^n = \frac{(1+i)^n - 1}{i},$$
$$\ddot{s}_{\overline{n}|} = \ddot{a}_{\overline{n}|}(1+i)^n = \frac{1-(1+i)^{-n}}{d}(1+i)^n = \frac{(1+i)^n - 1}{d}.$$

3.2 Deferred annuities

3.2.1 Definitions

For conventional immediate annuity and annuity-due payments starts at the first time interval $(0; 1)$ (in the end of the interval for immediate annuity and in the beginning of the interval for annuity-due). In some cases payments start later, i.e. are deferred. Corresponding cash flows are said to be *deferred annuities.*

To be more exact, consider n consecutive unit intervals

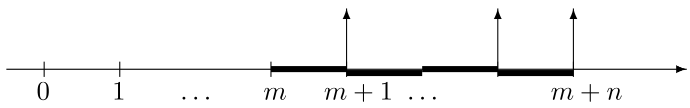
$$(m, m+1), \ldots, (m+n-1, m+n).$$

As usually, time $t_0 = 0$ means now, so that interval $(m, m+n)$ is deferred from the present time for m units of time.

A series of n level payments, each of amount 1, made at the **end** of these intervals, i.e. at times $m+1, m+2, \ldots, m+n$, is said to be *deferred immediate annuity* or *deferred annuity payable in arrear.*

This cash flow is shown on figure 3.3.

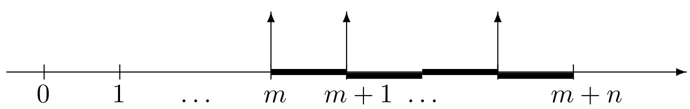
Figure 3.3:



A series of n level payments, each of amount 1, made at the **beginning** of these intervals, i.e. at times $m, m+1, \ldots, m+n-1$, is said to be *deferred annuity-due* or *deferred annuity payable in advance.*

This cash flow is shown on figure 3.4.

Figure 3.4:



3.2.2 The present values

The present value of the deferred immediate annuity at time $t_0 = 0$ is denoted by a symbol ${}_{m|}a_{\overline{n}|}$ and the present value of the deferred annuity-due at time $t_0 = 0$ is denoted by a symbol ${}_{m|}\ddot{a}_{\overline{n}|}$.

To calculate these values it is necessary to reduce each of n payments to the present time $t_0 = 0$ and then sum the present values:

$$ {}_{m|}a_{\overline{n}|} = v^{m+1} + ... + v^{m+n}, \tag{3.11}$$
$$ {}_{m|}\ddot{a}_{\overline{n}|} = v^{m} + ... + v^{m+n-1}. \tag{3.12}$$

Applying the formula for n terms of geometrical progression we have (if $v \neq 1$, i.e. $i \neq 0$):

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= \frac{v^{m+1} - v^{m+n+1}}{1-v} = v^{m+1}\frac{1-v^n}{1-v} \\ &= v^m \frac{1-v^n}{1/v-1} = v^m \frac{1-v^n}{i}, \end{aligned} \qquad (3.13)$$

$${}_{m|}\ddot{a}_{\overline{n}|} = \frac{v^m - v^{m+n}}{1-v} = v^m \frac{1-v^n}{d}. \qquad (3.14)$$

In the trivial case $i = 0$, when money do not change the value in a course of time, obviously we have: ${}_{m|}a_{\overline{n}|} ={}_{m|}\ddot{a}_{\overline{n}|} = n$.

Besides it is convenient to define ${}_{m|}a_{\overline{0}|}$ and ${}_{m|}\ddot{a}_{\overline{0}|}$ as 0. These definitions coordinate with a common agreement that sum which contains no summands equals 0.

3.2.3 Relation with conventional annuities

Comparing equations (3.3), (3.4) with equations (3.13), (3.14) we can express the present values of deferred annuities through the present values of conventional annuities (in the case $i \neq 0$):

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^m a_{\overline{n}|}, &(3.15)\\ {}_{m|}\ddot{a}_{\overline{n}|} &= v^m \ddot{a}_{\overline{n}|}. &(3.16) \end{aligned}$$

If $i = 0$, i.e. $v = 1$, these relations hold trivially.

Relations (3.15) (3.16) can be obtained directly from (3.11), (3.12); factoring out v^m, we get:

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= v^m(v + ... + v^n) = v^m a_{\overline{n}|}, \\ {}_{m|}\ddot{a}_{\overline{n}|} &= v^m(1 + ... + v^{n-1}) = v^m \cdot \ddot{a}_{\overline{n}|}. \end{aligned}$$

These relations can be also obtained with the help of the following reason. To find the present value (at time $t_0 = 0$) of the deferred annuity (either payable in advance or in arrear) we can

- Find the reduced value of this cash flow at time $t = m$. Since time $t = m$ is the beginning of the first interval from the list $(m, m+1), \dots, (m+n-1, m+n)$ with respect to the point $t = m$ the deferred annuity is the corresponding conventional annuity. Thus its value at time $t = m$ is $a_{\overline{n}|}$ or $\ddot{a}_{\overline{n}|}$) (depending on the case under consideration). This means that the deferred annuity can be replaced by the corresponding single payment at time $t = m$.

- Then find the present value of this payment at time $t_0 = 0$. According to equation (2.3), this is done by multiplication at the discount coefficient v^m (since the distance from $t_0 = 0$ to $t = m$ is m).

Besides, sums (3.11), (3.12) can be viewed as tails of sums (3.1), (3.2). Correspondingly, writing these sums as differences we can express the present value of deferred annuity as the difference of the present values of two conventional annuities:

$$\begin{aligned} {}_{m|}a_{\overline{n}|} &= (v + ... + v^m + v^{m+1} + ... + v^{m+n}) - (v + ... + v^m) \\ &= a_{\overline{m+n}|} - a_{\overline{m}|}, \qquad (3.17) \\ {}_{m|}\ddot{a}_{\overline{n}|} &= (1 + ... + v^{m-1} + v^m + ... + v^{m+n-1}) - (1 + ... + v^{m-1}) \\ &= \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}. \qquad (3.18) \end{aligned}$$

3.2.4 Accumulations

From the point of view of the final payment period deferred annuities and the corresponding conventional annuities cannot be distinguished one from the other. Thus, we need not any specific notations for the accumulations at the end of the final payment period for deferred annuities. To put this in other words, if we denote as ${}_{m|}s_{\overline{n}|}$ the value of the deferred immediate annuity at time $m + n$, then

$${}_{m|}s_{\overline{n}|} = s_{\overline{n}|}.$$

3.3 Increasing annuities

3.3.1 Definitions

Consider again sequence of n consecutive unit intervals

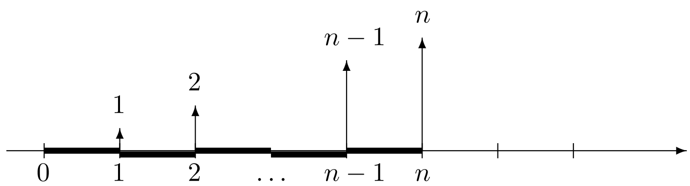
$$(0, 1), \dots, (n - 1, n).$$

As usual, $t_0 = 0$ means the present time.

A series of n payments $p_1 = 1, p_2 = 2, \dots, p_n = n$ made at the **end** of these intervals, i.e. at times $1, 2, \dots, n$, is called *increasing immediate annuity* or *increasing annuity payable in arrear.*

This cash flow is shown on figure 3.5.

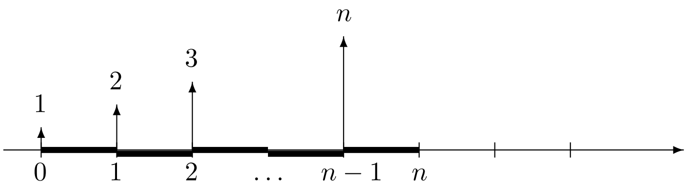
Figure 3.5:



A series of n payments $p_1 = 1, p_2 = 2, \dots, p_n = n$ made at the **beginning** of these intervals, i.e. at times $0, 1, \dots, n - 1$, is called *increasing annuity-due* or *increasing annuity payable in advance.*

This cash flow is shown on figure 3.6.

Figure 3.6:



3.3.2 The present values

The present value of the immediate increasing annuity at time $t_0 = 0$ is denoted by a symbol $(Ia)_{\overline{n}|}$ and the present value of the increasing annuity-due at time $t_0 = 0$ is denoted by a symbol $(I\ddot{a})_{\overline{n}|}$).

To calculate these values it is necessary to reduce each of n payments to the present time $t_0 = 0$ and then sum the present values:

$$(Ia)_{\overline{n}|} = v + 2v^2 + ... + nv^n, \tag{3.19}$$

$$(I\ddot{a})_{\overline{n}|} = 1 + 2v + 3v^2 + ... + nv^{n-1}. \tag{3.20}$$

The sum in the right-hand side of (3.20) is the derivative of the sum $v + v^2 + \cdots + v^n$ of n terms of geometrical progression with the common ratio v (in fact the later sum is the present value of the immediate annuity). If $v \neq 1$, then

$$v + v^2 + \cdots + v^n = \frac{v - v^{n+1}}{1 - v},$$

so that

$$(Ia)_{\overline{n}|} = \left(\frac{v - v^{n+1}}{1 - v}\right)'_v = \frac{1 - (n+1)v^n + nv^{n+1}}{(1-v)^2}. \tag{3.21}$$

The sum in the right-hand side of (3.19) can be written as

$$v(1 + 2v + ... + nv^{n-1}) = v(I\ddot{a})_{\overline{n}|},$$

so that applying (3.21) we get:

$$(I\ddot{a})_{\overline{n}|} = v\frac{1 - (n+1)v^n + nv^{n+1}}{(1-v)^2}. \tag{3.22}$$

In the trivial case $i = 0$, when money do not change the value in a course of time, obviously we have: $(Ia)_{\overline{n}|} = (I\ddot{a})_{\overline{n}|} = \frac{n(n+1)}{2}$.

Besides it is convenient to define $(Ia)_{\overline{0}|}$ and $(I\ddot{a})_{\overline{0}|}$ as 0.

3.3.3 Relation with the level annuities

Comparing equations (3.3), (3.4) with equations (3.21), (3.22), the present values of the increasing annuities can be expressed in terms of the present values of the corresponding level annuities (in the case $i \neq 0$[1]):

$$
\begin{aligned}
(Ia)_{\overline{n}|} &= \frac{1-v^n}{i(1-v)} - \frac{nv^n}{i} = \frac{a_{\overline{n}|}}{1-v} - \frac{nv^n}{i} = \frac{(1+i)a_{\overline{n}|} - nv^n}{i}, && (3.23)\\
(I\ddot{a})_{\overline{n}|} &= \frac{1-v^n}{(1-v)^2} - \frac{nv^n}{1-v} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{1-v} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d}. && (3.24)
\end{aligned}
$$

3.3.4 Accumulations

The symbol $(Is)_{\overline{n}|}$ is the value of the increasing immediate annuity at time $t_n = n$ when the final payment is made, and $(I\ddot{s})_{\overline{n}|}$ is the value of the increasing annuity-due at the same time $t_n = n$ (i.e. one unit of time after the final payment is made).

Formulas for the accumulations $(Is)_{\overline{n}|}$, $(I\ddot{s})_{\overline{n}|}$ can be obtained if we first calculate the present value of the corresponding annuity at time $t_0 = 0$ and then find the accumulation of the obtained value at time $t_n = n$:

$$
\begin{aligned}
(I\ddot{s})_{\overline{n}|} &= (1+i)^n (I\ddot{a})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{1-v} = \frac{\ddot{s}_{\overline{n}|} - n}{d}, && (3.25)\\
(Is)_{\overline{n}|} &= (1+i)^n (Ia)_{\overline{n}|} = \frac{s_{\overline{n}|}}{1-v} - \frac{n}{i} = \frac{s_{\overline{n}|}}{d} - \frac{n}{i}. && (3.26)
\end{aligned}
$$

3.4 Decreasing annuities

Consider again sequence of n consecutive unit intervals

$$(0,1), \ldots, (n-1, n).$$

As usual, $t_0 = 0$ means the present time.

A series of n payments $p_1 = n, p_2 = n-1, \ldots, p_n = 1$ made at the **end** of these intervals, i.e. at times $1, 2, \ldots, n$, is called *decreasing immediate annuity* or *decreasing annuity payable in arrear*. Its present value at time $t_0 = 0$ is denoted as $(Da)_{\overline{n}|}$.

A series of n payments $p_1 = n, p_2 = n-1, \ldots, p_n = 1$ made at the **end** of these intervals, i.e. at times $0, 1, \ldots, n-1$, is called *decreasing annuity-due* or *decreasing annuity payable in advance*. Its present value at time $t_0 = 0$ is denoted as $(D\ddot{a})_{\overline{n}|}$.

Union of the decreasing immediate annuity (or the decreasing annuity due) and the increasing immediate annuity (correspondingly, the increasing annuity-due) is the level immediate annuity (correspondingly, the level annuity-due) with the amount of the payment $n+1$. Thus,

$$
\begin{aligned}
(Da)_{\overline{n}|} + (Ia)_{\overline{n}|} &= (n+1)a_{\overline{n}|},\\
(D\ddot{a})_{\overline{n}|} + (I\ddot{a})_{\overline{n}|} &= (n+1)\ddot{a}_{\overline{n}|},
\end{aligned}
$$

[1]Later on, unless otherwise stated, we will not consider this trivial case

and equations (3.22), (3.23), (3.21), (3.24) yield:

$$\begin{aligned} (Da)_{\overline{n}|} &= \frac{(ni-1)a_{\overline{n}|}+nv^n}{i}, \\ (D\ddot{a})_{\overline{n}|} &= \frac{(ni-1)\ddot{a}_{\overline{n}|}+nv^{n-1}}{i}. \end{aligned}$$

From these relations for the accumulations $(Ds)_{\overline{n}|}$, $(D\ddot{s})_{\overline{n}|}$ at time $t=n$ we get:

$$\begin{aligned} (Ds)_{\overline{n}|} &= (1+i)^n(Da)_{\overline{n}|} = \frac{(ni-1)s_{\overline{n}|}+n}{i} \\ &= \frac{(ni-1)(1+i)^n+1}{i^2}, \\ (D\ddot{s})_{\overline{n}|} &= (1+i)^n(D\ddot{a})_{\overline{n}|} = \frac{(ni-1)\ddot{s}_{\overline{n}|}+n(1+i)}{i} \\ &= (1+i)(Ds)_{\overline{n}|}. \end{aligned}$$

3.5 Level annuities payable $p$thly

3.5.1 Definitions

Consider again sequence of n consecutive unit intervals

$$(0,1),\ldots,(n-1,n).$$

Let $t_0=0$ means the present time, and one unit of time is thought of as one year.

Divide each of n unit intervals into p equal subintervals, each of length $1/p$. The most interesting are the following cases:
$p=12$ (the subinterval of length $1/p$ corresponds to one month),
$p=4$ (the subinterval of length $1/p$ corresponds to one quarter),
$p=2$ (the subinterval of length $1/p$ corresponds to a half-year).

A series of np level payments, each of amount $\frac{1}{p}$, made at the **end** of these subintervals, i.e. at times

$$\begin{aligned} &1/p,\ldots,p/p=1;1+1/p,\ldots,1+p/p=2;\ldots; \\ &n-1+1/p,\ldots,n-1+p/p=n, \end{aligned}$$

is called *immediate annuity payable pthly* or *annuity payable pthly in arrear.*

A series of np level payments, each of amount $\frac{1}{p}$, made at the **beginning** of these subintervals, i.e. at times

$$\begin{aligned} &0,1/p,\ldots,(p-1)/p;1,1+1/p,\ldots,1+(p-1)/p;\ldots; \\ &n-1,n-1+1/p,\ldots,n-1+(p-1)/p, \end{aligned}$$

is called *pthly annuity-due* or *annuity payable pthly in advance.*

3.5.2 The present values

The present value (at time $t_0 = 0$) of the immediate annuity payable $p$thly is denoted by a symbol $a^{(p)}_{\overline{n}|}$, and value at time $t_n = n$ (the end of the final payment year), i.e. the accumulation, is noted as $s^{(p)}_{\overline{n}|}$.

The present value (at time $t_0 = 0$) of the $p$thly annuity-due is denoted by a symbol $\ddot{a}^{(p)}_{\overline{n}|}$, and value at time $t_n = n$ (the end of the final payment year), i.e. the accumulation, is noted as $\ddot{s}^{(p)}_{\overline{n}|}$.

Note that in these standard level annuities payable $p$thly each payment is of amount $1/p$, so that as a monetary unit we consider *algebraic* sum of all payments for the unit interval (in the typical case, for one year). For example, if over 5 years in the end of every month amount £100 is paid, then the unit sum is £1200, so that the present value of this cash flow is $1200a^{(12)}_{\overline{5}|}$.

Since the immediate annuity and the annuity due differs only at times $t_0 = 0$ and $t_n = n$, we have:

$$a^{(p)}_{\overline{n}|} = \ddot{a}^{(p)}_{\overline{n}|} - \frac{1}{p} + \frac{1}{p}v^n. \tag{3.27}$$

Indeed, to get the immediate annuity from the annuity-due we must take out the payment of amount $1/p$ at time $t_0 = 0$ and add the payment of amount $1/p$ at time $t_n = n$. However, since all payments are reduced to time $t_0 = 0$, the latter operation leads to the term $\frac{1}{p} \cdot v^n$, which is the present value (at time $t_0 = 0$) of amount $1/p$ due at time $t_n = n$.

Thus it is sufficient to get a formula for $\ddot{a}^{(p)}_{\overline{n}|}$.

With this goal introduce a new unit of time which is equal to pth part of the original unit interval (for example, if $p = 12$ and the original unit interval is one year, then the new unit interval is one month). The effective rate of interest for the new unit interval is $i^{(p)}_* = \frac{i^{(p)}}{p}$, where $i^{(p)}$ is the nominal rate of interest convertible $p$thly (see section 1.5), the new rate of discount $d^{(p)}_*$ is $\frac{d^{(p)}}{p}$, the new discount factor $v^{(p)}_* = 1 - d^{(p)}_* = (1 + i^{(p)}_*)^{-1}$ is

$$v^{(p)}_* = v^{\frac{1}{p}}.$$

Now we can consider the $p$thly annuity-due payable over interval $(0, n)$ as the conventional level annuity-due payable over interval $(0, np)$. Since the amount of each payment is $1/p$, we have:

$$\ddot{a}^{(p)}_{\overline{n}|@i} = \frac{1}{p} \cdot \ddot{a}_{\overline{np}|@i^{(p)}/p}, \tag{3.28}$$

where symbol $@i$ indicates the effective rate of interest for the interval which is considered as the uinit.

Applying first formula (3.4), and then formulas (2.13), (2.9), (3.3), we get (note that for the new unit interval parameters $i^{(p)}_*, d^{(p)}_*, v^{(p)}_*$ play the role of parameters i, d, v):

$$
\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|@i} &= \frac{1}{p} \cdot \frac{1-(v_*^{(p)})^{np}}{d_*^{(p)}} = \frac{1}{p} \cdot \frac{1-(1-d^{(p)}/p)^{np}}{d^{(p)}/p} \\
&= \frac{1-(1-d^{(p)}/p)^{np}}{d^{(p)}} = \frac{1-((1-d)^{1/p})^{np}}{d^{(p)}} \\
&= \frac{1-(1-d)^n}{d^{(p)}} = \frac{1-v^n}{d^{(p)}} \\
&= \frac{1-v^n}{d} \cdot \frac{d}{d^{(p)}} = \frac{d}{d^{(p)}} \ddot{a}_{\overline{n}|}. \qquad (3.29)
\end{aligned}
$$

Now from (3.27) for $a^{(p)}_{\overline{n}|}$ we get:

$$
\begin{aligned}
a^{(p)}_{\overline{n}|} &= \frac{1-v^n}{d^{(p)}} - \frac{1}{p} + \frac{1}{p}v^n = \frac{1-v^n}{d^{(p)}} - \frac{1-v^n}{p} \\
&= \frac{(1-v^n)(p-d^{(p)})}{pd^{(p)}}.
\end{aligned}
$$

With the help of (2.13), (1.14), (3.4) this relation can be reduced to the form:

$$
\begin{aligned}
a^{(p)}_{\overline{n}|} &= \frac{(1-v^n)(1-d)^{1/p}}{p(1-(1-d)^{1/p})} = \frac{1-v^n}{p((1-d)^{-1/p}-1)} \\
&= \frac{1-v^n}{p((1+i)^{1/p}-1)} = \frac{1-v^n}{i^{(p)}} \\
&= \frac{1-v^n}{i} \cdot \frac{i}{i^{(p)}} = \frac{i}{i^{(p)}} a_{\overline{n}|}. \qquad (3.30)
\end{aligned}
$$

We have defined $\ddot{a}^{(p)}_{\overline{n}|}$ $a^{(p)}_{\overline{n}|}$ only for integer n. It is easy to see that the use of the new unit of time allows to define in a natural way $\ddot{a}^{(p)}_{\overline{t}|}$ $a^{(p)}_{\overline{t}|}$ in the case $t = n + \frac{k}{p}$, $0 \le k \le n-1$. Namely,

$$
\begin{aligned}
\ddot{a}^{(p)}_{\overline{t}|@i} &= \frac{1}{p} \cdot \ddot{a}^{(p)}_{\overline{np+k}|@i^{(p)}/p} \\
&= \frac{1}{p} \cdot \frac{1-(v_*^{(p)})^{np+k}}{d_*^{(p)}} = \frac{1}{p} \cdot \frac{1-(1-d^{(p)}/p)^{np+k}}{d^{(p)}/p} \\
&= \frac{1-(1-d^{(p)}/p)^{np+k}}{d^{(p)}} = \frac{1-((1-d)^{1/p})^{np+k}}{d^{(p)}} \\
&= \frac{1-(1-d)^{n+k/p}}{d^{(p)}} = \frac{1-v^t}{d^{(p)}}. \qquad (3.31)
\end{aligned}
$$

Now

$$
a^{(p)}_{\overline{t}|@i} = \ddot{a}^{(p)}_{\overline{t}|@i} - \frac{1}{p} + \frac{1}{p}v^t = \frac{1-v^t}{i^{(p)}}. \qquad (3.32)
$$

3.5.3 The accumulations

Quantities $a_{\overline{n}|}^{(p)}$ and $s_{\overline{n}|}^{(p)}$, as well as the quantities $\ddot{a}_{\overline{n}|}^{(p)}$ and $\ddot{s}_{\overline{n}|}^{(p)}$, are the values of the same cash flow, but at different times ($t_0 = 0$ and $t_n = n$).

Thus, the following simple relations hold:

$$
\begin{aligned}
a_{\overline{n}|}^{(p)} &= s_{\overline{n}|}^{(p)} \cdot v^n, &(3.33)\\
s_{\overline{n}|}^{(p)} &= a_{\overline{n}|}^{(p)} \cdot (1+i)^n, &(3.34)\\
\ddot{a}_{\overline{n}|}^{(p)} &= \ddot{s}_{\overline{n}|}^{(p)} \cdot v^n, &(3.35)\\
\ddot{s}_{\overline{n}|}^{(p)} &= \ddot{a}_{\overline{n}|}^{(p)} \cdot (1+i)^n, &(3.36)
\end{aligned}
$$

and the formulas for the present values $a_{\overline{n}|}^{(p)}$ and $\ddot{a}_{\overline{n}|}^{(p)}$ allow to calculate the accumulations.

3.6 Continuous annuities

Consider immediate annuity and annuity due payable $p$thly over period $[0, n]$ and assume that $p \to \infty$. Using equations (1.15), (2.14), (3.29), (3.30), (3.3), (3.4) we get:

$$
\begin{aligned}
\lim_{p\to\infty} \ddot{a}_{\overline{n}|}^{(p)} &= \frac{d}{\delta}\ddot{a}_{\overline{n}|} = \frac{1-v^n}{\delta}, &(3.37)\\
\lim_{p\to\infty} a_{\overline{n}|}^{(p)} &= \frac{i}{\delta}a_{\overline{n}|} = \frac{1-v^n}{\delta}. &(3.38)
\end{aligned}
$$

Coincidence of the limits (3.37) and (3.38) can be explained as follows.

If $p \to \infty$ then both annuities consist of large number of small payments (of amount $1/p$ each) paid in short intervals of length $1/p$. Finally, when p is very close to ∞ the difference between the immediate annuity and annuity due becomes negligible, and both cash flows can be thought of as a continuous process similar to flow of fluid. The resulting process is called *the continuous annuity.*

Let $(t, t + \Delta)$ be a small period of time. It consists (approximately) of Δp intervals each of length $1/p$, so that over the period $(t, t+\Delta)$ amount $\frac{1}{p} \cdot (\Delta p) = \Delta$ will be paid. Thus in the continuous model the rate of payment at time t per unit time is $\rho(t) = \frac{\Delta}{\Delta} = 1$.

We can consider a more general continuously payable annuity when the rate of payment at time t is a general function $\rho(t)$. This means that the total amount of payment over small interval $(t, t + \Delta)$ is $\rho(t)\Delta + o(\Delta)$.

Clearly, the continuous annuity can be considered for any interval $(0, T)$, when T need not to be an integer.

The present value of the general continuous annuity payable with the rate $\rho(t)$ over interval $(0, T)$ can be calculated directly as follows.

With the help of points $t_0 = 0 < t_1 < t_2 < ... < t_k = T$ divide interval $[0, T]$ into large number n of small intervals $\Delta_j = (t_j, t_{j+1})$.

The total payment over the period Δ_j is approximately $\rho(t_j)\Delta_j$. Its present value at time $t_0 = 0$ is $v^{t_j}\rho(t_j)\Delta_j$, so that the total present value of the whole continuous annuity is approximately

$$\sum_j v^{t_j}\rho(t_j)\Delta_j. \tag{3.39}$$

This sum can be thought as the integral sum for the definite integral

$$\int_0^T v^t\rho(t)dt.$$

Thus,

$$\lim_{\max\Delta_j\to 0}\sum_j v^{t_j}\rho(t_j)\Delta_j = \int_0^T v^t\rho(t)dt. \tag{3.40}$$

On the other hand, as $\max\Delta_j \to 0$ the sum (3.39) gives the exact present value at time $t_0 = 0$ of the continuous annuity. Therefore, for the present value (PV) of the continuous annuity the following formula hold

$$\texttt{PV of the continuous annuity} = \int_0^T v^t\rho(t)dt. \tag{3.41}$$

Word for word repetition of the above arguments shows that for the continuous model of interest (introduced in section 1.4), when the force of interest is a general function of t,

$$\texttt{PV of the continuous annuity} = \int_0^T \rho(t)\exp\left(-\int_0^t \delta(u)du\right)dt. \tag{3.42}$$

If the rate of payment $\rho(t)$ is 1, the present value of the continuous annuity is denoted as $\overline{a}_{\overline{T}|}$. According to the (3.41):

$$\overline{a}_{\overline{T}|} = \int_0^n v^t dt = \int_0^T e^{-\delta t}dt = -\frac{1}{\delta}e^{-\delta t}\Big|_0^T = \frac{1-v^T}{\delta}. \tag{3.43}$$

Clearly, if $T = n$ is an integer, $\overline{a}_{\overline{T}|}$ can be calculated as either of two limits (3.37), (3.38).

If $T > 0$ is a general real, it can be approximated by a sequence of numbers $t = n + \frac{k}{p}$, $0 \le k \le n-1$, and then $\overline{a}_{\overline{T}|}$ can be calculated as either of two limits (3.31), (3.32).

The value of the continuous annuity payable over interval $(0, T)$ at time T is called *the accumulation*. To calculate the accumulation we can calculate the present value and then reduce this to time T.

For the conventional continuous annuity payable with the rate $\rho(t) = 1$ over interval $(0, T)$ the accumulation is denoted as $\overline{s}_{\overline{T}|}$. Applying this argument we have:

$$\overline{s}_{\overline{T}|} = \overline{a}_{\overline{T}|}\cdot(1+i)^T = \frac{(1+i)^T - 1}{\delta}. \tag{3.44}$$

Chapter 4

Assessment of investment projects

4.1 The internal rate of return

Consider an investment project, when at times $t'_1 < t'_2 < \cdots < t'_l$ an investor invests the sums $a'_1, a'_2, \ldots, a'_l$ accordingly, and then at times $t''_1 < t''_2 < \cdots < t''_m$ receives income $a''_1, a''_2, \ldots, a''_m$ accordingly.

Times $t'_1, t'_2, \ldots,$ when the investor invests his money can precede times $t''_1, t''_2, \ldots$ when the investor receives income. Often these times alternate.

Side by side with this investment project consider a bank account which earns compound interest at the rate i per annum. Assume that the bank offers to the investor the following deal: at times t'_k the amounts a'_k are deposited into the account, and at times t''_k the amounts a''_k are withdrawn from this auxiliary bank account. This auxiliary bank account is equivalent to the investment project under consideration in the sense that it requires from the investor the same outlays and generates the same income.

It should be noted that during some periods negative balance is possible, i.e. the investor owes money to the bank. We assume that during these period the amount of the debt grows according to the same rate i, i.e. the rate at which the bank credits the interest equals to the rate at which the bank lends money.

Let $T = \max_k\{t'_k, t''_k\}$ be the time when the project is completed. Correspondingly at this time the auxiliary bank account should be closed. If the deal offered by the bank is fair then the total value of all deposits must be equal to the total value of all withdrawals. Taking into account the fundamental concept of the present value we can write this condition as follows:

$$\sum_{k=1}^{l} a'_k v^{t'_k} = \sum_{k=1}^{m} a''_k v^{t''_k}, \tag{4.1}$$

where $v = \frac{1}{1+i}$ is the discount factor which corresponds to the rate of interest i.

Relating the investment project under consideration and the auxiliary bank account, it is natural to take as a measure of profitability of the investment project the rate of interest i for this bank account. This rate is called *the internal rate of return – IRR*, *money-weighted rate of return* or (with applications to fixed-interest securities) *the yield to the redemption.* The internal rate of return is a root of the equation (4.1); this equation is called *the yield equation.*

Sometimes it is convenient to join the sequences $t'_1, t'_2, \dots, t'_l$ and $t''_1, t''_2, \dots, t''_m$ into a single sequence $t_1 < t_2 < \cdots < t_n$ $(n = l+m)$ and consider the joint cash flow assuming that

1. if $t_k = t''_i$, then at time t_k the project generates income $c_k = a''_i$,
2. if $t_k = t'_j$, then at time t_k the project generates negative income $c_k = -a''_j$.

The sequence $(t_1, c_1), \dots, (t_n, c_n)$ is called *the net cash flow.*

Then the yield equation (4.1) can be written as

$$\sum_{k=1}^{n} c_k (1+i)^{-t_k} = 0. \tag{4.2}$$

4.2 Roots of the yield equation

In general, without any assumption about the structure of the flow of outlays (t'_k, a'_k) and the flow of income (t''_k, a''_k) we cannot say anything about roots of the yield equation (4.1). By definition, the yield is defined only when the yield equation has the only root and this root is positive (i.e. the investor really has some income from the project). Sometimes a weaker assumption, that this root is greater that -1, is made. If $i \in (-1, 0)$ then it means that the investor as a matter of fact lost some money he invested as the result of the transaction.

Under some natural assumption we can establish that the yield is well-defined. Assume, for example, that all outlays precede the receipt of any income, i.e.

$$t'_1 < t'_2 < \cdots < t'_l < t''_1 < t''_2 < \cdots < t''_m.$$

Rewrite the yield equation (4.1) as follows:

$$\sum_{k=1}^{l} a'_k (1+i)^{t'_l - t'_k} = \sum_{k=1}^{m} a''_k (1+i)^{t_l - t''_k}. \tag{4.3}$$

In the left-hand side of this equation all powers are positive (except for $t'_l - t'_k = 0$ when $k = l$). Thus, for $i \in (-1; +\infty)$ the right-hand side, as a function of i, either increases from a'_l to $+\infty$ or is the constant a'_l (if $l = 1$, i.e. the project involves the only outlay).

In the right-hand side of this equation all powers are negative. Thus, for $i \in (-1; +\infty)$ the right-hand side, as a function of i, decreases from $+\infty$ to 0.

Thus equation (4.3) has the only root i_0 on the interval $(-1, +\infty)$.

If, in addition, we assume that the total amount of all outlays is less than the total amount of income, i.e.

$$\sum_{k=1}^{l} a'_k < \sum_{k=1}^{m} a''_k, \tag{4.4}$$

then at the point $i = 0$ the left-hand side of (4.3) is less than the right-hand side. Thus the root i_0 is positive.

In general, to assess an investment project it is necessary to investigate the problem about the number of roots of the yield equation first. This step will help to calculate the yield if it exists.

Microsoft Excel has a function, IRR, which calculates the yield of an investment project described by the net cash flow $(t_1, c_1), \ldots, (t_n, c_n)$ where $t_1 = 0, t_1 = 1, \ldots, t_n = n$.

It should be noted that the internal rate of return is the simplest characteristics of the profitability of an investment project. In some cases the use of this characteristics can lead to a wrong conclusions and thus other characteristics of the profitability are needed.

Chapter 5

The loan schedule

5.1 A general scheme

Assume that one year is chosen as a unit of time and a loan of amount A is made at time $t = 0$ at the annual rate of interest i. The loan is to be repaid in n repayments, each of the same amount p, to be made at times $t_1 = 1, t_2 = 2, \ldots t_n = n$.

Important problems are

- to find the value of r;
- construct a schedule showing the division of each payment into capital and interest.

There are several ways to solve these problems.

5.1.1 Solution based on solving the functional equation

To solve this problem, introduce the sequence $x_1, x_2, \ldots, x_n$, where x_k is the loan outstanding just after the kth payment.

Obviously the following relations hold:

$$\begin{aligned} x_1 &= A(1+i) - p = x_0(1+i) - p, \text{ where } x_0 = A, \\ x_2 &= x_1(1+i) - p, \\ &\ldots \\ x_n &= x_{n-1}(1+i) - p, \end{aligned}$$

or in short

$$x_k = x_{k-1}(1+i) - p, \ 1 \leq k \leq n. \tag{5.1}$$

To solve this functional equation (i.e. to find the formula for the general term of the sequence x_k) introduce a new sequence y_k by the formula:

$$y_k = x_k - x \Leftrightarrow x_k = y_k + x,$$

where the parameter x will be defined later on.

For this new sequence recursive equation (5.1) becomes:

$$y_k + x = (y_{k-1} + x)\,(1+i) - p,\ 1 \le k \le n.$$

or equivalently

$$y_k = y_{k-1}(1+i) + ix - p,\ 1 \le k \le n. \tag{5.2}$$

If we chose x such that $ix - p = 0$, i.e $x = \frac{p}{i}$, then equation (5.2) becomes:

$$y_k = y_{k-1}(1+i),\ 1 \le k \le n.$$

This equation is exactly the equation which defines geometrical progression (which members are numbered starting from 0) with the common ratio $q = 1+i$, so that

$$y_k = y_0(1+i)^k,\ 0 \le k \le n.$$

For the main sequence x_k this result gives:

$$x_k - \frac{p}{i} = \left(x_0 - \frac{p}{i}\right)(1+i)^k,\ 0 \le k \le n,$$

so that

$$x_k = \left(A - \frac{p}{i}\right)(1+i)^k + \frac{p}{i},\ 0 \le k \le n. \tag{5.3}$$

Taking into account the boundary condition $x_n = 0$ (the loan must be repaid by the end of nth year) we find the amount of the regular payments p:

$$p = Ai\frac{(1+i)^n}{(1+i)^n - 1} = A\frac{i}{1 - v^n}, \tag{5.4}$$

where

$$v = \frac{1}{1+i}$$

is the discount factor.

Now for the loan outstanding after the kth payment (5.3) becomes:

$$x_k = A\frac{(1+i)^n - (1+i)^k}{(1+i)^n - 1} \equiv A\frac{1 - v^{n-k}}{1 - v^n},\ 0 \le k \le n. \tag{5.5}$$

The difference

$$c_k \equiv x_{k-1} - x_k = A\frac{(1+i)^k - (1+i)^{k-1}}{(1+i)^n - 1} = p \cdot v^{n-k+1} \tag{5.6}$$

is the decrease of the loan after kth payment and thus it is the share of kth payments which repays the capital, whereas

$$i_k = p - c_k = p \cdot \left(1 - v^{n-k+1}\right) \equiv Ai\frac{1 - v^{n-k+1}}{1 - v^n},\ 0 \le k \le n, \tag{5.7}$$

is interest content of kth payment.

Note that the value of i_k can also be obtained as ix_{k-1}.

Equation (5.1) can also be solved as follows. Divide both sides of this equation by $(1+i)^k$:

$$\frac{x_k}{(1+i)^k} = \frac{x_{k-1}}{(1+i)^{k-1}} - \frac{p}{(1+i)^k},\ 1 \le k \le n,$$

and then sum over $k = l+1, l+2, \ldots, n$. All terms $\frac{x_k}{(1+i)^k}$, except for $\frac{x_n}{(1+i)^n}$ in the left-hand side and $\frac{x_l}{(1+i)^l}$ in the right-hand side eliminate in pairs:

$$\frac{x_n}{(1+i)^n} = \frac{x_l}{(1+i)^l} - p\left(\sum_{k=l+1}^{n} \frac{1}{(1+i)^k}\right).$$

Taking into account the boundary condition $x_n = 0$ we get:

$$x_l = p \sum_{k=l+1}^{n} (1+i)^{l-k},\ 0 \le l \le n.$$

The sum in the right-hand side of this equation can be calculated with the help of the formula for the sum of geometrical progression, so that we have:

$$x_l = p\frac{\frac{1}{1+i} - \frac{1}{(1+i)^{n-l+1}}}{1 - \frac{1}{1+i}} = r\frac{1 - \frac{1}{(1+i)^{n-l}}}{i} = r\frac{1 - v^{n-l}}{i}, \tag{5.8}$$

In particular, if $l = 0$ we get:

$$A = p\frac{1 - v^n}{i},$$

which is equivalent to (5.4).

With the help of this result equation (5.8) becomes:

$$x_l = A\frac{1 - v^{n-l}}{1 - v^n},$$

which is exactly equation (5.5).

5.1.2 Financial approach

The main formula (5.4) can also be obtained with the help of the following financial considerations.

Consider the loan as a deposit of the capital A into an account 'A', which earns interest at the rate i. Repayment of the loan means regular level withdrawal of funds (the amount of each withdrawal is p) from this account. If we assume that the customer opens another similar account, say account 'B', which earns interest at the same rate i, and deposits every withdrawal into this new account, then for the customer the situation is as if he has the only account which earns interest at the rate i and made only the initial deposit of A at time $t = 0$. Thus, in total at time $t = n$ the customer will accumulate amount $A(1+i)^n$.

On the other hand, at time $t = n$ the account 'A' will have no funds at all (the loan if fully paid off), whereas the account 'B' will accumulate:

- the amount $p(1+i)^{n-1}$ from the first deposit,
- the amount $p(1+i)^{n-2}$ from the second deposit, and so on,
- the amount p from the final, nth, deposit,

i.e. in total the account 'B' will accumulate

$$p(1+i)^{n-1} + p(1+i)^{n-2} + \cdots + p \equiv p\sum_{k=1}^{n}(1+i)^{n-k}.$$

Thus,

$$A(1+i)^n = p\sum_{k=1}^{n}(1+i)^{n-k},$$

which is equivalent to (5.4).

5.1.3 Solution based on the principle of equivalence

A loan is an agreement between the lender and the borrower. According to this agreement the lender at time $t = 0$ gives to the borrower the amount A. In return, the borrower pays to the lender a level amount p at times $t_1 = 1, t_2 = 2, \ldots, t_n = n$.

The principle of equivalence states that the obligations of both parties of the deal must be identical. However, we cannot equate the sums paid by the lender and the borrower since the payments are make at different times. Thus consider the financial obligations of both parties at the time $t = 0$ when the agreement is concluded, i.i. consider the present values of these obligations.

Clearly, the present value of the lender's obligations is A, whereas the present value of the borrower's obligations is the present value of the level immediate annuity, i.e. $pa_{\overline{n}|}$. Thus, the principle of equivalence means requires that

$$A = pa_{\overline{n}|} \Leftrightarrow A = p\frac{1-v^n}{i},$$

which is equivalent to (5.4).

To find the amount x_k of the loan outstanding just after the kth payment note that it is exactly the obligations of the borrower at time k, i.e. the present value (at time k) of the level annuity payable at times $t_{k+1} = k+1, t_{k+2} = k+2, \ldots, t_n = n$. Thus

$$x_k = pa_{\overline{n-k}|} = p\frac{1-v^{n-k}}{i} = A\frac{1-v^{n-k}}{1-v^n}.$$

Chapter 6

Problems

6.1 Interest rates

Problem 6.1 [1] *A 90-days government bill is purchased for £*96 *at the time of issue and is sold after 45 days to another investor for £*97.90*. The second investor holds the bill until maturity and receives £*100*.*

Determine which investor receives the higher rate of return.

Solution. First note that both investors invested money for the same period of length 45 days.

For the first investor the effective rate of return for the period of 45 days is

$$i_1 = \frac{97.90 - 96}{96} = \frac{1.90}{96} \approx 1.979\%,$$

and for the second investor the effective rate of return for the period of 45 days is

$$i_2 = \frac{100 - 97.90}{97.90} \frac{2.10}{97.90} \approx 2.145\%.$$

Thus the second investor receives the higher rate of return. □

Problem 6.2 [2] *Calculate the time in days for £*$1,500$ *to accumulate to £*$1,550$ *at:*

(a) a simple rate of interest of 5% *per annum*

(b) a force of interest of 5% *per annum.*

Solution. (a) Assume that a simple interest is applied. Then in t (years) the initial value $P = 1,500$ of the fund becomes $A = P(1 + it) = 1500\,(1 + 0.05t)$. We are given that $A = 1550$. Thus we have the following equation for t:

$$1550 = 1500\,(1 + 0.05t) \Leftrightarrow t = \frac{2}{3} \text{ year } = 8 \text{ months } = 243 \text{ days.}$$

[1]The Institute of Actuaries, Exam CT1, September 2007, Problem 1

[2]The Institute of Actuaries, Exam CT1, September 2005, Problem 3

Here we assumed that the year consists of 12 months or, equivalently, 365 days.

(b) Assume that a compound interest is applied. Then in t (years) the initial value $P = 1,500$ of the fund becomes $A = Pe^{\delta t} = 1500e^{0.05t}$. We are given that $A = 1550$. Thus we have the following equation for t:

$$1550 = 1500e^{0.05t} \Leftrightarrow t = 20 \ln \frac{155}{150} \approx 0.655796 \text{ year } = 7.87 \text{ months } = 239 \text{ days.}$$

□

Problem 6.3 [3] *An investor purchases a share for 769p at the beginning of the year. Halfway through the year he receives a dividend, net of tax, of 4p and immediately sells the share for 800p. Capital gain tax of* 30% *is paid on the difference between the sale and the purchase price.*

Calculate the net annual effective rate of return the investor obtains on the investment.

Solution. The amount of the investment is $P = 769$.

The accumulation consists of three components:

- dividend: 4p;
- sale price of the share: 800p;
- capital gain tax: since the difference between the sale and the purchase price is $800 - 769 = 31$, the tax is $0.3 \cdot 31 = 9.3$.

Thus, the total amount received by the investor is 794.7p, i.e. income is $I = 25.7$p. Therefore the effective rate of return for a 6 months period is $i_* = \frac{25.7}{769} \approx 0.03342$.

The equivalent annual rate of return, i, is defined as the annual effective rate of interest for an auxiliary savings account which (according to the concept of compound interest) generates at the end of the period under consideration the same income as the investment project.

The equivalent annual rate of return (i.e. the internal rate of return), i, can be found from the equation

$$(1 + i)^{\frac{1}{2}} = 1 + i_*,$$

so that

$$i = (1 + i_*)^2 - 1 = 2i_* + i_*^2 \approx 6.7957\%.$$

Note that the corresponding nominal annual rate of return is $2i_* = 6.684\%$, i.e very close to the IRR (the relative error is less than 2%).

□

[3] The Institute of Actuaries, Exam CT1, September 2007, Problem 2

Problem 6.4 [4] *An investor has earned a money rate of return from a portfolio of bonds in a particular country of* 1% *per annum effective over a period of ten years. The country has experienced deflation (negative inflation) of* 2% *per annum effective during the period.*

Calculate the real rate of return per annum over the ten years.

Solution. If $i = 1\%$ is the effective money rate of return and $f = -2\%$ the rate of inflation, then the inflation adjusted rate of return is

$$\frac{i-f}{1+f} = \frac{0.01+0.02}{1+0.02} \approx 0.030612 = 3.0612\%.$$

□

Problem 6.5 [5] *An investment is discounted for 28 days at a simple rate of discount of* 4.5% *per annum. Calculate the annual effective rate of interest.*

Solution. First note that $d = 4.5\%$ per annum is the nominal discount rate. The effective discount rate for the 28 days period is $d_* = \frac{28}{365}d \approx 0.3452055\%$.

This means (by the definition of the effective discount rate) that investment of amount $1 - d_*$ at time 0 in 28 days will accumulate amount 1.

The corresponding annual effective rate of interest i is such a rate that savings account which earns this rate of interest per annum and credits the interest for a shorter period of time according to the principle of compound interest will have the same result, i.e.

$$(1-d_*)(1+i)^{\frac{28}{365}} = 1.$$

From this we have:

$$i = (1-d_*)^{-\frac{365}{28}} - 1 = \left(1 - \frac{28}{365}d\right)^{-\frac{365}{28}} - 1 \approx 0.046109 = 4.6109\%.$$

□

Stochastic interest rates

Problem 6.6 [6] *(i) In any given year, the interest rate per annum effective on monies invested with a given bank has mean value j and standard deviation s and is independent of the interest rates in all previous years.*

Let S_n be the accumulated amount after n years of a single investment of 1 at time $t = 0$.

(a) Show that $ES_n = (1+j)^n$.

(b) Show that $VarS_n = (1+2j+j^2+s^2)^n - (1+j)^{2n}$.

[4] The Institute of Actuaries, Exam CT1, September 2005, Problem 2
[5] The Institute of Actuaries, Exam CT1, April 2006, Problem 1
[6] The Institute of Actuaries, Exam CT1, April 2005, Problem 10

(ii) The interest rate per annum effective in (i), in any year, is equally likely to be i_1 or i_2 ($i_1 > i_2$). No other values are possible.

(a) Derive expressions for j and s^2 in terms of i_1 and i_2.

(b) The accumulated value at time $t = 25$ years of £1 million invested with the bank at time $t = 0$ has expected value £5.5 million and standard deviation £0.5 million.

Calculate the values of i_1 and i_2.

Solution. (i)

(a) Let ξ_k be the interest rate in the kth year. We are given that $\xi_1, \xi_2, \ldots, \xi_n$ are independent and identically distributed (i.i.d.) random variables (r.v.) with the mean values $E\xi_k = j$ and variances $Var\xi_k = s^2$.

The accumulated amount after n years is given by the formula

$$S_n = (1+\xi_1)(1+\xi_2)\ldots(1+\xi_n),$$

so that it is a random variable.

For its mean value we have:

$$ES_n = E\left[(1+\xi_1)(1+\xi_2)\ldots(1+\xi_n)\right].$$

Since random variables ξ_k, $1 \le k \le n$, are independent, the mean value of the product is the product of the mean values:

$$ES_n = E(1+\xi_1)\cdot E(1+\xi_2)\cdot\dots\cdot E(1+\xi_n).$$

Since random variables ξ_k, $1 \le k \le n$, are identically distributed, for any k we have:

$$E(1+\xi_k) = 1 + E\xi_k = 1 + j,$$

so that for ES_n we finally have:

$$ES_n = (1+j)^n.$$

(b) To find $VarS_n = ES_n^2 - (ES_n)^2$ we must find ES_n^2:

$$\begin{aligned} ES_n^2 &= E\left[(1+\xi_1)^2(1+\xi_2)^2\ldots(1+\xi_n)^2\right] \\ &= E(1+\xi_1)^2\cdot E(1+\xi_2)^2\cdot\dots\cdot E(1+\xi_n)^2. \end{aligned}$$

Since random variables ξ_k, $1 \le k \le n$, are identically distributed, for any k we have:

$$\begin{aligned} E(1+\xi_k)^2 &= E(1+2\xi_k+\xi_k^2) = 1 + 2E\xi_k + E\xi_k^2 \\ &= 1 + 2E\xi_k + Var\xi_k + (E\xi_k)^2 = 1 + 2j + s^2 + j^2, \end{aligned}$$

so that for ES_n^2 we finally have:

$$ES_n = (1 + 2j + j^2 + s^2)^n$$

and correspondingly for $VarS_n$:

$$VarS_n = (1 + 2j + j^2 + s^2)^n - (1 + j)^{2n}.$$

(ii) We are given that for any k

$$P(\xi_k = i_1) = \frac{1}{2},\ P(\xi_k = i_2) = \frac{1}{2}.$$

Thus,

$$j \equiv E\xi_k = \frac{1}{2}i_1 + \frac{1}{2}i_2 = \frac{i_1 + i_2}{2},$$

and

$$E\xi_k^2 = \frac{1}{2}i_1^2 + \frac{1}{2}i_2^2 = \frac{i_1^2 + i_2^2}{2},$$

so that

$$s^2 = Var\xi_k = E\xi_k^2 - (E\xi_k) = \frac{i_1^2 + i_2^2}{2} - \left(\frac{i_1 + i_2}{2}\right)^2 = \frac{(i_1 - i_2)^2}{4}.$$

In this particular case the above formulas for ES_n and $VarS_n$ becomes:

$$\begin{aligned} ES_n &= \left(\frac{k_1 + k_2}{2}\right)^n, \\ VarS_n &= \left(\frac{k_1^2 + k_2^2}{2}\right)^n - \left(\frac{k_1 + k_2}{2}\right)^{2n}, \end{aligned}$$

where $k_1 = 1 + i_1$, $k_2 = 1 + i_2$.

Next, we are given that the accumulated value at time $t = 25$ years of £1 million invested with the bank at time $t = 0$ has expected value £5.5 million and standard deviation £0.5 million. It means that $n = 25$, $ES_n = 5.5$, $VarS_n = 0.25$ (we take £1 million as a new monetary unit). Using the above formulas we get the following set of equations:

$$\begin{cases} \left(\frac{k_1+k_2}{2}\right)^{25} = 5.5, \\ \left(\frac{k_1^2+k_2^2}{2}\right)^{25} - \left(\frac{k_1+k_2}{2}\right)^{50} = 0.25. \end{cases} \Leftrightarrow \begin{cases} k_1 + k_2 = 2a \\ k_1^2 + k_2^2 = 2b \end{cases}$$

where $a = \sqrt[25]{5.5}$, $b = \sqrt[25]{30.5}$. This set of two equations with two unknowns can be easily solved; it has the unique solution such that $k_1 > k_2$:

$$k_1 = a + \sqrt{b - a^2},\ k_2 = a - \sqrt{b - a^2}.$$

In our case, we have:

$$k_1 \approx 1.08995,\ k_2 \approx 1.051142,$$

so that

$$i_1 \approx 0.089995,\ i_2 \approx 0.051142.$$

□

Problem 6.7 [7] *£80,000 is invested in a bank account which pays interest at the end of each year. interest is always reinvested in the account. The rate of interest is determined at the beginning of each year and remains unchanged until the beginning of the next year. The rate of interest applicable in any one year is independent of the rate applicable in any other year.*

During the first year, the annual effective rate of interest will be one of 4%, 6% or 8% with equal probability.

During the second year, the annual effective rate of interest will be either 7% with probability 0.75 or 5% with probability 0.25.

During the third year, the annual effective rate of interest will be either 6% with probability 0.7 or 4% with probability 0.3.

(i) Derive the expected accumulated amount in the bank account at the end of three years.

(ii) Derive the variance of the accumulated amount in the bank account at the end of three years.

(iii) Calculate the probability that the accumulated amount is more than £97,000 at the end of three years.

Solution. Let ξ_1, ξ_2, ξ_3 be the annual interest rates for the first, second, third year correspondingly. If we take the initial investment £80,000 as the new monetary unit, then the accumulation at the end of the third year is $A = (1+\xi_1)(1+\xi_2)(1+\xi_3)$. It is a random variable, which takes 12 values:

- $1.04 \cdot 1.07 \cdot 1.06 = 1.179568$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{7}{10} = \frac{21}{120}$;
- $1.04 \cdot 1.07 \cdot 1.04 = 1.157312$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{3}{10} = \frac{9}{120}$;
- $1.04 \cdot 1.05 \cdot 1.06 = 1.157520$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{7}{10} = \frac{7}{120}$;
- $1.04 \cdot 1.05 \cdot 1.04 = 1.135680$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{10} = \frac{3}{120}$;
- $1.06 \cdot 1.07 \cdot 1.06 = 1.202252$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{7}{10} = \frac{21}{120}$;
- $1.06 \cdot 1.07 \cdot 1.04 = 1.179568$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{3}{10} = \frac{9}{120}$;
- $1.06 \cdot 1.05 \cdot 1.06 = 1.179780$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{7}{10} = \frac{7}{120}$;
- $1.06 \cdot 1.05 \cdot 1.04 = 1.157520$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{10} = \frac{3}{120}$;
- $1.08 \cdot 1.07 \cdot 1.06 = 1.224936$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{7}{10} = \frac{21}{120}$;
- $1.08 \cdot 1.07 \cdot 1.04 = 1.201824$ with the probability $\frac{1}{3} \cdot \frac{3}{4} \cdot \frac{3}{10} = \frac{9}{120}$;
- $1.08 \cdot 1.05 \cdot 1.06 = 1.202040$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{7}{10} = \frac{7}{120}$;
- $1.08 \cdot 1.05 \cdot 1.04 = 1.179360$ with the probability $\frac{1}{3} \cdot \frac{1}{3} \cdot \frac{3}{10} = \frac{3}{120}$.

Note that

[7] The Institute of Actuaries, Exam CT1, April 2007, Problem 11

- the value $A = 1.157520$ is obtained when $\xi_1 = 0.04$, $\xi_2 = 0.05$, $\xi_3 = 0.06$ and when $\xi_1 = 0.06$, $\xi_2 = 0.05$, $\xi_3 = 0.04$;
- the value $A = 1.179568$ is obtained when $\xi_1 = 0.04$, $\xi_2 = 0.07$, $\xi_3 = 0.06$ and when $\xi_1 = 0.06$, $\xi_2 = 0.07$, $\xi_3 = 0.04$,

so that strictly speaking the random variable A takes 10 values and

- the probability of the value $A = 1.157520$ is the sum $\frac{7}{120} + \frac{3}{120} = \frac{10}{120}$;
- the probability of the value $A = 1.179568$ is the sum $\frac{21}{120} + \frac{9}{120} = \frac{30}{120}$.

Now the expectation EA can be found with the help of the general formula $EA = \sum_x xP(A = x)$:

$$EA = 1.1898606 \text{ (new monetary units) } = £95\ 188.848.$$

To calculate $VarA$ first find the second moment EA^2 by the general formula $EA^2 = \sum_x x^2 P(A = x)$:

$$EA^2 = 1.416305 \text{ (new monetary units}^2\text{)} ,$$

so that

$$VarA = EA^2 - (EA)^2 = 0.0005367301 \text{ (new monetary units}^2\text{) } = 3\,435\,072.82(£^2).$$

More interesting measure of the dispersion is the standard deviation $s = \sqrt{VarA}$:

$$s = £1\ 853.4.$$

The value $£97,000$ in the new monetary units is $\frac{97\ 000}{80\ 000} = 1.2125$. Among possible values of the accumulated amount at the end of three years only one value exceeds the level 1.2125; it is 1.224936 which corresponds to $\xi_1 = 0.08$, $\xi_2 = 0.07$, $\xi_3 = 0.06$. The probability of this outcome is $\frac{21}{120} = \frac{7}{40} = 0.175$.

Note that EA and $VarA$ could be calculated with the help of the approach used to solve the problem 6.6.

□

Problem 6.8 [8] *The expected effective annual rate of return from a bank's investment portfolio is* 6% *and the standard deviation of annual effective returns is* 8%. *The annual effective returns are independent and* $(1+i_t)$ *are lognormally distributed, where* i_t *is the return in year* t.

Deriving any necessary formulae:

(i) calculate the expected value of an investment of £2 *million after 10 years.*

(ii) calculate the probability that the accumulation of the investment will be less than 80% *of the expected value.*

[8] The Institute of Actuaries, Exam CT1, September 2007, Problem 9

Solution.

(i) Consider £2 million as a new monetary unit. Then the accumulated amount after $n = 10$ years is given by the formula

$$S = (1 + i_1)(1 + i_2) \dots (1 + i_{10}),$$

so that it is a random variable.

For its mean value we have:

$$ES = E\left[(1 + i_1)(1 + i_2) \dots (1 + i_{10})\right].$$

Since random variables i_k, $1 \leq k \leq 10$, are independent, the mean value of the product is the product of the mean values:

$$ES = E(1 + i_1) \cdot E(1 + i_2) \cdot \ldots \cdot E(1 + i_{10}).$$

Besides, since random variables i_k, $1 \leq k \leq n$, are identically distributed, all values $E(1 + i_k)$ are identical and equal to $1 + Ei_k = 1.06$. Thus,

$$ES = 1.06^{10} \approx 1.790848 \text{ (new monetary units) } = \text{£}3.581695 \text{ (million)}.$$

(ii) First recall that a positive random variable η has a log-normal distribution with the parameters a and σ iff $\ln \eta$ has a normal distribution with some parameters (a, σ):

$$P(\ln \eta < x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-a)^2}{2\sigma^2}} dt.$$

Thus the assumption that the annual effective returns are independent and $(1 + i_t)$ are lognormally distributed, means that the forces of interest $\delta_1 = \ln(1 + i_1), \dots, \delta_{10} = \ln(1 + i_{10})$ are independent and have the same normal distribution with the identical parameters (a, σ).

Obviously, if random variables $\eta_1, \dots, \eta_n$ are independent and have log-normal distributions some parameters $(a_1, \sigma_1), \dots, (a_n, \sigma_n)$ correspondingly, then their product has a log-normal distribution with the parameters $a = a_1 + \cdots + a_n$ and $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

Thus random variable S also has a log-normal distribution with the parameters $10a$, $\sqrt{10}\sigma$, where a and σ are the parameters of the log-normal random variable $\eta_t = 1 + i_t$.

If η has a log-normal log-normal distribution with the parameters a and σ then its mean value is given by

$$E\eta = e^{a + \frac{\sigma^2}{2}}$$

and the coefficient of variation $c = \frac{\sqrt{Var \eta}}{E\eta}$ is given by

$$c = \sqrt{e^{\sigma^2} - 1}. \tag{6.1}$$

In particular, ES can be written in the form

$$ES = e^{10a+5\sigma^2}. \tag{6.2}$$

We must calculate the probability $P = P(S < 0.8ES)$. First note that the events $\{S < x\}$ and $\{\ln S < \ln x\}$ are identical. Besides, since $\ln S \sim N(10a, \sqrt{10}\sigma)$, the shifted and scaled random variable $\frac{\ln S - 10a}{\sqrt{10}\sigma}$ has the standard $(0, 1)$ normal distribution. Thus,

$$P = P(S < 0.8ES) = P(\ln S < \ln(0.8ES)) = \Phi\left(\frac{\ln(0.8ES) - 10a}{\sqrt{10}\sigma}\right).$$

Taking into account (6.2) we have:

$$P = \Phi\left(\frac{\ln 0.8 + 5\sigma^2}{\sqrt{10}\sigma}\right). \tag{6.3}$$

Since $E(1 + i_t) = 1 + Ei_t = 1.06$. $Var(1 + i_t) = Vari_t = 0.64$, the coefficient of variation of $\eta_t = 1 + i_t$ is $\frac{0.08}{1.06} = \frac{4}{53}$. On the other hand, it is given by (6.1). Thus

$$\sqrt{e^{\sigma^2} - 1} = \frac{4}{53} \Leftrightarrow \sigma^2 = \ln\frac{2825}{2809},$$

so that the argument of the function Φ in (6.2) is

$$\frac{\ln 0.8 + 5\ln\frac{2825}{2809}}{\sqrt{10\ln\frac{2825}{2809}}} \approx -0.8171428$$

and therefore $P \approx 0.2069$.

It should be noted that Microsoft Excel has a function

$$LOGNORMDIST(x, MEAN, DEVIATION)$$

which allows calculation of the value at a given point x of the distribution function of the log-normal random variable with a given mean value $MEAN$ and standard deviation $DEVIATION$. With the help of this function the problem could be solved as follows:

- enter in a cell $A1$ the value **0.06** (the mean value of i);
- enter in a cell $A2$ the value **0.08** (the value of $\sqrt{Vari}$);
- enter in a cell $A3$ a formula **=(1+A1)^10** to calculate ES;
- enter in a cell $A4$ a formula **=LN((A2/(1+A1))^2+1)** to calculate σ^2;
- enter in a cell $A5$ a formula **=-A4/2+LN(1+A1)** to calculate a;
- enter in a cell $A6$ a formula **=LOGNORMDIST(0.8*A3,10*A5,SQRT(10*A4))** to get the answer.

□

Problem 6.9 [9] *An insurance company has just written contracts that require it to make payments to policyholders of £1,000,000 in five years' time. The total premiums paid by policyholders amounted to £850,000. The insurance company is to invest half the premium in fixed interest securities that provide a return of 3% per annum effective. The other half of the premium income is to be invested in assets that have an uncertain return. The return from these assets in year t, i_t, has a mean value of 3.5% per annum effective and a standard deviation of 3% per annum effective. $(1+i_t)$ are independently and lognormally distributed.*

(i) Deriving all necessary formulae, calculate the mean and standard deviation of the accumulation of the premiums over the five-year period.

(ii) A director of the company suggests that investing all the premiums in the assets with an uncertain return would be preferable because the expected accumulation of the premiums would be greater that the payments due to the policyholders.

Explain why this still may be a more risky investment policy.

Solution. (i) Consider £1,000,000 as a new monetary unit. Then the premium collected is 0.85. The amount invested in fixed interest securities (at 3% annual interest) is 0.425, so that in five years it accumulates to

$$S_f = 0.425(1+i)^5 = 0.492691482.$$

The amount invested in assets is 0.425 and in five years it accumulates to

$$S_a = 0.425(1+i_1)(1+i_2)(1+i_3)(1+i_4)(1+i_5).$$

Since the rates $i_1, \ldots, i_5$ are random variables, so is the accumulation S_a.

For its mean value we have:

$$\begin{aligned} ES_a &= 0.425E\left[(1+i_1)\ldots(1+i_5)\right] = 0.425E(1+i_1)\ldots E(1+i_5) \\ &= 0.425\left(1+Ei_t\right)^5 = 0.50476668. \end{aligned}$$

The total accumulation of the premiums over the five-years period is $S = S_f + S_a$. Thus

$$ES = S_f + ES_a = 0.997458161,$$

i.e. $ES = £997\,458.16$. It means, in particular, that on average the accumulation is not sufficient to cover obligations of the insurer.

To find the standard deviation σ_S of the accumulation of the premiums over the five-year period, note that

$$\sigma_S^2 = VarS = Var(S_f + S_a) = VarS_a = ES_a^2 - \left(ES_a\right)^2.$$

[9] The Institute of Actuaries, Exam CT1, September 2005, Problem 8

We have already calculated ES_a, and

$$\begin{aligned} ES_a^2 &= 0.425^2 \cdot E\left[(1+i_1)^2 \ldots (1+i_5)^2\right] = 0.425^2 \left(E(1+i_t)^2\right)^5 \\ &= 0.425^2 \left(1 + 2Ei_t + Ei_t^2\right)^5 = 0.425^2 \left(1 + 2Ei_t + Vari_t + (Ei_t)^2\right)^5 \\ &= 0.425^2 \left(1 + 0.07 + 0.0009 + 0.001225\right)^5 \approx 0.25586152, \end{aligned}$$

so that

$$VarS_a \approx 0.001072119$$

and $\sigma_S \approx 0.032743226$, i.e. $\sigma_S \approx$ £32 743.23.

(ii) The answer to the second question depends on the criterion adopted.

1. Decisions are compared according to the expected accumulation.

Under the decision of the director, the accumulation over the five-year period is (in pounds)

$$S' = 0.85(1+i_1)\ldots(1+i_5),$$

so that

$$ES' = 0.85\left(E(1+i_t)\right)^5 = 0.85 \cdot 1.035^5 \approx 1.00953336,$$

i.e. £1 009 533.36, whereas the first strategy gives $ES =$ £997 458.16. Thus in this case the director's decision is better.

2. Decisions are compared according to the probability of loss, i.e the probability that the total accumulation is less than the obligations of the company.

In the first case this probability is

$$\begin{aligned} P(S<1) &= P(S_f + S_a < 1) = P(S_a < 1 - S_f) \\ &= P(0.425\xi < 1 - 0.425 \cdot 1.035^5) \approx P(\xi < 1.165254871) \\ &= F_\xi(1.165254871). \end{aligned}$$

where $\xi = (1+i_1)\ldots(1+i_5)$ and $F_\xi(x)$ is the distribution function of ξ.

In the second case this probability is

$$P(S'<1) = P(0.85\xi < 1) \approx P(\xi < 1.176470588) = F_\xi(1.176470588).$$

Since distribution function of any continuous random variable is increasing, $P(S' < 1) > P(S < 1)$. Thus the second decision is more risky. It should be noted that the arguments of F_ξ are very close, so that the difference between both decisions is not too high.

More important is the following remark. Since we know the distribution of the random annual rate of return of investment into risky assets, we can calculate both probabilities of ruin.

Let a and σ be the mean and the standard deviation of (normally distributed) random variable $\ln \eta_t = \ln(1+i_t)$.

Then,

$$\begin{aligned} E\eta_t &= e^{a+\frac{\sigma^2}{2}} \\ c_{\eta_t} &= \sqrt{e^{\sigma^2} - 1}. \end{aligned}$$

We are given that $E\eta_t = 1.035$, $c_{\eta_t} = \frac{0.03}{1.035}$. Thus,

$$
\begin{aligned}
e^{\sigma^2} &= \left(\frac{30}{1035}\right)^2 + 1, \\
e^a &= \frac{1.035}{\sqrt{\left(\frac{30}{1035}\right)^2 + 1}},
\end{aligned}
$$

so that

$$
\begin{aligned}
\sigma &= \sqrt{\ln\left(\left(\frac{30}{1035}\right)^2 + 1\right)} \approx 0.028979422, \\
a &= \ln 1.035 - \frac{1}{2}\ln\left(\left(\frac{30}{1035}\right)^2 + 1\right) \approx 0.033981523.
\end{aligned}
$$

Random variable $\ln\xi$ as a sum of 5 independent and identically distributed random variables $\ln(1+i_1), \ldots, \ln(1+i_5)$ has a normal distribution with the mean $5a$ and the variance $5\sigma^2$.

Thus for $P(S<1)$ we have:

$$
\begin{aligned}
P(S<1) &= P(\xi < 1.165254871) = P(\ln\xi < \ln 1.165254871) \\
&= P\left(\frac{\ln\xi - 5a}{\sqrt{5}\sigma} < \frac{\ln 1.165254871 - 5a}{\sqrt{5}\sigma}\right) \\
&= \Phi\left(\frac{\ln 1.165254871 - 5a}{\sqrt{5}\sigma}\right) = \Phi(-0.26184863) \approx 0.4.
\end{aligned}
$$

Similar calculations gives that

$$
P(S'<1) = \Phi\left(\frac{\ln 1.176470588 - 5a}{\sqrt{5}\sigma}\right) = \Phi(-0.26184863) \approx 0.45.
$$

These values of the ruin probability are unacceptable, so that as a matter of fact the insurer should not write contracts.

□

6.2 Present values. Valuing cash flows

Problem 6.10 [10]

(i) Calculate the present value of £100 over ten years at the following rates of interest/discount:

(a) *a rate of interest of 5% per annum convertible monthly*

(b) *a rate of discount of 5% per annum convertible monthly*

[10] The Institute of Actuaries, Exam CT1, September 2005, Problem 5

(c) *a force of interest of* 5% *per annum*

(ii) A 91-day treasury bill is bought for \$98.91 *and is redeemed at* \$100. *Calculate the annual effective rate of interest obtained from the bill.*

Solution. (i)

(a) We are given that the nominal rate of interest $i^{(12)}$ is 0.05. Thus the effective monthly rate of interest is $i_*^{(12)} = \frac{1}{12} i^{(12)} = \frac{1}{240}$. Consider one month as a new unit of time. Then ten years has length $n = 10 \times 12 = 120$ (units). Thus, the present value of £100 is

$$PV_1 = 100 \cdot \left(1 + i_*^{(12)}\right)^{-120} = 100 \cdot \left(1 + \frac{1}{240}\right)^{-120} \approx 60.72. \tag{6.4}$$

(b) We are given that the nominal rate of discount $d^{(12)}$ is 0.05. Thus the effective monthly rate of discount is $d_*^{(12)} = \frac{1}{12} d^{(12)} = \frac{1}{240}$. Again consider one month as a new unit of time. Then ten years has length $n = 10 \times 12 = 120$ (units). Thus, the present value of £100 is

$$PV_2 = 100 \cdot \left(1 - d_*^{(12)}\right)^{120} = 100 \cdot \left(1 - \frac{1}{240}\right)^{120} \approx 60.59. \tag{6.5}$$

(c) We are given that the force of interest δ is 0.05. Thus, the present value of £100 is

$$PV_3 = 100 \cdot e^{-10\delta} = 100 \cdot e^{-0.5} = \frac{100}{\sqrt{e}} \approx 60.65. \tag{6.6}$$

If we rewrite the formulae (6.4) and (6.5) as

$$PV_1 = \frac{100}{\sqrt{\left(1 + \frac{1}{240}\right)^{240}}}$$

and

$$PV_2 = \frac{100}{\sqrt{\left(1 - \frac{1}{240}\right)^{-240}}}$$

respectively, then we can apply the classic limit

$$\lim_{x \to 0} (1 + x)^{\frac{1}{x}} = e.$$

As $\pm\frac{1}{240}$ "is close to 0", both PV_1 and PV_2 are close to

$$\frac{100}{\sqrt{e}},$$

which is exactly PV_3.

(ii) Treasury bills are government securities issued to borrow money for a short period (4, 13, 36 or 52 weeks). As opposite to conventional bonds, which are issued for a longer periods and regularly pay interest (the coupon), they do not pay any interest. Upon maturity an investor gets only the par value of the bill, so that investor buys the bill at some discount of the face value. From this point of view treasury bills can be considered as a sort of zero-coupon bonds.

The bill under consideration is issued for $h = 13$ week ($= 13 \times 7 = 91$ days). The par value is \$100 and the purchase price is \$98.91. Thus the effective rate of discount for the 13 weeks period is

$$d_* = \frac{100 - 98.91}{100} = 1.09\%$$

and the effective rate of interest for the 13 weeks period is

$$i_* = \frac{100 - 98.91}{98.91} \approx 1.102\%.$$

The equivalent annual rate of return, i, is defined as the annual effective rate of interest for an auxiliary savings account which (according to the concept of compound interest) generates at the end of the period under consideration the same income as the treasury bill. Thus, assuming that the year consists of 365 days, the equivalent annual rate of return can be found from the equation

$$(1+i)^{\frac{91}{365}} = 1 + i_*,$$

so that

$$i = (1+i_*)^{\frac{365}{91}} - 1 = \left(\frac{100}{98.91}\right)^{\frac{365}{91}} - 1 \approx 4.494\%.$$

□

Problem 6.11 [11] *On January 1, 2002, Pat, age 40, purchases a 5-payment, 10-year term insurance of 100,000:*

(i) *Death benefits are payable at the moment of death.*

(ii) *Contract premiums of 4000 are payable annually at the beginning of each year for 5 years.*

(iii) $i = 0.05$

(iv) *L is the loss random variable at time of issue.*

Calculate the value of L if Pat dies on June 30, 2004.

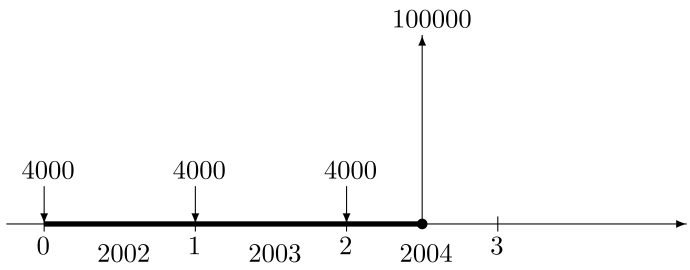
Solution. The situation described in the problem can be represented with the help of the figure 6.1.

Since the future life time T_{40} is known exactly:

$$T_{40} = 2.5,$$

[11] *Course/Exam 3 – Actuarial Models,* The Society of Actuaries and the Casualty Actuarial Society, May 2001, Problem 2.

Figure 6.1:



in the situation under consideration there is no element of uncertainty.

The insurer's obligations are to pay amount 100 000 at the time $T_{40} = 2.5$ (we take the time when the policy was issued as the initial). The present value of these obligations is

$$100\ 000 \cdot (1.05)^{-2.5} \approx 88\ 517.$$

We know that the insured paid exactly 3 premiums (4000 each): on 1 January 2002, on 1 January 2003, on 1 January 2004. The present value of this cash flow is

$$4\ 000 \cdot (1 + 1.05^{-1} + 1.05^{-2}) \approx 11\ 438.$$

Thus the present value of the company's loss equals 77 079. □

Problem 6.12 [12] *An eleven month forward contract is issued on 1 March 2008 on a stock with a price of £10 per share at that date. Dividends of 50 pence per share are expected to be paid on 1 April and 1 October 2008.*

Calculate the forward price at issue, assuming a risk-free rate of interest of 5% per annum effective and no arbitrage.

Solution. First remind the notions used in the text of the problem. *The risk-free interest rate* is the rate which can be obtained by investing into assets without any risk of default. The most common example is US treasury bills (the US government always can print as many new dollar banknotes as needed to pay the obligations nominated in US dollars).

The term *arbitrage* describes the situation when a person/institution can earn money without any risk using market unbalance. For example, assume that one bank buys US dollars at the rate 0.63 (£ per dollar) and sells US dollars at the rate 0.64 (£ per dollar) and another bank buys US dollars at the rate 0.60 (£ per dollar) and sells US dollars at the rate 0.62 (£ per dollar). Then a person having USD100 can sell this amount to the first bank and get £63. Then this amount is sold to the second bank and the person receives $\frac{63}{0.62} = 101.61$ (USD). The profit USD1.61 is earned without any risk (unless the

[12]The Institute of Actuaries, Exam CT1, April 2008, Problem 1

second bank change its rates before the transaction is completed). Obviously, in this situation many people will sell dollars to the first bank and then buy dollars from the second bank. This excessive demand for pounds in the first bank and dollars in the second bank will cause pressure to reduce the purchase price (of dollars) in the first bank and increase sell price in the second bank to the levels when the opportunity to make money from "air" disappears.

The "no arbitrage" assumption means that this and similar situations are not possible (which is theoretically the case in financial markets). In particular, it means that *if two financial instruments have identical cash flows, then they must have identical prices.*

Now consider the notion of *forward contract.*

We know that on 1 March 2008 (time $t_0 = 0$) one share under consideration has a price of $S = £10$. This price assumes that if at time $t_0 = 0$ someone (the investor) pays to the owner of the share this amount in cash, then in return he immediately, on the spot, receives this share. Correspondingly this price is said to be the cash price or the spot price.

In the problem, however, other situation is discussed. It is assumed that the seller agrees to sell the share on 1 February 2009 (time $T > 0$) and the buyer agrees to buy it. It should be noted that in this deal money and shares will change hands at time T rather than today (at time 0), so that in fact at time 0 (when the deal is discussed) "the seller" need not to own any share and "the buyer" need not to have any money. The seller of the share is said to hold a short forward position and the buyer of the share is said to hold a long forward position.

Clearly nobody knows the market price of the share in 11 month time from now. Thus, both parties must reach an agreement about the price K in their specific deal. This price is said to be the forward price.

To calculate this price consider two scenarios of behavior for an investor.

1. At time $t_0 = 0$ buy one share at the spot price S and enter a forward contract to sell one share at forward price K at time T (in our problem $T = \frac{11}{12}$).

During the interval $[0, T]$ the share provides to the shareholder dividends: $D_1 = 0.50$ at time $t_1 = \frac{1}{12}$ and $D_2 = 0.50$ at time $t_1 = \frac{7}{12}$. At time T according to the forward contract the shareholder will sell the share for the forward price and receive amount K.

Thus, in return for the initial investment of S this policy produces the following cash flow:

$$D_1 \text{ at time } t_1,\ D_2 \text{ at time } t_2,\ K \text{ at time } T. \tag{6.7}$$

2. At time $t_0 = 0$ invest an amount $Kv^T + D_1 v^{t_1} + D_2 v^{t_2}$ (where $v = frac11 + i$ is the discount factor which corresponds to the risk-free rate of interest i) in the risk free-investment.

By time t_1 this amount accumulates to

$$\left(Kv^T + D_1 v^{t_1} + D_2 v^{t_2}\right)(1+i)^{t_1} = Kv^{T-t_1} + D_1 + D_2 v^{t_2 - t_1}.$$

Then at time t_1 the investor should withdraw amount D_1 and reinvest the rest, i.e. amount $Kv^{T-t_1} + D_2 v^{t_2-t_1}$.

By time t_2 this amount accumulates to

$$\left(Kv^{T-t_1} + D_2 v^{t_2-t_1}\right)(1+i)^{t_2-t_1} = Kv^{T-t_2} + D_2.$$

Then at time t_2 the investor should withdraw amount D_2 and reinvest the rest, i.e. amount Kv^{T-t_2}.

By time T this amount accumulates to

$$\left(Kv^{T-t_2}\right)(1+i)^{T-t_2} = K,$$

so that at this time the investor can receive amount K.

Thus, in return for the initial investment of $Kv^T + D_1 v^{t_1} + D_2 v^{t_2}$ this policy produces the following cash flow:

$$D_1 \text{ at time } t_1, \; D_2 \text{ at time } t_2, \; K \text{ at time } T. \tag{6.8}$$

Since cash flows (6.7) and (6.8) are identical, the "no arbitrage" assumption yields that prices of both instruments must be identical:

$$S = Kv^T + D_1 v^{t_1} + D_2 v^{t_2},$$

and thus the forward price K is given by the following formula:

$$K = S(1+i)^T - D_1(1+i)^{T-t_1} - D_2(1+i)^{T-t_2}. \tag{6.9}$$

To put this in other words, the forward price is the accumulation of the spot price (at the risk-free rate of interest) minus the total accumulation from all dividends.

In our case this result gives:

$$K = 10 \cdot 1.05^{\frac{11}{12}} - 0.5 \cdot 1.05^{\frac{10}{12}} - 0.5 \cdot 1.05^{\frac{4}{12}} \approx 9.43.$$

The main formula (6.9) can be rewritten as

$$K = \left(S - D_1 v^{t_1} - D_2 v^{t_2}\right) \cdot (1+i)^T = P \cdot (1+i)^T,$$

where $P = S - D_1 v^{t_1} - D_2 v^{t_2}$ is the present value of the cash flow formed by the spot price and (negative) dividends. In this form it is convenient for the use of Microsoft Excel.

First in cells A1,A2,A3 enter the dates 1 March 2008, 1 April 2008, 1 October 2008 (note that these cells should be formatted accordingly) and in cells B1, B2, B3 the corresponding cash flows: 10, -0.5, -0.5.

Then the present value P can be calculated with the help of function XNPV. To do this in a cell, say C1, enter the formula: `=XNPV(0.05,B1:B3,A1:A3)` and press Enter. Then in the cell C1 you will see the value of P; in our case: 9.016167859.

To calculate the forward price as the accumulation of P over 11 months, enter in a cell, say C2, the formula: `=C1*1.05^(11/12)` and press Enter. Then in the cell C2 you will see the value of K; in our case: 9.43.

It should be noted that if we calculate the value of P as

$$10 - 0.5 \cdot 1.05^{-\frac{1}{12}} - 0.5 \cdot 1.05^{-\frac{7}{12}},$$

then we get a slightly different result: $P = 9.016058662$. The difference is connected with the different ways of calculating the length of time intervals: if we think that interval between 1 March and 1 October contains 214 days (as it is assumed by XNPV function) then it is $\frac{214}{365} = 0.58630137$ of year, if we think that interval between 1 March and 1 October contains 213 days then it is $\frac{213}{365} = 0.583561644$ of year, if we think of this interval as 7 months then it is $\frac{7}{12} = 0.583333333$ of year.
□

Problem 6.13 [13] *A share currently trades at £*10 *and will pay a dividend of 50p in one month's time. A six-month forward contract is available on the share for £*9.70*. Show that an investor can make a risk-free profit if the risk-free force of interest is* 3% *per annum.*

Solution. First an investor should borrow £10 at the risk-free rate of interest, i.e. at 3% per annum effective, for 6 months. As a consequence, he must return in six months time the amount $10e^{\frac{1}{2}\cdot 0.03} \approx 10.1511$.

To get this amount he must buy one share £10. Owning this share he can inter the forward contract to sell one share for £9.70 in six moths time without any risk. Besides, he will get a dividend of £)0.50 in one month's time, which should be invested at the risk-free rate of interest for 5 months. As a result he will get $0.5e^{\frac{5}{12}\cdot 0.03} \approx 0.5063$. With £9.70 for the share he will get approximately £10.2063.

This will allow him to repay 10.15, so that the net income is approximately 0.0552. This profit can be obtained without any risk.
□

Problem 6.14 [14] *A one-year forward contract is issued on 1 April 2007 on a share with a price of 900p at this date. Dividends of 50p per share are expected on 30 September 2007 and 31 March 2008. The 6-month and 12-month spot, risk-free rates of interest are $i_1 = 5\%$ and $i_2 = 6\%$ per annum effective respectively on 1 April 2007.*

Calculate the forward price at issue, stating any assumptions.

Solution. Let 1 April 2007 be time $t_0 = 0$, so that 30 September 2007 is time $t_1 = \frac{1}{2}$ and 31 March 2008 is the time $T = 1$.

To calculate the forward price at issue consider two scenarios of behavior for an investor.

1. At time $t_0 = 0$ buy one share at the spot price $S = £9$ and enter a forward contract to sell one share at forward price K at time $T = 1$.

During the interval $[0, T]$ the share provides to the shareholder dividends: $D_1 = 0.50$ at time $t_1 = \frac{1}{2}$ and $D_2 = 0.50$ at time $T = 1$. At time $T = 1$

[13] The Institute of Actuaries, Exam CT1, April 2006, Problem 5

[14] The Institute of Actuaries, Exam CT1, September 2007, Problem 5

according to the forward contract the shareholder will sell the share for the forward price and receive amount K.

Thus, in return for the initial investment of $S = £9$ this policy produces the following cash flow:

$$D_1 \text{ at time } t_1 = \tfrac{1}{2},\ D_2 + K \text{ at time } T = 1. \tag{6.10}$$

2. At time $t_0 = 0$ invest

- the amount $D_1 v_1^{t_1}$ (where $v_1 = \frac{1}{1+0.05}$ is the discount factor which corresponds to 6-month spot, risk-free rates of interest) for 6 months
- the amount $Kv_2^T + D_2 v_2^T$ (where $v_2 = \frac{1}{1+0.06}$ is the discount factor which corresponds to the 12-month spot, risk-free rate of interest) for 1 year.

At time t_1 the first investment will provide the amount D_1 and at time T the second investment will provide the amount $K + D_2$.

Thus, in return for the initial investment of $D_1 v_1^{t_1} + Kv_2^T + D_2 v_2^T$ this policy produces the following cash flow:

$$D_1 \text{ at time } t_1,\ D_2 + K \text{ at time } T. \tag{6.11}$$

Since cash flows (6.10) and (6.11) are identical, the "no arbitrage" assumption yields that prices of both instruments must be identical:

$$S = Kv_2^T + D_1 v^{t_1} + D_2 v^T,$$

and thus the forward price K is given by the following formula:

$$K = S(1 + i_2) - D_1 \frac{1 + i_2}{\sqrt{1 + i_1}} - D_2 \approx 8.52277. \tag{6.12}$$

□

6.3 Annuities

Problem 6.15 [15] *A bank offers two repayment alternatives for a loan that is to be repaid over ten years. The first requires the borrower to pay £1,200 per annum quarterly in advance and the second requires the borrower to make payments at an annual rate of £1,260 every second year in arrears.*

Determine which term would provide the best deal for the borrower at a rate of interest of 4% per annum effective.

Solution. Divide the ten year period into 5 sub-periods:

$$(0, 2], (2, 4], (4, 6], (6, 8], (8, 10].$$

[15] The Institute of Actuaries, Exam CT1, September 2008, Problem 5

According to the first alternative, during the first sub-period the borrower makes 8 payments, £300 each, at the beginning of each quarter. If we consider one quarter as the basic unit of time, then this cash flow is the standard annuity-due; corresponding technical rate of interest is $i_*^{(4)} = (1+i)^{\frac{1}{4}} - 1$, where $i = 0.04$ is the main technical rate of interest. By the end of sub-period $(0, 2])$ these payments accumulate to

$$S = 300\ddot{s}_{\overline{8}|} = 300\frac{(1+i_*^{(4)})^8 - 1}{i_*^{(4)}} = 300\frac{2i + i^2}{\sqrt[4]{1+i} - 1} \approx 2484.42.$$

Now we can replace 8 payments , £300 each, at the beginning of each quarter, by a single payment of £2484.42 at the end of the sub-period.

Now the first alternative assumes 5 payments of £2484.42 at times $2, 4, 6, 8, 10$, whereas the second alternative assumes 5 payments of £2520 at the same times. Clearly, the borrower should prefer the first alternative.

It should be noted that the answer strongly depends on the technical rate of interest i used to evaluate cash flows. Say, if $i = 6\%$, then $S = 2526.94$ and the borrower should prefer the second alternative.

If $i = 5.67404795\%$ then $S = 2520$, so that both options are equivalent.

□

Problem 6.16 [16] *An annuity certain with payments of £*150 *at the end of each quarter is to be replaced by an annuity with the same term and present value, but with payments at the beginning of each month instead.*

Calculate the revised payments, assuming annual force of interest of 10%.

Solution. Let n (years) is the duration of the original annuity under consideration. If $i = 10\%$ is annual effective rate of interest, then the effective rate of interest per quarter is $i_*^{(4)} = (1+i)^{\frac{1}{4}} - 1$. Consider one quarter as a new unit if time. Then this annuity certain with payments of £150 at the end of each quarter is the standard immediate annuity with $4n$ payments of £150 each. Thus its present value is

$$150\, a_{\overline{4n}|}\Big|_{@i_*^{(4)}} = 150\frac{1 - (1+i_*^{(4)})^{-4n}}{i_*^{(4)}} = 150\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{4}} - 1}.$$

Now consider the revised annuity. Let £X be the amount of a level monthly payment. The effective rate of interest per month is $i_*^{(12)} = (1+i)^{\frac{1}{12}} - 1$. Consider one month as a new unit if time. Then this annuity certain with payments of £150 at the beginning of each month is the standard annuity due with $12n$ payments of £X each. Thus its present value is

$$X\, \ddot{a}_{\overline{4n}|}\Big|_{@i_*^{(12)}} = X(1+i_*^{(12)})\frac{1-(1+i_*^{(12)})^{-12n}}{i_*^{(12)}} = X(1+i)^{\frac{1}{12}}\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{12}} - 1}.$$

[16] The Institute of Actuaries, Exam CT1, April 2006, Problem 2

Since both annuities must have the same present value,

$$150\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{4}}-1} = X(1+i)^{\frac{1}{12}}\frac{1-(1+i)^{-n}}{(1+i)^{\frac{1}{12}}-1},$$

which yields that

$$X = 150(1+i)^{-\frac{1}{12}}\frac{(1+i)^{\frac{1}{12}}-1}{(1+i)^{\frac{1}{4}}-1} \approx 49.211 \text{ (pounds).} \tag{6.13}$$

It is interesting to note that the answer does not depend on the duration of the original annuity under consideration (although it is not obvious a priori).

This solution can be written shorter with the help of the formulae for annuities payable $p$thly.

The present value of the first annuity is $600 \cdot a^{(4)}_{\overline{n}|} = 600\frac{1-v^n}{i^{(4)}}$ (here $600 = 4 \cdot 150$ is the algebraic value of all annual payments).

The present value of the second annuity is $12X \cdot \ddot{a}^{(12)}_{\overline{n}|} = 12X\frac{1-v^n}{d^{(12)}}$ (here $12X$ is the algebraic value of all annual payments).

Since

$$600\frac{1-v^n}{i^{(4)}} = 12X\frac{1-v^n}{d^{(12)}},$$

we can write X as follows:

$$X = 50\frac{d^{(12)}}{i^{(4)}} = 150\frac{1-(1+i)^{-\frac{1}{12}}}{(1+i)^{\frac{1}{4}}-1},$$

which is identical to (6.13). □

Problem 6.17 [17] *An investor pays 400 every half-year in advance into a 25-year savings plan.*

Calculate the accumulated fund at the end of the term if the interest rate is 6% per annum convertible monthly for the first 15 years and 6% per annum convertible half-yearly for the final 10 years.

Solution.

Consider the first 15 years of the project and take 1 month as a unit of time and £400 as a new monetary unit.

The effective rate of interest for this new unit of time is $i_* \equiv i_*^{(12)} = \frac{1}{12}i^{(12)}$, where $i^{(12)}$ is the annual nominal rate of interest convertible monthly:

$$i_* = 0.5\%.$$

Correspondingly the accumulation factor $k_* = 1 + i_*$ is 1.005.

[17] The Institute of Actuaries, Exam CT1, April 2007, Problem 1

The length of the first period of the project is $n = 12 \cdot 15 = 180$ (units of time), so that the cash flow during the first 15 years can be thought of as an annuity due with $n = 180$ payments. Its value at the end the 15 years period is

$$\ddot{s}_{\overline{n}|} = k_*^n + k_*^{n-1} + \cdots + k_* = \frac{k_*^n - 1}{d_*},$$

where $d_* = \frac{i_*}{1+i_*} = \frac{0.005}{1.005}$ the corresponding effective rate of discount.

Thus the accumulation by the end of the first 15 years is

$$A_1' = 1.005\frac{1.005^{180} - 1}{0.005} = 292.2728 \text{ (new monetary units) } \equiv \pounds 116\ 909.12.$$

Now consider the final 10 years of the project and take 6 month as a unit of time. The effective rate of interest for this new unit of time is $i_* \equiv i_*^{(2)} = \frac{1}{2}i^{(2)}$, where $i^{(2)}$ is the annual nominal rate of interest convertible half-yearly:

$$i_* = 3\%.$$

Correspondingly the accumulation factor $k_* = 1 + i_*$ is 1.03.

The length of the second period of the project is $n = 2 \cdot 10 = 20$ (units of time), so that the amount $A_1' = \pounds 116\ 909.12$ by by the end of the final 10 years becomes

$$A_1 = A_1' \cdot k_*^{20} = \pounds 211\ 150.88.$$

The cash flow of the regular payments of the investor during the final 10 years can be thought of as an annuity due with $n = 20$ payments. Its value at the end the 10 years period is

$$\ddot{s}_{\overline{n}|} = k_*^n + k_*^{n-1} + \cdots + k_* = \frac{k_*^n - 1}{d_*},$$

where $d_* = \frac{i_*}{1+i_*} = \frac{0.03}{1.03}$ the corresponding effective rate of discount.

Thus the accumulation of these payments by the end of the final 10 years is

$$A_2 = 1.03\frac{1.03^{20} - 1}{0.03} = 27.67649 \text{ (new monetary units) } \equiv \pounds 11\ 070.59.$$

The total accumulated fund at the end of the term is $A_1 + A_2 = 222\ 221.47$. □

Problem 6.18 [18] *A pension fund purchased an office block nine months ago for £5 million.*

The pension fund will spend a further £900,000 on refurbishment in two months time.

A company has agreed to occupy the office block six months from now. The lease agreement states that the company will rent the office block for fifteen years

[18] The Institute of Actuaries, Exam CT1, April 2009, Problem 6

and will then purchase the property at the end of the fifteen year rental period for £6 million.

It is further agreed that rents will be paid quarterly in advance and will be increased every three years at the rate of 4% per annum compound. The initial rent has been set at £800,000 per annum with the first rental payment due immediately on the date of occupation.

Calculate, as at the date of purchase of the office block, the net present value of the project to the pension fund assuming an effective rate of interest of 8% per annum.

Solution. Consider £1 million as a new monetary unit and the date of purchase of the office block as time $t = 0$. and assume that time is measured in years.

The cash flow under consideration consists of negative payments, i.e. expenses, and positive payments, i.e. income from sale the property and rental income.

There are two negative payments:

1. $c_1 = -5$ at time $t_1 = 0$;
2. $c_2 = -0.9$ and time $t_2 = \frac{11}{12}$.

The present value of these payments is (below $v = \frac{1}{1+i} = \frac{1}{1.08}$ is a discount factor)

$$a_1 = c_1 v^{t_1} + c_2 v^{t_2} = -5 \cdot v^0 - 0.9 \cdot v^{\frac{11}{12}} = -5.838695025.$$

Now consider positive payments.

The company will occupy the office block six months from now, i.e. 15 months from the date $t = 0$ of purchase of the office block, and will buy the property at the end of the fifteen year rental period, i.e. at time $t_s = 15\frac{15}{12} = 16.25$. Since the property will be sold for the amount 6, the present value of this payment is

$$6 \cdot v^{16.25} = 1.717968608.$$

Rental payments form a specific annuity payable from time $t_0 = \frac{15}{12}$. It can be divided into 5 three year periods with the level annual rents:

1. from time $t_0 = \frac{15}{12}$ till time $t_1 = t_0 + 3$ annual payment is 0.8, so that quarterly payment is 0.2;
2. from time t_1 till time $t_2 = t_1 + 3$ annual payment is $0.8 \cdot 1.04^3$, so that quarterly payment is $0.2 \cdot 1.04^3$;
3. from time t_2 till time $t_3 = t_2 + 3$ annual payment is $0.8 \cdot 1.04^6$, so that quarterly payment is $0.2 \cdot 1.04^6$;
4. from time t_3 till time $t_4 = t_3 + 3$ annual payment is $0.8 \cdot 1.04^9$, so that quarterly payment is $0.2 \cdot 1.04^9$;
5. from time t_4 till time $t_5 = t_4 + 3$ annual payment is $0.8 \cdot 1.04^{12}$, so that quarterly payment is $0.2 \cdot 1.04^{12}$.

During each such period rental payments form a standard annuity due payable quarterly. Thus its present value at the beginning of the corresponding period is (below p is the annual payment for the corresponding period)

$$p \cdot \ddot{a}^{(4)}_{\overline{3}|} = p \frac{1 - v^3}{4\left(1 - v^{\frac{1}{4}}\right)}.$$

If we replace rental payments during such a period by its present value payable at the beginning of the period, then the flow of rental income will consists of 5 payments:

1. payment of $0.2 \frac{1-v^3}{1-v^{\frac{1}{4}}}$ at $t_0 = \frac{15}{12}$;
2. payment of $0.2 \cdot 1.04^3 \frac{1-v^3}{1-v^{\frac{1}{4}}}$ at $t_1 = 3\frac{15}{12}$;
3. payment of $0.2 \cdot 1.04^6 \frac{1-v^3}{1-v^{\frac{1}{4}}}$ at $t_2 = 6\frac{15}{12}$;
4. payment of $0.2 \cdot 1.04^9 \frac{1-v^3}{1-v^{\frac{1}{4}}}$ at $t_3 = 9\frac{15}{12}$;
5. payment of $0.2 \cdot 1.04^{12} \frac{1-v^3}{1-v^{\frac{1}{4}}}$ at $t_4 = 12\frac{15}{12}$.

The present value of this cash flow is

$$0.2 \frac{1 - v^3}{1 - v^{\frac{1}{4}}} v^{\frac{15}{12}} \left(1 + 1.04^3 v^3 + 1.04^6 v^6 + 1.04^9 v^9 + 1.04^{12} v^{12}\right) \approx 7.936165561.$$

Thus, the present value of positive payments is

$$1.717968608 + 7.936165561 = 9.654134168,$$

so that the present value of the project is

$$-5.838695025 + 9.654134168 = 3.815439143 \text{ (new monetary units)},$$

i.e. £3 815 439.14.

□

Problem 6.19 [19] *Jim began saving money for his retirement by making monthly deposits of 200 into a fund earning 6% interest compounded monthly. The first deposit occurred on January 1, 1985. Jim became unemployed and missed making deposits 60 through 72. He then continued making monthly deposits of 200.*

How much did Jim accumulate in his fund on December 31, 1999?

[19] *Course/Exam 2 – Economics, Finance, and Interest Theory,* The Society of Actuaries and the Casualty Actuarial Society, May 2000, problem No.47

Solution. Let one month is a new unit of time, and \$200 is a new monetary unit. The effective rate of interest for this unit interval is $i_* = \frac{1}{12} \cdot 0.06 = 0.005$.

Monthly deposits from 1 January 1985 till 1 November 1989 (inclusive) can be thought of as the standard level annuity. Since the total number of deposits is $n = 59$, the value of this annuity at the time of the final payment (1 November 1989) is

$$\begin{aligned} s_{\overline{59|}} &= (1+i_*)^{58} + (1+i_*)^{57} + \cdots + (1+i_*) + 1 \\ &= \frac{(1+i_*)^{59} - 1}{i_*} \approx 68.42789. \end{aligned}$$

Period from 1 November 1989 till 31 December 1999 consists of 122 months. Thus by the end of 1999 this amount accumulates to

$$s_{\overline{59|}} \cdot (1+i_*)^{122} \approx 125.7455688.$$

Monthly deposits from 1 January 1991 till 1 December 1999 (inclusive) can be thought of as the standard level annuity. Since the total number of deposits is $n = 108$, the value of this annuity at the time of the final payment (1 December 1999) is

$$\begin{aligned} s_{\overline{108|}} &= (1+i_*)^{107} + (1+i_*)^{106} + \cdots + (1+i_*) + 1 \\ &= \frac{(1+i_*)^{108} - 1}{i_*} \approx 142.7399. \end{aligned}$$

For the period from 1 December 1999 till 31 December 1999 this amount grows to

$$s_{\overline{108|}} \cdot (1+i_*) \approx 143.453599.$$

Thus on 31 December 1999 Jim accumulates in his fund 269.199168 (new monetary units), i.e. in absolute figures approximately \$53 839.83. □

Problem 6.20 [20] *An actuarial student has created an interest rate model under which the annual effective rate of interest is assumed to be fixed over the whole of the next ten years. The annual effective rate is assumed to be 2%, 4% and 7% with probabilities 0.25, 0.55 and 0.2 respectively.*

(a) Calculate the expected accumulated value of an annuity of £800 per annum payable annually in advance over the next ten years.

(b) Calculate the probability that the accumulated value will be greater than £10,000.

Solution. The annuity under consideration is annuity due with annual payment £800 and the number of payments $n = 10$. Thus its accumulated value is (in pounds)

$$A = 800\ddot{s}_{\overline{10|}} = 800(1+i)\frac{(1+i)^{10} - 1}{i}.$$

[20] The Institute of Actuaries, Exam CT1, April 2006, Problem 6

Since the effective rate of interest i is a random variable, so is A. This random variable takes tree values:

$A_1 = 800(1+i)\frac{(1+i)^{10}-1}{i}\Big|_{@i=2\%} \approx 8\ 934.97$ with probability 0.25;

$A_2 = 800(1+i)\frac{(1+i)^{10}-1}{i}\Big|_{@i=4\%} \approx 9\ 989.08$ with probability 0.55;

$A_3 = 800(1+i)\frac{(1+i)^{10}-1}{i}\Big|_{@i=7\%} \approx 11\ 826.88$ with probability 0.2.

Thus,

$$EA = A_1 \cdot 0.25 + A_2 \cdot 0.55 + A_3 \cdot 0.2 \approx 10\ 093.11.$$

Random variable A is greater than 10 000 only when $i = 7\%$, so that

$$P(A > 10\ 000) = P(i = 7\%) = 0.2.$$

□

Problem 6.21 [21] *An individual wishes to receive an annuity which is payable monthly in arrears for 15 years. The annuity is to commence in exactly 10 years at an initial rate of* £$12,000$ *per annum. The payments increase at each anniversary by* 3% *per annum. The individual would like to buy the annuity with a single premium 10 years from now.*

(i) *Calculate the single premium required in 10 years' time to purchase the annuity assuming an interest rate of* 6% *per annum effective.*

The individual wishes to invest a lump sum immediately in an investment product such that, over the next 10 years, it will have accumulated to the premium calculated in (i). The annual effective returns from the investment product are independent and $(1+i_k)$ *is lognormally distributed, where* i_k *is the return in the* k*th year. The expected annual effective rate of return is* 6% *and the standard deviation of annual returns is* 15%*.*

(ii) *Calculate the lump sum which the individual should invest immediately in order to have a probability of 0.98 that the proceeds will be sufficient to purchase the annuity in 10 years' time.*

(iii) *Comment your answer to (ii).*

Solution. Consider one annuity year. During this period the individual will receive 12 payments, £1 000 each, at the end of every month. If we consider one month as the basic unit of time, then this cash flow is a standard immediate annuity (to be evaluated at the rate $i_*^{(12)} = (1+i)^{\frac{1}{12}} - 1$), so that its present value at the beginning of the year under consideration is

$$\begin{aligned} a &= 1000 a_{\overline{12}|} = 1000 \frac{1-(1+i_*^{(12)})^{-12}}{i_*^{(12)}} \\ &= 1000 \frac{i}{1+i} \frac{1}{\sqrt[12]{1+i}-1} \approx 11\ 628.80. \end{aligned}$$

[21] The Institute of Actuaries, Exam CT1, April 2009, Problem 11

Now we can replace 12 monthly payments during the annuity year by one payment of $a = 11\ 628.80$ at the beginning of the year. However, since the payments increase at each anniversary by 3% per annum, as a matter of fact the kth payment, $k = 1, \dots, 15$, is $1.03^{k-1}a$.

Thus the present value of the flow of payments at the time of commencement of the annuity is

$$P = \sum_{k=1}^{15} a1.03^{k-1}1.06^{-k+1} = a\sum_{l=0}^{14}\left(\frac{103}{106}\right)^l = a\frac{1-\left(\frac{103}{106}\right)^{15}}{1-\frac{103}{106}} \approx 143\ 774.42.$$

This is exactly the single premium required in 10 years' time to purchase the annuity.

If the individual invest a lump sum X immediately in an investment product which earns interest at the rate i_k in year k, then in 10 years he will have accumulation $S = X\prod_{k=1}^{10}(1+i_k)$.

The mean value of the accumulation depends only on the mean rate of return $i = Ei_k$ and is

$$ES = X\prod_{k=1}^{10} E(1+i_k) = X\prod_{k=1}^{10}(1+Ei_k) = X(1+i)^{10} = X1.06^{10} \approx 1.8X.$$

Thus, if the individual invest a lump sum

$$X_0 = \frac{P}{(1+i)^{10}} \approx 80282.89,$$

then *expected* accumulation will be exactly the single premium $P = 143\ 774.42$ required in 10 years' time to purchase the annuity.

However, since the rates i_k are random variables, so is the accumulation factor $A = \prod_{k=1}^{10}(1+i_k)$. Its distribution function is (for positive values of x)

$$F_A(x) = P(A < x) = P(\ln A < \ln x) = P\left(\sum_{k=1}^{10}\ln(1+i_k) < \ln x\right).$$

Random variables $\ln(1+i_k)$ are independent and have identical normal distribution with some parameters a and σ. We know that in this case

$$\begin{aligned} e^{a+\frac{\sigma^2}{2}} &= E(1+i_k), \\ e^{\sigma^2} - 1 &= \frac{Var(1+i_k)}{(E(1+i_k))^2}, \end{aligned}$$

so that

$$\begin{aligned} \sigma &= \sqrt{\ln\left(\frac{Vari_k}{(1+i)^2}+1\right)}, \\ a &= \ln(1+i) - \frac{1}{2}\ln\left(\frac{Vari_k}{(1+i)^2}+1\right). \end{aligned}$$

In our case, when $i = 0.03$ and $Var i_k = 0.15$, we have:

$$\begin{aligned} \sigma &\approx 0.140808587, \\ a &\approx 0.048355379. \end{aligned}$$

Random variable $\sum_{k=1}^{10} \ln(1 + i_k)$ has a normal distribution with the mean $10a$ and the standard deviation $\sqrt{10}\sigma$. Thus shifted and scaled random variable

$$\frac{\sum_{k=1}^{10} \ln(1 + i_k) - 10a}{\sqrt{10}\sigma}$$

is the standard $N(0, 1)$ Gaussian variable.

It yields that

$$F_A(x) = \Phi\left(\frac{\ln x - 10a}{\sqrt{10}\sigma}\right).$$

Thus, the probability that the proceeds will be sufficient to purchase the annuity in 10 years' time is

$$P(S \geq P) = P(XA \geq P) = P\left(A \geq \frac{P}{X}\right) = 1 - F_A\left(\frac{P}{X}\right) = 1 - \Phi\left(\frac{\ln P - \ln X - 10a}{\sqrt{10}\sigma}\right).$$

If we are given the value α of this probability (α is close to 1), then

$$\frac{\ln P - \ln X - 10a}{\sqrt{10}\sigma} = z_{1-\alpha} = -z_\alpha,$$

where z_α is α percentile of the standard Gaussian distribution. Correspondingly,

$$X = Pe^{-10a + z_\alpha \sqrt{10}\sigma} = \frac{P}{(1+i)^{10}} e^{5\sigma^2 + z_\alpha \sqrt{10}\sigma} = X_0 e^{5\sigma^2 + z_\alpha \sqrt{10}\sigma},$$

where $X_0 = \frac{P}{(1+i)^{10}} \approx 80282.89$, is the lump sum for which *expected* accumulation would be exactly the single premium $P = 143\ 774.42$ required in 10 years' time to purchase the annuity.

The coefficient $e^{5\sigma^2 + z_\alpha \sqrt{10}\sigma}$ is 2.75549832, so that $X \approx 221\ 219.36$.

The value of the coefficient $K = e^{5\sigma^2 + z_\alpha \sqrt{10}\sigma}$ shows that the initial investment which would practically guarantee sufficient accumulation must be almost triple investment calculated on average scenario. This result is a consequence of the high standard deviation of i_k. It is well-known that if ξ is a normal random variable with the mean a and variance σ^2 that for sure (with probability 99.5%) we can say only that ξ is somewhere between $a - 3\sigma$ and $a + 3\sigma$. In our case it means that the force of interest $\delta_k = \ln(1 + i_k)$ lies somewhere between -0.374070383 and 0.470781141, or equivalently the annual rate of return i_k lies somewhere between -0.312071508 and 0.601244503. Thus, although the average rate of return is 6% it can be as high as 60%, but can be -31%, which means losses for the investor.

However, if $Var i_k$ is small, then the coefficient K is closer to 1, so that X is closer to X_0. Say, if the standard deviation $\sqrt{Var i_k}$ is 1% then $K \approx 1.06$.

□

6.4 Continuous models

Problem 6.22 [22] *The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:*

$$\delta(t) = \begin{cases} 0.04, & \text{if } 0 \le t \le 5, \\ 0.008t, & \text{if } 5 < t \le 10, \\ 0.005t + 0.0003t^2, & \text{if } 10 < t. \end{cases}$$

(i) calculate the present value of a unit sum of money due at time $t = 12$.

(ii) Calculate the effective annual rate of interest over the 12 years.

(iii) Calculate the present value at time $t = 0$ of a continuous payment stream that is paid at the rate of $e^{-0.05t}$ per unit time between time $t = 2$ and time $t = 5$.

Solution. (i) If $\delta(t)$ is the force of interest, then the present value at time $t_0 = 0$ of a unit sum of money due at time $t = 12$ is:

$$PV = \exp\left\{-\int_0^{12} \delta(t)dt\right\}.$$

Since the function $\delta(t)$ is piece-wise, it is convenient to replace the integral $\int_0^{12} \delta(t)dt$ as the sum:

$$\begin{aligned} \int_0^{12} \delta(t)dt &= \int_0^5 \delta(t)dt + \int_5^{10} \delta(t)dt + \int_{10}^{12} \delta(t)dt \\ &= \int_0^5 0.04dt + \int_5^{10} 0.008tdt + \int_{10}^{12} \left(0.005t + 0.0003t^2\right) dt. \end{aligned}$$

All integrals in the right-hand side can be easily calculated:

$$\begin{aligned} \int_0^5 0.04dt &= 0.04 \cdot (5 - 0) = 0.2, \\ \int_5^{10} 0.008tdt &= 0.004t^2\Big|_5^{10} = 0.3, \\ \int_{10}^{12} \left(0.005t + 0.0003t^2\right) dt &= 0.0025t^2 + 0.0001t^3\Big|_{10}^{12} = 0.1828, \end{aligned}$$

so that

$$PV = \exp(-0.6828) \approx 0.5052.$$

(ii) If i is effective annual rate of interest, then $PV = (1+i)^{-12}$, i.e.

$$i = \frac{1}{\sqrt[12]{PV}} - 1 \approx 5.855\%.$$

[22] The Institute of Actuaries, Exam CT1, April 2006, Problem 9

(iii) According to the general formula,

$$PV = \int_0^5 \rho(t)V(t)dt.$$

If a continuous payment stream is paid only between time $t = 2$ and time $t = 5$, then $\rho(t) = 0$ for $t \notin [2, 5]$, so that

$$PV = \int_2^5 \rho(t)V(t)dt.$$

Besides, on the interval $[0, 5]$ the force of interest is constant: $\delta(t) = 0.04$. Thus, for $t \in [0, 5]$,

$$V(t) \equiv \exp\left(-\int_0^t \delta(u)du\right) = e^{-0.04t}.$$

Then,

$$\begin{aligned} PV &= \int_2^5 \rho(t)V(t)dt = \int_2^5 e^{-0.05t}e^{-0.04t}dt = \int_2^5 e^{-0.09t}dt \\ &= \frac{1}{-0.09}e^{-0.09t}\Big|_2^5 = \frac{e^{-0.18} - e^{-0.45}}{0.09} \approx 2.196023. \end{aligned}$$

□

Problem 6.23 [23] *The force of interest, $\delta(t)$, is a function of time and at any time t, measured in years, is given by the formula:*

$$\delta(t) = \begin{cases} 0.06, & \text{if } 0 \le t \le 4, \\ 0.10 - 0.01t, & \text{if } 4 < t \le 7, \\ 0.01t - 0.04, & \text{if } 7 < t. \end{cases}$$

(i) calculate the value at time $t = 5$ of 1,000 due for payment at time $t = 10$.

(ii) Calculate the constant rate of interest per annum convertible monthly which leads to the same result as in (i) being obtained.

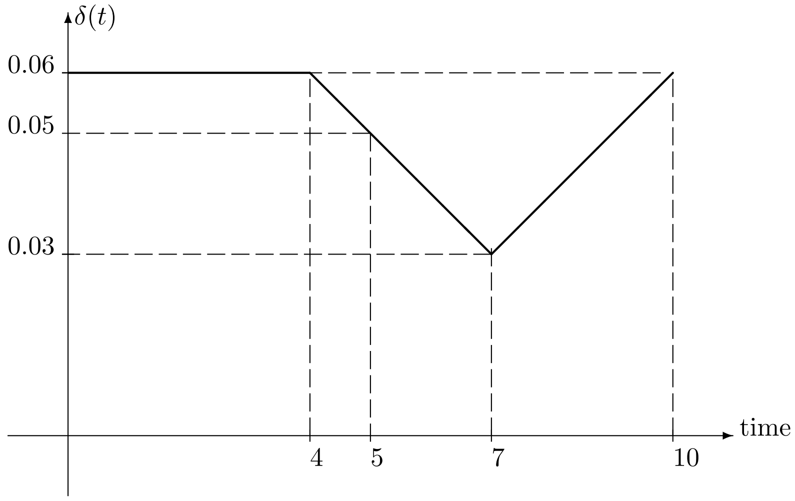
(iii) Calculate the accumulated amount at time $t = 12$ of a payment stream, paid continuously from time $t = 0$ to $t = 4$, under which the rate of payment at time t is $\rho(t) = 100e^{0.02t}$.

Solution. If $\delta(t)$ is the force of interest, then the discounted value at time t_1 of C due at time t_2 is:

$$PV = C \exp\left\{-\int_{t_1}^{t_2} \delta(t)dt\right\}.$$

In our case $\int_{t_1}^{t_2} \delta(t)dt \equiv \int_5^{10} \delta(t)dt$. This integral can be viewed as the area under the graph of the function $\delta(t)$.

Figure 6.2:



The graph of this function is shown on figure 6.2. Note that $\delta(5) = 0.05$, $\delta(7) = 0.03$, $\delta(10) = 0.06$.

This area, in turn, is the union of two trapezia. Thus,

$$\int_5^{10} \delta(t)dt = \frac{0.05 + 0.03}{2} \cdot 2 + \frac{0.03 + 0.06}{2} \cdot 3 = 0.215,$$

so that

$$PV = 1000 \cdot e^{-0.215} = 806.54.$$

(ii) If $i^{(12)}$ is the nominal rate of interest per annum convertible monthly which leads to the same result as in (i) being obtained, then

$$PV = 1000 \left(1 + \frac{i^{(12)}}{12}\right)^{-60},$$

i.e.

$$i^{(12)} = 12 \left(\sqrt[60]{\frac{1000}{PV}} - 1 \right) \approx 4.3077\%.$$

(iii) According to the general formula, the accumulated amount is

$$A = \int_0^{12} \rho(t) \exp\left(\int_t^{12} \delta(u)du \right) dt.$$

[23]The Institute of Actuaries, Exam CT1, April 2008, Problem 9

If a continuous payment stream is paid only between time $t = 0$ and time $t = 4$, then $\rho(t) = 0$ for $t \notin [0,4]$, so that

$$A = \int_0^4 \rho(t) \exp\left(\int_t^{12} \delta(u)du\right) dt.$$

Besides, for $t \in [0,5]$,

$$\begin{aligned}
\int_t^{12} \delta(u)du &= \int_t^4 \delta(u)du + \int_4^7 \delta(u)du + \int_7^{12} \delta(u)du \\
&= (4-t)\cdot 0.06 + \frac{0.06+0.03}{2}\cdot 3 + \frac{0.03+0.08}{2}\cdot 5 \\
&= 0.6 - 0.06t.
\end{aligned}$$

Then,

$$\begin{aligned}
A &= \int_0^4 100e^{0.02t}e^{0.6-0.06t}dt = 100e^{0.6}\int_0^4 e^{-0.04t}dt \\
&= \frac{100e^{0.6}}{-0.04}e^{-0.04t}\Big|_0^4 = 2500\left(e^{0.6} - e^{0.44}\right) \approx 673.529.
\end{aligned}$$

□

6.5 Assessment of investment projects

Problem 6.24 [24]

An ordinary share pays annual dividends. The next dividend is due in exactly eight months' time. This dividend is expected to be £1.10 per share. Dividends are expected to grow at a rate of 5% per annum compound from this level and are expected to continue in perpetuity. Inflation is expected to be 3% per annum. The price of the share is £21.50.

Calculate the expected effective annual real rate of return for an investor who purchases the share.

Solution. Let one year is taken as the basic unit of time. Then in return for the initial payment of the price $p_0 = £21.50$ at time $t_0 = 0$ the share will pay the first dividend $p_1 = £1.10$ at time $t_1 = \frac{8}{12} = \frac{2}{3}$, the second dividend $p_2 = £1.10 \cdot 1.05$ at time $t_2 = 1\frac{2}{3}$, the third dividend $p_3 = £1.10 \cdot 1.05^2$ at time $t_3 = 2\frac{2}{3}$, and so on, the kth dividend $p_k = £1.10 \cdot 1.05^{k-1}$ at time $t_k = (k-1) + \frac{2}{3} = k - \frac{1}{3}$ $(k = 1, 2, \dots)$. Thus the equation of value is

$$-p_0 + \sum_{k=1}^{+\infty} p_k v^{k-\frac{1}{3}} = 0 \Leftrightarrow -21.50 + 1.10\frac{v^{\frac{2}{3}}}{1-1.05v} = 0 \quad (\text{for } v < \tfrac{1}{1.05}). \quad (6.14)$$

[24] The Institute of Actuaries, Exam CT1, September 2007, Problem 3

Let $x = v^{\frac{1}{3}}$. Then this equation becomes

$$\frac{1.10x^2}{1 - 1.05x^3} = 21.50. \tag{6.15}$$

The function $f(x) = \frac{1.10x^2}{1-1.05x^3}$ is defined for $x < \frac{1}{\sqrt[3]{1.05}}$ (this condition is equivalent to $v < \frac{1}{1.05}$) and is increasing from $f(0) = 0$ to $+\infty$ when $x \in \left[0, \frac{1}{\sqrt[3]{1.05}}\right)$. Thus equation (6.15) has the unique root $x_0 \in \left[0, \frac{1}{\sqrt[3]{1.05}}\right)$.

This root can be found with the help of Microsoft Excel using the Goal Seek feature. To do this,

- Open a new, blank worksheet.
- In cell A1 enter any value for x, say, 0.
- In cell A2 enter the formula which defines $f(x)$; in our case:

$$= 1.1 * A1^2/(1 - 1.05 * A1^3)$$

- On the **Data** tab, in the **Data Tools** group, click **What-If Analysis**, and then click **Goal Seek**.
- In the **Set cell** box, enter the reference for the cell that contains the formula, i.e. A2.
- In the **To value** box, enter 21.50.
- In the **By changing cell** box, enter the reference for the cell that contains the value of x, i.e. A1.
- Click **OK**.

After this **Goal Seek** runs and produces the following result:

Goal Seek Status:
Goal Seeking with Cell A2 found a solution
Target value: 21.5
Current value: 21.500006

After you click **OK**, in the cell A1 you will see the root $x_0 = 0.967891178$.

Correspondingly, the equation of value (6.14) has the unique root $v_0 \in \left[0, \frac{1}{1.05}\right)$ which is $v_0 = x_0^3 = 0.906733361$. Now the rate of return i_0 can be found as $i_0 = \frac{1}{v_0} - 1 = 0.10286005$.

It should be noted that this is money rate of return, since it does not take into account the inflation. To find the real rate of return i_r, note that the money rate of return i_0 means the following: if we invest at time $t = 0$ amount 1, then we get amount $1 + i_0$ at time $t = 1$. However, purchase power of 1 at time $t = 0$ is the same as purchase power of $1 + f$ at time $t = 1$ ($f = 3\%$ is the inflation rate) or, equivalently, purchase power of $\frac{1}{1+f}$ at time $t = 0$ is the same as purchase power of 1 at time $t = 1$. Thus, in real money at time $t = 1$ we receive amount $\frac{1+i_0}{1+f}$ in return for investment of 1 at time $t = 0$. Thus,

$$i_r = \frac{1+i_0}{1+f} - 1 = \frac{i_0 - f}{1+f} = 0.0707.$$

□

Problem 6.25 [25] *An investor deposits 1000 on January 1, 2006, and deposits another 1000 on January 1 2008 into a fund that matures on January 1 2010. The interest rate on the fund differs every year and is equal to the annual effective rate of growth of the gross domestic product (GDP) during the 4th quarter of the previous year. The following are the relevant GDP values for the past 4 years:*

Table 6.1:

year	III quarter	IV quarter
2005	800.0	808.0
2006	850.0	858.5
2007	900.0	918.0
2008	930.0	948.6

What is the internal rate of return earned by the investor over the 4 year period?

Solution.

The *quarterly* effective rate of growth of the gross domestic product during the 4th quarter of 2005, 2006, 2007, 2008 is 1%, 1%, 2% and 2% respectively. Thus the annual interest rate on the fund is is $(1.01^4 - 1) \approx 4.06\%$ in 2006 and 2007, and $(1.02^4 - 1) \approx 8.24\%$ in 2008 and 2009. By 1 January 2010 investor's deposits accumulate to

$$\Big(1000 \cdot \big(1.01^4\big)^2 + 1000\Big) \cdot \big(1.02^4\big)^2 \approx 2440.40.$$

The net cash flow which describes this investment project is:

$$\begin{aligned} &t_1 = 0, t_2 = 1, t_3 = 2, t_4 = 3, t_5 = 4 \\ &c_1 = -1000, c_2 = 0, c_3 = -1000, c_4 = 0, c_5 = 2440.40. \end{aligned}$$

Thus the equation of value

$$\sum_k c_k (1+i)^{-t_k} = 0$$

becomes:

$$-1000 - 1000 \cdot (1+i)^{-2} + 2440.40 \cdot (1+i)^{-4} = 0.$$

[25] *Course 2 – Interest Theory, Economics and Finance,* The Society of Actuaries and the Casualty Actuarial Society, November 2000, problem No.51 – a revised version

This equation can be solved with the help of the new unknown $x = (1+i)^{-2}$; it has two real roots:

$$i_1 \approx 0.067822,\ i_2 \approx -2.067822.$$

The root we are interested in, must be greater than -1, so that the internal rate of return earned by the investor over the 4 year period is 6.7822%.

□

6.6 Loans

Problem 6.26 [26] *A loan is amortized over five years with monthly payments at a nominal interest rate of 9% compounded monthly. The first payment is 1000 and is to be paid one month from the date of the loan. Each succeeding monthly payment will be 2% lower than the prior payment.*

Calculate the outstanding loan balance immediately after the 40*th payment is made.*

Solution. Let one month is a unit of time. Then the effective rate of interest for this period, $i_*^{(12)}$, which corresponds to the nominal rate of interest $i^{(12)} = 9\%$ is

$$i_*^{(12)} = \frac{1}{12} i^{(12)} = 0.75\% = 0.0075.$$

Although we do not know the amount of the loan, the amounts of successive payments varies according to a simple law and can be easily found. The amount of the nth payment, p_n, is given by the formula:

$$p_n = 1000 \cdot (1-0.02)^{n-1},\ n = 1, \ldots, 60.$$

Thus lets calculate the outstanding loan balance immediately after the 40th payment is made by prospective method. At time 40 it is necessary to make 20 payments

$$1000 \cdot (0.98)^{40},\ 1000 \cdot (0.98)^{41},\ \ldots,\ 1000 \cdot (0.98)^{59}$$

at times $41, 42, \ldots, 60$ correspondingly. The present value of this cash flow at time 40, immediately after 40th payment, is

$$\begin{aligned} & \sum_{n=1}^{20} 1000 \cdot (0.98)^{39+n} (1.0075)^{-n} \\ = \quad & 1000 \frac{(0.98)^{40}(1.0075)^{-1} - (0.98)^{60}(1.0075)^{-21}}{1 - \frac{0.98}{1.0075}} \approx 6889.11. \end{aligned}$$

□

[26] *Course 2 – Interest Theory, Economics and Finance,* The Society of Actuaries and the Casualty Actuarial Society, November 2001, problem No.9

Problem 6.27 [27]

(i) *State the features of a eurobond.*

(ii) *An investor purchases a eurobond on the date of issue at a price of £97 per £100 nominal. Coupons are paid annually in arrear. The bond will be redeemed at par twenty years from the issue date. The rate of return from the bond is 5% per annum effective.*

(a) *Calculate the annual rate of coupon paid by the bond.*

(b) *Calculate the duration of the bond.*

Solution. Eurobond is a contract under which the issuer agrees to pay to the bond holder the par value (i.e. the amount stated on the face of the bond) at the maturity date, plus regular (usually, annual) interest payments (coupons) which are expressed as a percentage of the face value of the bond.

The purchase price of the eurobond is the sum of the present value of the face amount and the present value of cash flow of the interest payments calculated according to the rate of return from the bond (as a matter of fact, it is the internal rate of return for the transaction).

In the case under consideration:

- the face value is £100;
- the maturity date is $n = 20$ (from the date of issue);
- the interest payments form an immediate annuity with $n = 20$ level coupon payments (the amount x of each interest payment is unknown);
- the purchase price is £97;
- the present values are calculated according to the interest rate $i = 5\%$ (per annum).

Thus, the following equation holds:

$$\text{The purchase price} = \text{The face value} \cdot v^n + \text{The amount of each coupon} \cdot a_{\overline{n}|},$$

where the discount factor v and the present value of the standard immediate annuity $a_{\overline{n}|}$ are calculated at the interest rate $i = 5\%$ (per annum), or equivalently,

$$97 = 100v^{20} + xa_{\overline{20}|}.$$

From this:

$$x = \frac{97 - 100v^{20}}{a_{\overline{20}|}} = i\frac{97(1+i)^{20} - 100}{(1+i)^{20} - 1} \approx 4.75927. \tag{6.16}$$

[27] The Institute of Actuaries, Exam CT1, September 2005, Problem 6

To calculate the duration τ of the bond, note that the bond holder will receive 20 payments of coupons at times $t_1 = 1, t_2 = 2, \ldots, t_{20} = 20$ (the amount of each coupon is x) and the final payments of the face value £100 at time $t_f = 20$.

Since the present value of all these payments is £97 (the purchase price), the weights of the payments are as follows:

- for the kth coupon (due at time $t_k = k$) the weight is $w_k = \frac{xv^k}{97}$, $k = 1, 2, \ldots, 20$;
- for the final payments of the face value the weight is $w_f = \frac{100v^{20}}{97}$.

By the definition, the duration of the bond is the weighted average payment time

$$\tau = \sum_{k=1}^{20} w_k t_k + w_f t_f.$$

Thus,

$$\begin{aligned}
\tau &= \frac{1}{97}\left(x\sum_{k=1}^{20} kv^k + 2000v^{20}\right) \\
&= \frac{1}{97}\left(x\frac{v}{(1-v)^2}(1 - 21v^{20} + 20v^{21}) + 2000v^{20}\right) \\
&\approx 13.21467(\text{years}) \approx 13(\text{years})\ 2(\text{months})\ 18(\text{days}).
\end{aligned}$$

□

Problem 6.28 [28] *A loan is to be repaid by an annuity payable annually in arrear. The annuity starts at a rate of £300 per annum and increases each year by £30 per annum. The annuity is to be paid for 20 years.*

Repayments are calculated using a rate of interest of 7% per annum effective. Calculate:

(i) *The amount of the loan.*

(ii) *The capital outstanding immediately after the 5th payment has been made.*

(iii) *The capital and interest components of the final payment.*

Solution. (i) Since the annuity starts at a rate of £300 per annum and increases each year by £30 per annum, then the amount p_k of kth payment (at time $t_k = k$) is given by the formula $p_k = 270 + 30k$.

The amount L of the loan is the present value of the annuity; this present value is to be evaluated at the technical rate of interest $i = 0.07$. Since the amount to be paid off at time k is $270 + 30k$, we have:

$$\begin{aligned}
L &= \sum_{k=1}^{20}(270 + 30k)v^k = 270\sum_{k=1}^{20} v^k + 30v\sum_{k=1}^{20} v^{k-1} \\
&= 270f(v) + 30vf'(v),
\end{aligned}$$

[28] The Institute of Actuaries, Exam CT1, April 2009, Problem 3

where

$$f(v) = \sum_{k=1}^{20} v^k = a_{\overline{20}|} = \frac{v - v^{21}}{1 - v},$$

so that

$$f'(v) = \frac{1 - 21v^{20} + 20v^{21}}{(1-v)^2}.$$

Thus,

$$L = \frac{v}{(1-v)^2}\left(300 - 270v - 900v^{20} + 870v^{21}\right) \approx 5\ 503.476.$$

This present value can be calculated with the help of Microsoft Excel with the help of the function NPV$(i, p_1, p_2, \ldots, p_n)$. This function returns the present value of the series of payments $p_1, p_2, \ldots, p_n$ to be made at times $t_1 = 1, t_2 = 2, \ldots, t_n = n$ discounted according to the effective rate of interest per period i. In our case one should enter in a cell the following formula:

=NPV(0.07,300,330,360,390,420,450,480,510,540,570,600,630,660,690,720,750,780,810,840,870)

and then press "Enter".

Immediately after this in the cell we will see the value 5 503.48 (if the cell is formatted to display 2 decimal places).

(ii) The capital outstanding immediately after the 5th payment has been made is equal to the obligations of the borrower at this time. At time $t = 5$ the borrower still has to pay amounts $p_6, \ldots, p_{20}$ at time $t_6 = 6, \ldots, t_{20} = 20$ (in total, 15 payments). Thus

$$x_5 = \sum_{k=6}^{20} (270 + 30k) v^{k-5}.$$

To calculate the sum introduce a new index of summation $l = k - 5$:

$$\begin{aligned} x_5 &= \sum_{l=1}^{15} (270 + 30(l+5)) v^l = \sum_{l=1}^{15} (420 + 30l) v^l = 420 \sum_{k=1}^{15} v^l + 30v \sum_{l=1}^{15} v^{l-1} \\ &= 420 g(v) + 30 v g'(v), \end{aligned}$$

where

$$g(v) = \sum_{l=1}^{15} v^l = a_{\overline{15}|} = \frac{v - v^{16}}{1 - v},$$

so that

$$g'(v) = \frac{1 - 16v^{15} + 15v^{16}}{(1-v)^2}.$$

Thus,

$$x_5 = \frac{v}{(1-v)^2}\left(450 - 420v - 900v^{15} + 870v^{16}\right) \approx 5\ 671.94.$$

This present value can be calculated with the help of Microsoft Excel; the function NPV. In our case one should enter in a cell the following formula:

=NPV(0.07,450,480,510,540,570,600,630,660,690,720,750,780,810,840,870)

and then press "Enter".

Immediately after this in the cell we will see the value 5 671.94 (if the cell is formatted to display 2 decimal places).

(iii) The capital outstanding immediately after the 19th payment has been made is equal to the obligations of the borrower at this time. At time $t = 19$ the borrower still has to pay the final amount $p_{20} = 870$ at time $t_{20} = 20$. Thus

$$x_{19} = 870v = \frac{870}{1+i} \approx 813.08.$$

This is exactly the capital component of the final payment.

To find the interest components of the final payment we may either deduct x_{19} from the final payment: $870 - 813.08 = 56.92$, or calculate the interest due as $x_{19}i = 813.08 \cdot 0.07 \approx 56.92$.

□

Chapter 7

Appendix 1

7.1 The Actuarial Profession

In United Kingdom *The Actuarial Profession* is a term which is used to denote two (closely connected) professional organizations: The Institute of Actuaries and the Faculty of Actuaries.

By definition an actuary is a member of either The Institute of Actuaries or the Faculty of Actuaries. The actuarial qualification allows a person to get very well-paid job in an insurance company, a bank, a consulting company, etc. (which cannot be occupied without actuarial qualification).

To become an Associate or a Fellow of the Faculty or Institute of Actuaries it is necessary to make a study of a number of courses ordered by the Profession and prove the quality of the study by passing the examinations (as a rule).

7.2 Actuarial Examinations

The actuarial examinations usually are held twice a year (in April and September).

The exams are offered for those who has a status of a student member of the Institute.

The courses ordered by the Profession are divided into four groups (stages in terminology of the Profession):

1. Core Technical Stage
2. Core Applications Stage
3. Specialist Technical Stage
4. Specialist Applications Stage

The Core Technical Stage includes 9 courses:

1. Financial Mathematics (Course CT1)
2. Finance and Financial Reporting (Course CT2)
3. Probability and Mathematical Statistics (Course CT3)
4. Models (Course CT4)
5. Contingencies (Course CT5)
6. Statistical Methods (Course CT6)
7. Business Economics (Course CT7)
8. Financial Economics (Course CT8)
9. Business Awareness Module (Course CT9)

The Core Application Stage includes 3 courses:

1. Actuarial Risk Management (Course CA1)
2. Model documentation , analysis and reporting (Course CA2)
3. Communications (Course CA3)

The Specialist Technical Stage includes 9 courses:

1. Health and Care (Course ST1)
2. Life Insurance (Course ST2)
3. General Insurance (Course ST3 - in 2010 this subject has been replaced by subjects ST7 and ST8)
4. Pensions and Other Benefits (Course ST4)
5. Finance and Investment (Course ST5)
6. Finance and Investment (Course ST6)
7. General Insurance - Reserving and capital modeling (Course ST7)
8. General Insurance - Pricing (Course ST8)
9. Enterprize Risk Management (Course ST9)

The Specialist Applications Stage includes 6 courses:

1. Health and Care (Course SA1)
2. Life Insurance (Course SA2)
3. General Insurance (Course SA3)

4. Pensions and Other Benefits (Course SA4)

5. Finance (Course SA5)

6. Investment (Course SA6)

For Associateship students must gain passes in all the Core Technical (CT1-CT9) and Core Applications subjects (CA1-CA3). Students must also satisfy any other conditions required by Councils for Associateship.

For Fellowship, in addition, candidates must gain passes two Specialist Technical subjects (from the list ST1-ST9) and one Specialist Applications subject (from the list SA1-SA6). Candidates must also satisfy any other conditions required by Council of the Faculty of Actuaries or the Council of the Institute of Actuaries for Fellowship.

From April 2009 the Actuarial Profession is offering for any university student to take the first exam, CT1 (Financial Mathematics). In the case of success, the candidate receives a Certificate in Financial Mathematics.

The students who complete or are exempted from all of the Core Technical stage subjects (CT1–CT9) automatically receive The Diploma in Actuarial Techniques.

The students of the Faculty and Institute of Actuaries who complete or are exempted from the courses CT1, CT2, CT4, CT7, CT8, CT9 and CA1 automatically receive The Certificate in Finance and Investment.

During the exams candidates may use actuarial tables including a list of standard formulae and electronic calculators, but only the following calculators are permitted: Casio FX85, Hewlett Packard HP9S, Hewlett Packard HP 12C, Sharp EL531, Texas Instruments BA II Plus, Texas Instruments TI-30 (all the models with or without any suffix).

7.3 Exemptions

To attract students to join the actuarial profession the Institute and Faculty of Actuaries offer exemptions from some examinations for students of the universities which offer undergraduate and postgraduate actuarial, statistical, mathematical, economical programmes, particularly programmes accredited by the Profession. The syllabus coverage on university accredited programmes is equivalent but not necessarily identical to the Profession's syllabus. In UK the accredited programs are offered by University of Cambridge, University of Oxford Said Business School, City University London, London School of Economics, Imperial College Business School in London, University College Dublin, Heriot-Watt University in Edinburgh, University of Kent, University of Leicester, University of Manchester, Queen's University Belfast, University of Southampton, Swansea University.

Overseas the accredited programs are offered by:

In Ireland by University College Cork, Dublin City University

In Canada by University of Waterloo

In Australia by Australian National University, Macquarie University Sydney, University of Melbourne, University of New South Wales

In South Africa by University of Cape Town, University of the Free State (Bloemfontein), University of Kwazulu-Natal, North West University (Potchefstroom), University of Pretoria, Stellenbosch University, University of the Witwatersrand (Johannesburg)

In Hong Kong by University of Hong Kong

In Singapore by Nanyang Technological University

In Egypt by Cairo University

Usually students of an university offering actuarial programs may expect exemption from blocks of exams; the most common block consists of 8 Core Technical Stage subjects (CT1-CT8). The get exemption the whole program should be completed, i.e. it is not possible to get subject by subject exemptions. But, say, in the case of University of Cambridge, programme M. Phil in Statistical Science, only exemption from subjects CT3 and CT6 may be granted. Sometimes, depending on modules taken, exemption from other courses may be granted.

The exemption is not granted automatically; it requires an application and payment of fees. The exemption fees for CT subjects is £150 per subject, for ST subjects is £210 per subject, for CA subjects is from £400 to £210 per subject (for overseas students the rates are much lower, approximately $25-35\%$ of the UK rates). Besides, a student must join either the Faculty or Institute of Actuaries as a student member.

Chapter 8

Appendix 2

8.1 Syllabus for Subject CT1

The Actuarial Profession[1]

Aim

The aim of the Financial Mathematics subject is to provide a grounding in financial mathematics and its simple applications.

Objectives

On completion of the subject the trainee actuary will be able to:

1. Describe how to use a generalized cashflow model to describe financial transactions.
   (a) For a given cashflow process, state the inflows and outflows in each future time period and discuss whether the amount or the timing (or both) is fixed or uncertain.
   (b) Describe in the form of a cashflow model the operation of a zero coupon bond, a fixed interest security, an index-linked security, cash on deposit, an equity, an "interest only" loan, a repayment loan, and an annuity certain.
2. Describe how to take into account the time value of money using the concepts of compound interest and discounting.
   (a) Accumulate a single investment at a constant rate of interest under the operation of simple/compound interest

[1]This is an official syllabus for the 2010 examinations of the Institute of Actuaries and the Faculty of Actuaries

(b) Define the present value of a future payment.

(c) Discount a single investment under the operation of simple (commercial) discount at a constant rate of discount.

(d) Describe how a compound interest model can be used to represent the effect of investing a sum of money over a period.

3. Show how interest rates or discount rates may be expressed in terms of different time periods.

   (a) Derive the relationship between the rates of interest and discount over one effective period arithmetically and by general reasoning.

   (b) Derive the relationships between the rate of interest payable once per effective period and the rate of interest payable p times per time period and the force of interest.

   (c) Explain the difference between nominal and effective rates of interest and derive effective rates from nominal rates.

   (d) Calculate the equivalent annual rate of interest implied by the accumulation of a sum of money over a specified period where the force of interest is a function of time.

4. Demonstrate a knowledge and understanding of real and money interest rates.

5. Calculate the present value and the accumulated value of a stream of equal or unequal payments using specified rates of interest and the net present value at a real rate of interest, assuming a constant rate of inflation.

   (a) Discount and accumulate a sum of money or a series (possibly infinite) of cashflows to any point in time where: the rate of interest or discount is constant, the rate of interest or discount varies with time but is not a continuous function of time, either or both the rate of cashflow and the force of interest are continuous functions of time

   (b) Calculate the present value and accumulated value of a series of equal or unequal payments made at regular intervals under the operation of specified rates of interest where the first payment is deferred for a period of time/not deferred

6. Define and use the more important compound interest functions including annuities certain.

   (a) Derive formulae in terms of i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$ for $a_{\overline{n}|}$, $s_{\overline{n}|}$, $a^{(p)}_{\overline{n}|}$, $s^{(p)}_{\overline{n}|}$, $\ddot{a}_{\overline{n}|}$, $\ddot{s}_{\overline{n}|}$, $\ddot{a}^{(p)}_{\overline{n}|}$, $\ddot{s}^{(p)}_{\overline{n}|}$, $\overline{a}_{\overline{n}|}$ and $\overline{s}_{\overline{n}|}$.

   (b) i, v, n, d, δ, $i^{(p)}$ and $d^{(p)}$ for ${}_{m|}a_{\overline{n}|}$, ${}_{m|}a^{(p)}_{\overline{n}|}$, ${}_{m|}\ddot{a}_{\overline{n}|}$, ${}_{m|}\ddot{a}^{(p)}_{\overline{n}|}$ and ${}_{m|}\overline{a}_{\overline{n}|}$.

   (c) Derive formulae in terms of i, v, n, δ , $a_{\overline{n}|}$ and $\ddot{a}_{\overline{n}|}$ for $(Ia)_{\overline{n}|}$, $(I\ddot{a})_{\overline{n}|}$, $(I\overline{a})_{\overline{n}|}$, $(\overline{I}\overline{a})_{\overline{n}|}$ and the respective deferred annuities.

7. Define an equation of value.
   (a) Define an equation of value, where payment or receipt is certain.
   (b) Describe how an equation of value can be adjusted to allow for uncertain receipts or payments.
   (c) Understand the two conditions required for there to be an exact solution to an equation of value.
8. Describe how a loan may be repaid by regular instalments of interest and capital.
   (a) Describe flat rates and annual effective rates.
   (b) Calculate a schedule of repayments under a loan and identify the interest and capital components of annuity payments where the annuity is used to repay a loan for the case where annuity payments are made once per effective time period or p times per effective time period and identify the capital outstanding at any time.
9. Show how discounted cashflow techniques can be used in investment project appraisal.
   (a) Calculate the net present value and accumulated profit of the receipts and payments from an investment project at given rates of interest.
   (b) Calculate the internal rate of return implied by the receipts and payments from an investment project.
   (c) Describe payback period and discounted payback period and discuss their suitability for assessing the suitability of an investment project.
   (d) Determine the payback period and discounted payback period implied by the receipts and payments from an investment project.
   (e) Calculate the money-weighted rate of return, the time-weighted rate of return and the linked internal rate of return on an investment or a fund.
10. Describe the investment and risk characteristics of the following types of asset available for investment purposes: fixed interest government borrowings, fixed interest borrowing by other bodies, shares and other equity-type finance, derivatives
11. Analyze elementary compound interest problems.
    (a) Calculate the present value of payments from a fixed interest security where the coupon rate is constant and the security is redeemed in one instalment.
    (b) Calculate upper and lower bounds for the present value of a fixed interest security that is redeemable on a single date within a given range at the option of the borrower.

(c) Calculate the running yield and the redemption yield from a fixed interest security (as in 1.), given the price.
(d) Calculate the present value or yield from an ordinary share and a property, given simple (but not necessarily constant) assumptions about the growth of dividends and rents.
(e) Solve an equation of value for the real rate of interest implied by the equation in the presence of specified inflationary growth.
(f) Calculate the present value or real yield from an index-linked bond, given assumptions about the rate of inflation.
(g) Calculate the price of, or yield from, a fixed interest security where the investor is subject to deduction of income tax on coupon payments and redemption payments are subject to the deduction of capital gains tax.
(h) Calculate the value of an investment where capital gains tax is payable, in simple situations, where the rate of tax is constant, indexation allowance is taken into account using specified index movements and allowance is made for the case where an investor can offset capital losses against capital gains.

12. Calculate the delivery price and the value of a forward contract using arbitrage free pricing methods.

    (a) Define "arbitrage" and explain why arbitrage may be considered impossible in many markets.
    (b) Calculate the price of a forward contract in the absence of arbitrage assuming: no income or expenditure associated with the underlying asset during the term of the contract, a fixed income from the asset during the term, a fixed dividend yield from the asset during the term.
    (c) Explain what is meant by "hedging" in the case of a forward contract.
    (d) Calculate the value of a forward contract at any time during the term of the contract in the absence of arbitrage, in the situations listed in (b) above.

13. Show an understanding of the term structure of interest rates.

    (a) Describe the main factors influencing the term structure of interest rates.
    (b) Explain what is meant by the par yield and yield to maturity.
    (c) Explain what is meant by, derive the relationships between and evaluate: discrete spot rates and forward rates, continuous spot rates and forward rates
    (d) Define the duration and convexity of a cashflow sequence, and illustrate how these may be used to estimate the sensitivity of the value of the cashflow sequence to a shift in interest rates.

(e) Evaluate the duration and convexity of a cashflow sequence.

(f) Explain how duration and convexity are used in the (Redington) immunization of a portfolio of liabilities.

14. Show an understanding of simple stochastic models for investment returns.

    (a) Describe the concept of a stochastic interest rate model and the fundamental distinction between this and a deterministic model.

    (b) Derive algebraically, for the model in which the annual rates of return are independently and identically distributed and for other simple models, expressions for the mean value and the variance of the accumulated amount of a single premium.

    (c) Derive algebraically, for the model in which the annual rates of return are independently and identically distributed, recursive relationships which permit the evaluation of the mean value and the variance of the accumulated amount of an annual premium.

    (d) Derive analytically, for the model in which each year the random variable $(1 + i)$ has an independent log-normal distribution, the distribution functions for the accumulated amount of a single premium and for the present value of a sum due at a given specified future time.

    (e) Apply the above results to the calculation of the probability that a simple sequence of payments will accumulate to a given amount at a specific future time.

Bibliography

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