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Computer simulation of population dynamics inside the urban environment

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Abstract. In this paper using a mathematical model of the so-called "space-dynamic" approach we investigate the problem of development and temporal dynamics of different urban population groups. For simplicity we consider an interaction of only two population groups inside a single urban area with axial symmetry. This problem can be described qualitatively by a system of two non-stationary nonlinear differential equations of the diffusion type with boundary conditions of the third type. The results of numerical simulations show that with a suitable choice of the diffusion coefficients and interaction functions between different population groups we can receive different scenarios of population dynamics: from complete displacement of one population group by another (originally more "aggressive") to the "peaceful" situation of co-existence of them together.

Keywords: mathematical modeling, the urban population, the non-linear differential equations of the diffusion type, spatiotemporal dynamics of urban population, stable and unstable regimes

1. Introduction

In recent years models of spatial economy have been widely used for describing different urban processes.

Processes of urbanization along with urban consolidation have been observed for a long time in both developed and developing countries, but modern cities are more and more characterized by decentralization tendencies. At present we can frequently observe heterogeneous rather than homogeneous urban formations.

The examples of urban formations of both established and still establishing forms may be such metropolises as Tokyo, New York, Mexico City, Moscow, Shanghai, Singapore etc.

It is well known that distribution of different population groups according to living areas represents quite complicated function of spatial variables and time period.

The given work presents computerized results which were obtained by means of "soft" mathematical model ([1]; Arnold) of interaction of two groups of urban population with non-linear spatial diffusion taken into account.

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2. Mathematical model

The described mathematical model presents only two groups of urban population according to their social and economic indicators.

For example, urban population may be classified by their income level (the “rich” and the “poor”), according to their level of education (“more educated” and “less educated”) and by a number of other factors.

In a number of economically and socially developed countries coexistence of population groups belonging to different social layers brings about serious social problems. Therefore, at present such processes are intensively studied from different viewpoints including mathematical models within spatial economy.

Interaction of similar population groups within a single urban agglomeration may be described at qualitative level using non-standard equations in partial derivatives of the diffusion type.

For simplicity and visibility the given work considers only two groups of population characterized by density of distribution $u = u(x, y, t)$ and $v = v(x, y, t)$ in point (x, y) of the fixed dimensional area D at a given period of time t.

The system of non-linear differential equations describing time dynamics of these population groups looks formal as:

$$\begin{cases} \dfrac{\partial u}{\partial t} = L_1[u] + F_1(u, v), \\ \dfrac{\partial v}{\partial t} = L_2[v] + F_2(u, v), \end{cases} \tag{1}$$

where elliptical differential operator of the second order $L_s[\omega] = \nabla \cdot (K_s \nabla \omega)$, $s = \overline{1,2}$, describes spatial diffusion of density functions of u and v population groups. In general, coefficients of diffusion K_s may have non-linear dependence on the unknown functions u and v.

Functions $F_s(u, v)$, $s = \overline{1,2}$, describe “interaction” of different population groups living in the same area. Quite realistic seems the following type of these functions:

$$\begin{aligned} F_1(u,v) &= u\,(a_1 - b_1 u - c_1 v) - d_1\,u\,v, \\ F_2(u,v) &= v\,(a_2 - b_2 v - c_2 u) - d_2\,u\,v. \end{aligned} \tag{2}$$

In this work we consider the simple area as a two dimensional area D, though this is not a necessary restriction, but it leads to a better understanding of the processes under study, namely rectangle $D = \{(x, y) : (0 \le x \le l_1) \times (0 \le y \le l_2)\}$ on the plane of variables (x, y). At the boundary $\Gamma = \partial D$ are given heterogeneous boundary conditions (boundary conditions of the 3rd type):

$$\left. h_s \frac{\partial w}{\partial n} + p_s w \right|_{\Gamma} = \varphi_s(x, y, t),\ s = \overline{1,2}, \tag{3}$$

where $\dfrac{\partial .}{\partial n}$ shows a derivative along the single outer normal at the points of boundary Γ.

It is necessary to add original conditions (the initial Cauchy conditions for equations (1) and boundary conditions (3)) and, as a result, we have the following boundary value problem consisting of two non-linear differential diffusion (parabolic) equations:

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(K_1(u,v)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_1(u,v)\frac{\partial u}{\partial y}\right) + F_1(u,v), \\ \frac{\partial v}{\partial t} = \frac{\partial}{\partial x}\left(K_2(u,v)\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_2(u,v)\frac{\partial v}{\partial y}\right) + F_2(u,v), \\ h_1\frac{\partial u}{\partial n} + p_1 u\Big|_\Gamma = \varphi_1(x,y,t),\ (x,y)\in\Gamma, \\ h_2\frac{\partial v}{\partial n} + p_2 v\Big|_\Gamma = \varphi_2(x,y,t),\ (x,y)\in\Gamma, \\ u(x,y,t)\big|_{t=0} = u_0(x,y),\ (x,y)\in D, \\ v(x,y,t)\big|_{t=0} = v_0(x,y),\ (x,y)\in D. \end{cases} \tag{4}$$

This system of equations (4) was solved numerically by finite difference scheme [2; Samarskii] using the well-known Peaceman & Rachford method (or Alternating direction implicit (ADI) method) [3; Peaceman & Rachford]. The program code was realized in the MatLab development environment.

While conducting computational experiments both the initial conditions (functions) $u_0(x,y)$ and $v_0(x,y)$ and the parameters for diffusion coefficients $K_s(u,v)$ as well as functions of the so called interaction $F_s(u,v)$, $s=\overline{1,2}$, were varied.

3. Computer simulation results

Suppose that the relations between two groups of population, chosen by us, may be both “friendly” and “neutral” or even “unfriendly”. A similar approach was proposed earlier in [4; Zhang].

In Zhang model numerical coefficients are $a_1 = a_2 = a$, $b_1 = b_2 = b$ and $c_1 = c_2 = c$, the summand of the form $u(a - bu - cv)$serves to describe the reaction of the population to existing economic conditions of living within urban environment.

It is possible to interpret coefficient a as physical capacity of urban space at a point (x_1, x_2). When parameter a is constant, physical capacity is homogeneous in space. If we suppose that value $(bu + cv)$ is a quantitative measure of space occupied by both groups, the value $(a - bu - cv)$ may be considered as excessive space for physical capacity.

When this value at some point exceeds zero, the living place becomes more attractive for population. It is evident when it equals zero and notion $(-d_1uv)$ and diffusion effects are negligibly small, population migration stops. Notion $(-d_1uv)$ serves to measure interaction of groups.

Coefficient d_1 may be both positive and negative (or zero). If it is positive, population “group 1” does not feel like living with population “group 2”. If $d_1 = 0$, there are no prejudices between groups. If coefficient d_1 is negative high density of “group 2” attracts population of “group 1”.

In the given model boundary conditions (3) are homogeneous and look the following way:

$$\begin{cases} h_1\frac{\partial u}{\partial n} + p_1 u\Big|_\Gamma = 0,\ (x,y)\in\Gamma, \\ h_2\frac{\partial v}{\partial n} + p_2 v\Big|_\Gamma = 0,\ (x,y)\in\Gamma. \end{cases} \tag{5}$$

As initial distribution density for both interacting population groups let’s take “symmetrical” functions, so that the whole urban area was partially occupied by this or that population group (Fig. 1). It may be shown how urban structure for different parameters of the modeled system will change. For example, when relations between two groups of the population are of “aggressive” character and

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diffusion coefficients depend on the opposite functions, i.e. $K_1(u,v)=v$ and, accordingly $K_2(u,v)=u$, the obtained calculation results (Fig. 2, a-d) show that both population groups equally fill the given area, but due to “hostility” one group (the more “aggressive” one) dislodges the other one as a result, and, therefore, the urban structure of the population becomes practically homogeneous.

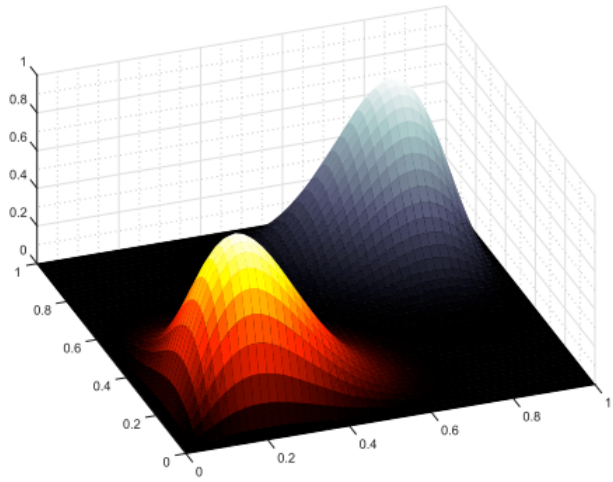

Figure 1. Initial density of distribution for functions u and v.

If coefficients $d_1<0$ and $d_2=0$, diffusion coefficients K_1 and K_2 depend only on function u, one can see the following picture of spatial-time diffusion of the population (Fig. 3, a-f).

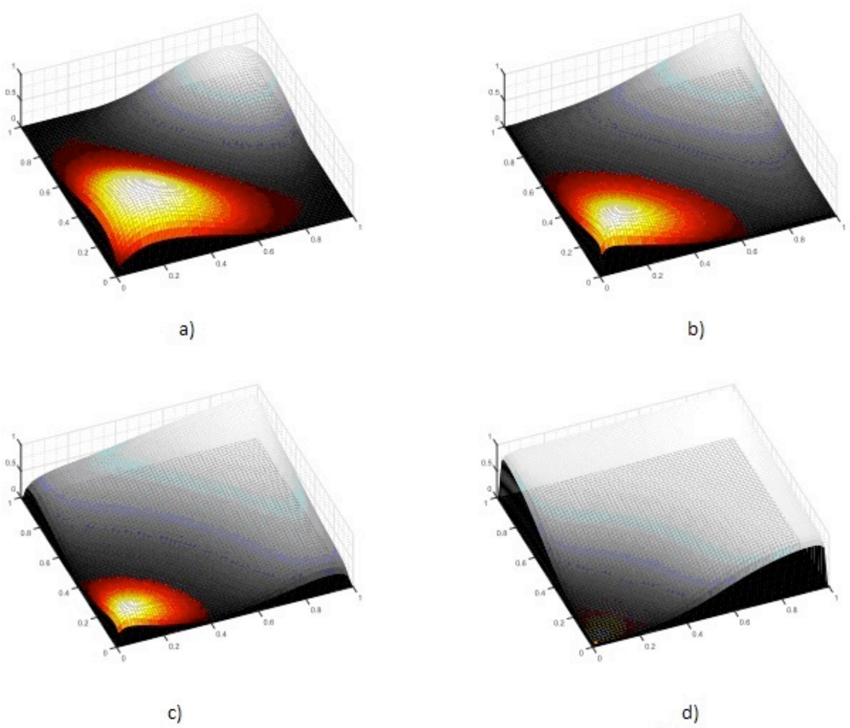

Figure 2. Effect of “geographical” diffusion of the population under “hostile” relations between population groups.

This example can be classified according to educational level when less educated people tend to live in areas with more educated population. It can also be said that areas with more favourable facilities are preferable for life which defines evident population migration from one area to another.

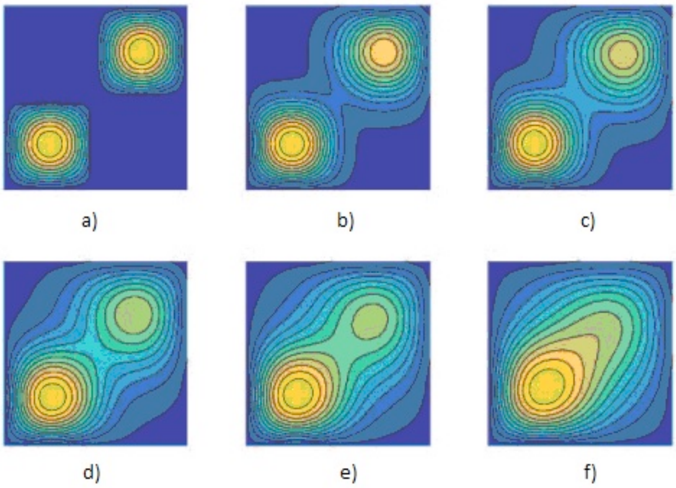

Figure 3. Redistribution of population with neutral or friendly relations ($K_1 = u, K_2 = u$).

As an example, suppose that interaction between population groups can be classified as purely competitive. Diffusion coefficients in equations are considered to have sedate dependence. Under such supposition one population group dominates over the other, dislodging it outside the city borders (Fig. 4, a-f).

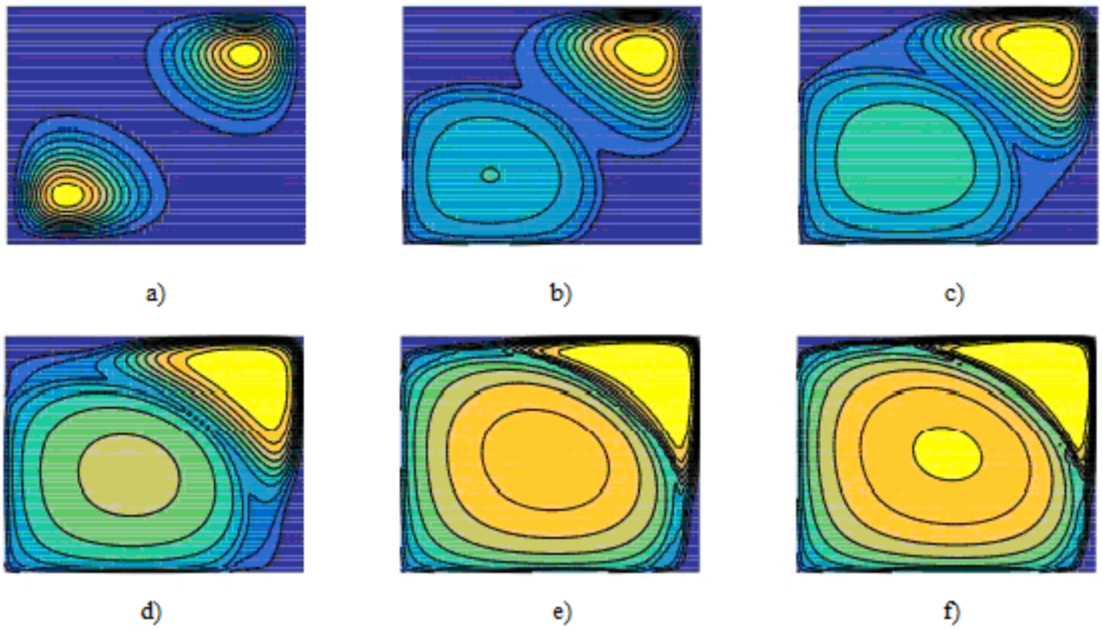

Figure 4. Redistribution of population under competitive relations ($K_1 = u^k, K_2 = u^\sigma, k < \sigma$).

4. Conclusions

The present work considers the mathematical model aimed at describing the mechanism of interaction between different population groups leaving in the same urban area.

This problem is formulated in terms of non-linear differential equations of the diffusion type. It is no use applying the so-called "hard" model in such kind of mathematical simulation. At present we can speak only about "soft" mathematical modeling.

The term "soft" mathematical model implies only the parameters and factors that are essential or available. Having a confirmed and verified mathematical model one can proceed to solve purely applied problems.

In his time (to be more exactly in 1997) an outstanding Russian mathematician, a member of the Russian Academy of Sciences Vladimir I. Arnold noted that "… success comes not so much with application of ready-made recipes ("hard" models), but rather with mathematical approach to the problems of the real world phenomena" [1; Arnold].

5. Appendix

To confirm the accuracy of the difference schemes and the convergence rate of the numerical solutions we made the corresponding analysis using precise analytical solutions of non-linear differential equations of the diffusion type.

The similar analysis was made on the example of the system of non-linear equations of "reaction-diffusion" type [5; Polyanin & Zaitsev]:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial u}{\partial x}\right) + u\, f(bu - cw) + g(bu - cw), \\ \dfrac{\partial w}{\partial t} = \dfrac{a}{x^n}\dfrac{\partial}{\partial x}\left(x^n \dfrac{\partial w}{\partial x}\right) + w\, f(bu - cw) + h(bu - cw), \end{cases} \tag{A.1}$$

where parameter $n = 1$ defines problem with the axial symmetry and parameter $n = 2$ defines problem with the central symmetry.

In literature function u it is customary to call 'activator' and the second function w, which leads to slowing down the described diffusion process, is called 'inhibitor'.

The system of equations (A.1) has precise solution of the type:

$$\begin{cases} u(x,t) = \phi(t) + c \cdot exp\left(\int f(b\phi - c\psi)dt\right)\theta(x,t), \\ w(x,t) = \psi(t) + b \cdot exp\left(\int f(b\phi - c\psi)dt\right)\theta(x,t), \end{cases}$$

where functions $\phi = \phi(t)$ and $\psi = \psi(t)$ are determined by the system of standard differential equations:

$$\begin{cases} \dfrac{d\varphi}{dt} = \varphi\, f(b\phi - c\psi) + g(b\phi - c\psi), \\ \dfrac{d\psi}{dt} = \psi\, f(b\phi - c\psi) + h(b\phi - c\psi), \end{cases}$$

and function $\theta = \theta(x,t)$ satisfies the linear partial derivatives equation:

$$\frac{\partial \theta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial \theta}{\partial x}\right). \tag{A.2}$$

Let's multiply the first equation of system (A.1) by 'b' and add it to the second one, multiplied by '$-c$'. Then we get the following equation:

$$\frac{\partial \zeta}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial \zeta}{\partial x}\right) + \zeta\, f(\zeta) + b\, g(\zeta) - c\, h(\zeta), \tag{A.3}$$

where $\zeta = bu - cw$.

This equation may be obtained by transforming the first equation of the initial system of the type:

$$\frac{\partial u}{\partial t} = \frac{a}{x^n}\frac{\partial}{\partial x}\left(x^n \frac{\partial u}{\partial x}\right) + u\, f(\zeta) + g(\zeta). \tag{A.4}$$

Equation (A.3) may be transformed by division method. If solution $\zeta = \zeta(x,t)$ of the equation (A.3) is given, function $u = u(x,t)$ may be defined by solving linear equation (A.4) and function $w = w(x,t)$ according to formula $w = (bu - \zeta)/c$.

Let`s note two more important solutions of equation (A.3):

- in general case, equation (A.2) implies stationary solution $\zeta = \zeta(x)$, while the corresponding precise solution of equation (A.4) has the form $u = u_0(x) + \sum e^{\beta_n t} u_n(x)$;
- if the condition $\zeta \cdot f(\zeta) + b \cdot g(\zeta) - c \cdot h(\zeta) = k_1 \cdot \zeta + k_0$ takes place, the equation (A.2) is linear and takes the form $\frac{\partial \zeta}{\partial t} = \frac{a}{x^n} \frac{\partial}{\partial x}\left(x^n \frac{\partial \zeta}{\partial x} \right) + k_1 \zeta + k_0$ or is transformed to the linear equation (A.2) by the change of variables $\zeta = e^{k_1 t} \overline{\zeta} - k_0 k_1^{-1}$.

Of particular, interest is the stationary case, when system (A.1) is brought to the system of non-linear elliptic equations:

$$\begin{cases} \Delta u = u\, f(bu - cw) + g(bu - cw), \\ \Delta w = w\, f(bu - cw) + h(bu - cw), \end{cases} \tag{A.5}$$

where $\Delta . = \frac{\partial^2 .}{\partial x^2} + \frac{\partial^2 .}{\partial y^2}$ is Laplace operator.

Precise solution of system (A.5) may be given as:

$$\begin{cases} u = \phi(x) + c\,\theta(x,y), \\ w = \psi(x) + b\,\theta(x,y), \end{cases}$$

with function $\phi = \phi(x)$ and $\psi = \psi(x)$ are defined from the system of usual differential equations:

$$\begin{cases} \frac{d^2\phi}{dx^2} = \phi\, f(b\phi - c\psi) + g(b\phi - c\psi), \\ \frac{d^2\psi}{dx^2} = \psi\, f(a\phi - b\psi) + h(a\phi - b\psi), \end{cases}$$

where function $\theta = \theta(x,y)$ satisfies Schrödinger linear differential equation of special type:

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = F(x)\theta(x,y)$$

with function $F(x) = f(bu - cw)$.

It can be solved by method of separation of variables. If we multiply the first equation of system (A.4) by 'b' and add the second equation, multiplied by '$-c$', we get the following equation:

$$\frac{\partial^2 \zeta}{\partial x^2} + \frac{\partial^2 \zeta}{\partial y^2} = \zeta \cdot f(\zeta) + b \cdot g(\zeta) - c \cdot h(\zeta), \tag{A.6}$$

where $\zeta = bu - cw$.

This equation may be transformed by means of the first equation of the initial system (A.1). As a result we obtain the equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = u\, f(\zeta) + g(\zeta). \tag{A.7}$$

Equation (A.7) can be resolved consistently by separation technique. Existence of exact solutions for such kinetic functions as $F(\zeta) = \zeta \cdot f(\zeta) + b \cdot g(\zeta) - c \cdot h(\zeta)$ can be found in [5].

In conclusion with respect to this equation we mention two important special cases:

- in the first case, equation (A.7) gives solution of the running wave $\zeta = \zeta(z)$, where $z = k_1 x + k_2 y$ and k_1, k_2 – arbitrary constants;

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- in the second case, if condition $\zeta \cdot f(\zeta) + b \cdot g(\zeta) - c \cdot h(\zeta) = c_1 \cdot \zeta + c_0$ is fulfilled, equation (A.6) is a well-known Helmholtz linear differential equation.

Thus, for a wide class of non-linear differential systems of the form (A.1), exact solutions can be found, and so the study of which makes it possible to investigate the general properties of such systems.

6. References

[1] Arnold V.I. “Hard” and “soft” mathematical models // Published by Moscow Center for Continuous Mathematical Education (MCCME, Moscow). 2004. 32 P. (in Russian).
[2] Alexannder A. Samarskii. The theory of difference schemes // New York – Basel. Marcel Dekker, Inc. 2001. 761 P.
[3] Peaceman D.W., Rachford Jr. H.H. The numerical solution of parabolic and elliptic differential equations // Journal of the Society for industrial and Applied Mathematics. 1955. Vol. 3. No 1. P.P. 28-41.
[4] Zhang W.-B. Synergetic economics: Time and change in nonlinear economics (Springer series in Synergetics) // Berlin, Germany: Springer-Verlag. 1991. 246 P.
[5] Polyanin A.D., Zaisev V.F. Handbook of Nonlinear Partial Differential Equations (second edition). Published by Chapman & Hall/CRC Press. 2011. 1912 P.