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MECHANICS

Nonexistence of Liouvillian Solutions in the Problem of Motion of a Rotationally Symmetric Ellipsoid on a Perfectly Rough Plane

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Abstract—Using the Kovacic algorithm, the nonexistence of liouvillian solutions in the problem of motion of a rotationally symmetric ellipsoid on a perfectly rough horizontal plane for almost all values of parameters of the problem is proven.

Keywords: rotationally symmetric ellipsoid, Kovacic algorithm, liouvillian solutions.

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INTRODUCTION

Let us consider the problem of motion of a rotationally symmetric ellipsoid on a perfectly rough horizontal plane. In [1–3], it has been shown that the problem solving is reduced to integration of a second-order linear differential equation over the projection of the angular velocity of the ellipsoid onto its axis of symmetry, which is denoted by r. This equation is as follows

$$\frac{d^2r}{d\theta^2} + h_1(\theta)\frac{dr}{d\theta} + h_2(\theta)r = 0,$$

$$h_1(\theta) = \frac{\cos\theta}{\sin\theta} - \frac{4a_3^2\cos\theta}{(a_1^2\sin^2\theta + a_3^2\cos^2\theta)\sin\theta} + \frac{3(A_1a_1^2 - A_3a_3^2)ma_1^2a_3^2\sin\theta\cos\theta}{(a_1^2\sin^2\theta + a_3^2\cos^2\theta)((A_1 + ma_3^2)A_3a_3^2\cos^2\theta + (A_3 + ma_1^2)A_1a_1^2\sin^2\theta)}, \tag{1}$$

$$h_2(\theta) = -\frac{ma_1^2((a_3^2 - a_1^2)^2A_3\sin^4\theta + a_3^2(A_1a_1^2 - A_3a_3^2)(1 + \cos^2\theta))}{(a_1^2\sin^2\theta + a_3^2\cos^2\theta)((A_1 + ma_3^2)A_3a_3^2\cos^2\theta + (A_3 + ma_1^2)A_1a_1^2\sin^2\theta)}.$$

Here, m is the mass of the ellipsoid, a_1 and a_3 are the lengths of its semiaxes, A_1 and A_3 are its equatorial and axial major central moments of inertia, and θ is the angle between the axis of dynamical symmetry of the ellipsoid and the vertical. In the case when one manages to obtain the general solution of Eq. (1), the further problem solving is reduced to quadratures.

Using the so-called Kovacic algorithm [4], the nonexistence of liouvillian solutions of differential equation (1) for almost all parameter values is proven in this work. The paper is structured in the following way. In the first section, the Kovacic algorithm is described and it is shown how liouvillian solutions of a second-order linear differential equation can be found with its help. In the second section, the algorithm is applied to study the existence of liouvillian solutions of differential equation (1).

1. DESCRIPTION OF THE KOVACIC ALGORITHM

Let us consider the differential field $\mathbb{C}(x)$ of rational functions of one complex variable x. We accept the standard notations $\mathbb{Z}$ and $\mathbb{Q}$ for the sets of integer and rational numbers, respectively. Our goal is to find a solution of the differential equation

$$z'' + a(x)z' + b(x)z = 0, \tag{2}$$

where $a(x)$ and $b(x) \in \mathbb{C}(x)$. In [4] an algorithm has been proposed that allows one to explicitly find so-called liouvillian solutions of differential equation (2), that is, solutions that can be expressed in terms of liouvillian functions. In turn, liouvillian functions are elements of a liouvillian field, which is defined in the following way.

Definition. Let F be a differential field of functions of one complex variable x that contains $\mathbb{C}(x)$; namely, F is a field of characteristic zero with the operation of differentiation ()' that acts on elements of this field by the rules $(a + b)' = a' + b'$ and $(ab)' = a'b + ab'$ for any a and b from F. The field F is called *liouvillian* if there exists a sequence (tower) of finite field extensions

$$\mathbb{C}(x) = F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = F,$$

obtained by adjoining one element such that for any $i = 1, 2, \ldots, n$ we have

$$F_i = F_{i-1}(\alpha) \quad \text{with} \quad \frac{\alpha'}{\alpha} \in F_{i-1}$$

(that is, F_i is formed by adjoining the exponential of an indefinite integral over F_{i-1}); or

$$F_i = F_{i-1}(\alpha) \quad \text{with} \quad \alpha' \in F_{i-1}$$

(that is, F_i is formed by adjoining an integral over F_{i-1}); or F_i is a finite algebraic extension over F_{i-1} (that is, $F_i = F_{i-1}(\alpha)$ and α satisfies a finite-degree polynomial equation of the form

$$a_0 + a_1\alpha + \ldots + a_n\alpha^n = 0,$$

where $a_j \in F_{i-1}$, $j = 0, 1, 2, \ldots, n$, and not all of these are equal to zero). □

Thus, liouvillian functions are constructed sequentially from rational functions by using algebraic operations and the operation of indefinite integration and by taking the exponential of a given expression. A solution of Eq. (2) that is expressed in terms of liouvillian functions most closely corresponds to the notion of a *closed-form solution* or a *solution in quadratures*. The main advantage of the Kovacic algorithm lies in the fact that it allows one not only to establish the existence or nonexistence of such a solution but also to explicitly find it in the case of its existence.

To reduce differential equation (2) to a simpler form, we change the variables by the formula

$$y(x) = z(x)e^{\frac{1}{2}\int a(x)dx}. \tag{3}$$

Then Eq. (2) takes the form

$$y'' = h(x)y, \quad h(x) = \frac{1}{2}a' + \frac{1}{4}a^2 - b, \quad h(x) \in \mathbb{C}(x). \tag{4}$$

Hereinafter, it is assumed that a second-order differential equation with which the Kovacic algorithm deals is written in form (4). The following theorem, which has been proven by Kovacic [4], determines the structure of a solution of this differential equation.

Theorem 1. *For differential equation* (4), *only the following four cases are true*:

(i) *Differential equation* (4) *has a solution of the form* $\eta = e^{\int \omega(x)dx}$, *where* $\omega(x) \in \mathbb{C}(x)$ (*liouvillian solution of type* 1).

(ii) *Differential equation* (4) *has a solution of the form* $\eta = e^{\int \omega(x)dx}$, *where* $\omega(x)$ *is an algebraic function of degree* 2 *over* $\mathbb{C}(x)$, *and item* (i) *is not the case* (*liouvillian solution of type* 2).

(iii) *All solutions of differential equation* (4) *are algebraic over* $\mathbb{C}(x)$, *and items* (i) *and* (ii) *are not the case. In this situation, a solution of differential equation* (4) *has the form* $\eta = e^{\int \omega(x)dx}$, *where* $\omega(x)$ *is an algebraic function of degree* 4, 6, *or* 12 *over* $\mathbb{C}(x)$ (*liouvillian solution of type* 3).

(iv) *Differential equation* (4) *has no liouvillian solutions*.

The following theorem, which has been proven in [4], specifies conditions that are necessary for one of the first three cases listed in Theorem 1 to be true.

Theorem 2. *For differential equation* (4), *the following conditions are necessary for one of the first three cases listed in Theorem* 1 *to be true, that is, for Eq.* (4) *to have a liouvillian solution of a type specified in description of the corresponding case*:

(i) *Each pole of the function $h(x)$ is of order* 1 *or of an even order. The order of $h(x)$ at $x = \infty$ is even or larger than* 2.

(ii) *The function $h(x)$ has at least one pole of order* 2 *or of an odd order larger than* 2.

(iii) *The function $h(x)$ has no pole of an order larger than* 2. *The order of $h(x)$* at ∞ *equals at least* 2. *If the decomposition of the function $h(x)$ into the sum of partial fractions has the form*

$$h(x) = \sum_i \frac{\alpha_i}{(x - c_i)^2} + \sum_j \frac{\beta_j}{x - d_j},$$

then for any i we have

$$\sqrt{1 + 4\alpha_i} \in \mathbb{Q}, \quad \sum_j \beta_j = 0$$

and, *moreover*,

$$\sqrt{1 + 4\gamma} \in \mathbb{Q}, \quad \textit{where} \quad \gamma = \sum_i \alpha_i + \sum_j \beta_j d_j.$$

To find a liouvillian solution of type 1 of differential equation (4), the Kovacic algorithm is stated in the following way (see [4] for details). We assume that the necessary conditions for the existence of a solution in case (i) are satisfied and denote the set of finite poles of the function $h(x)$ by Γ.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we determine a rational function $\left[\sqrt{h}\right]_c$ and two complex numbers α_c^+ and α_c^-:

(c_1) If $c \in \Gamma$ and c is a pole of order 1, then $\left[\sqrt{h}\right]_c = 0$, $\alpha_c^+ = \alpha_c^- = 1$.

(c_2) If $c \in \Gamma$ and c is a pole of order 2, then $\left[\sqrt{h}\right]_c = 0$.

Let b be the coefficient at $1/(x - c)^2$ in the decomposition of the function $h(x)$ into partial fractions. Then we have

$$\alpha_c^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4b}.$$

(c_3) If $c \in \Gamma$ and c is a pole of order $2\nu \geq 4$ (of a necessary even order, in view of the corresponding conditions of Theorem 2), then the function $\left[\sqrt{h}\right]_c$ is a sum of terms that include all the fractions $1/(x - c)^i$ for $2 \leq i \leq \nu$ in the Laurent series expansion of the function $\sqrt{h}$ in the vicinity of c. There are two values of this function $\left[\sqrt{h}\right]_c$ that differ in sign; one of them can be chosen. So, we obtain

$$\left[\sqrt{h}\right]_c = \frac{a}{(x - c)^{\nu}} + \ldots + \frac{d}{(x - c)^2}.$$

Let b be the coefficient at $1/(x - c)^{n+1}$ for the function $h - \left[\sqrt{h}\right]_c^2$. Then we have

$$\alpha_c^{\pm} = \frac{1}{2}\left(\pm\frac{b}{a} + \nu\right).$$

(∞_1) If the order of the function $h(x)$ at $x = \infty$ is larger than 2, then $\left[\sqrt{h}\right]_\infty = 0$, $\alpha_\infty^+ = 0$, and $\alpha_\infty^- = 1$.

(∞_2) If the order of the function $h(x)$ at $x = \infty$ is equal to 2, then $\left[\sqrt{h}\right]_\infty = 0$.

Let b be the coefficient at $1/x^2$ in the Laurent series expansion of $h(x)$ in the vicinity of $x = \infty$. Then we have

$$\alpha_\infty^{\pm} = \frac{1}{2} \pm \frac{1}{2}\sqrt{1 + 4b}.$$

(∞_3) If the order of the function $h(x)$ at $x = \infty$ is equal to $-2\nu \le 0$ (and is even, in view of the conditions of Theorem 2), then the function $\left[\sqrt{h}\right]_\infty$ is a sum of power terms x^i, $0 \le i \le \nu$, from the Laurent series expansion of $\sqrt{h}$ at $x = \infty$ (one of the two possible variants can be chosen). Then we obtain

$$\left[\sqrt{h}\right]_\infty = ax^\nu + \ldots + d.$$

Let b be the coefficient at $x^{\nu-1}$ for the function $h - \left(\left[\sqrt{h}\right]_\infty\right)^2$. Then we have

$$\alpha_\infty^\pm = \frac{1}{2}\left(\pm\frac{b}{a} - \nu\right).$$

Step 2. For each family $s = (s(c))_{c\in\Gamma\cup\{\infty\}}$, where $s(c)$ can be either sign+ or sign−, we put

$$d = \alpha_\infty^{s(\infty)} - \sum_{c\in\Gamma}\alpha_c^{s(c)}. \tag{5}$$

If d is a nonnegative integer number, then the function

$$\theta = \sum_{c\in\Gamma}\left(s(c)\left[\sqrt{r}\right]_c + \frac{\alpha_c^{s(c)}}{x-c}\right) + s(\infty)\left[\sqrt{r}\right]_\infty$$

is a candidate for the role of θ. If d is not a nonnegative integer number, then the corresponding family s is removed from consideration. If all possible families s are removed from consideration in such a way, it would mean that item (i) is not the case.

Step 3. For each family s, we search for a polynomial P of degree d (the constant d is determined by (5)) that satisfies the differential equation

$$P'' + 2\theta P' + (\theta' + \theta^2 - h)P = 0.$$

If such a polynomial is found for some family s, then $\eta = Pe^{\int\theta(x)dx}$ is a needed solution of differential equation (4). If such a polynomial P is found for none of the families s, then item (i) is not the case for differential equation (4).

Now we state the Kovacic algorithm to search for a solution of type 2 of differential equation (4). We denote the set of finite poles of the function $h(x)$ by Γ.

Step 1. For each $c \in \Gamma \cup \{\infty\}$, we determine the set E_c in the following way:

(c_1) If $c \in \Gamma$ is a pole of order 1, then $E_c = \{4\}$.

(c_2) If $c \in \Gamma$ is a pole of order 2 and b is the coefficient at $1/(x-c)^2$ in the decomposition of the function $h(x)$ into partial fractions, then

$$E_c = \left\{(2 + k\sqrt{1+4b}) \cap \mathbb{Z}\right\}, \quad k = 0, \pm 2.$$

(c_3) If $c \in \Gamma$ is a pole of order $\nu > 2$, then $E_c = \{\nu\}$.

(∞_1) If $h(x)$ is of order > 2 at $x = \infty$, then $E_\infty = \{0, 2, 4\}$.

(∞_2) If $h(x)$ is of order 2 at $x = \infty$ and b is the coefficient at $1/x^2$ in the Laurent series expansion of the function h at $x = \infty$, then

$$E_\infty = \left\{(2 + k\sqrt{1+4b}) \cap \mathbb{Z}\right\}, \quad k = 0, \pm 2.$$

(∞_3) If $h(x)$ is of order $\nu < 2$ at $x = \infty$, then $E_\infty = \{\nu\}$.

Step 2. Let us consider the families $s = (e_\infty, e_c)$, $c \in \Gamma$, where $e_c \in E_c$, $e_\infty \in E_\infty$, and at least one of these numbers is odd. We put

$$d = \frac{1}{2}\left(e_\infty - \sum_{c\in\Gamma} e_c\right). \tag{6}$$

If d is a nonnegative integer number, then the corresponding family should be stored. Otherwise, it should be rejected.

Step 3. For each family stored at step 2, we construct a rational function

$$\theta = \frac{1}{2}\sum_{c\in\Gamma}\frac{e_c}{x-c} \tag{7}$$

and attempt to find a polynomial of degree d (where d is determined by (6)) such that

$$P''' + 3\theta P'' + (3\theta^2 + 3\theta' - 4h)P' + (\theta'' + 3\theta\theta' + \theta^3 - 4h\theta - 2h')P = 0. \tag{8}$$

If we are a success and the corresponding polynomial is obtained, we put

$$\varphi = \theta + \frac{P'}{P};$$

let ω be a solution of a quadratic equation (an algebraic equation of degree 2) of the form

$$\omega^2 - \varphi\omega + \frac{1}{2}\varphi' + \frac{1}{2}\varphi^2 - h = 0.$$

Then $\eta = e^{\int \omega(x)dx}$ is a solution of differential equation (4). If the corresponding polynomial P is not found, item (ii) is not the case.

The Kovacic algorithm is stated similarly to search for liouvillian solutions of type 3 of differential equation (4).

2. BASIC RESULTS

We ask ourselves whether liouvillian solutions of differential equation (1) exist. We change the independent variable in it by the formula $\cos^2\theta = x$ and introduce the notations

$$\frac{A_3}{A_1} = A \in (0,2), \quad \frac{ma_1^2}{A_1} = B, \quad \frac{a_3^2}{a_1^2} = C, \quad \frac{1}{1-C} = x_1, \quad \frac{A+B}{A+B-(1+BC)AC} = x_2.$$

As a result, Eq. (1) is rewritten to become

$$\frac{d^2r}{dx^2} + d_1(x)\frac{dr}{dx} + d_2(x)r = 0,$$

$$d_1(x) = \frac{3x-1}{2x(x-1)} - \frac{2(x_1-1)}{(x-1)(x-x_1)} + \frac{3(x_2-x_1)}{2(x-x_1)(x-x_2)}, \tag{9}$$

$$d_2(x) = \frac{(x_2-x_1)(Ax^2 - ((A-1)(x_1^2-2x_1+3)-x_1+3)x - ((A-1)(x_1-2)-1)x_1)}{4(x_1-1)((A-1)x_1-A)(x-1)(x-x_1)(x-x_2)x}.$$

Using a change of variables of form (3), we reduce differential equation (9) to the form

$$\frac{d^2y}{dx^2} = E(x)y,$$

$$E(x) = \frac{c_0}{x} + \frac{b_0}{x^2} + \frac{c_1}{x-1} + \frac{b_1}{(x-1)^2} + \frac{c_{x_1}}{x-x_1} + \frac{b_{x_1}}{(x-x_1)^2} + \frac{c_{x_2}}{x-x_2} + \frac{b_{x_2}}{(x-x_2)^2},$$

$$b_1 = \frac{3}{4}, \quad b_0 = b_{x_1} = b_{x_2} = -\frac{3}{16},$$

$$c_0 = \frac{(2x_1x_2 - x_1 - 3x_2)(A-1)}{8x_2((A-1)x_1 + A)} + \frac{A((x_2-1)(x_1+1) - (x_1-1)^2(2x_2-1))}{8x_1x_2(x_1-1)((A-1)x_1 - A)}, \tag{10}$$

$$c_1 = -\frac{x_1x_2 - 2x_1 - 4x_2 + 5}{4(x_1-1)(x_2-1)},$$

$$c_{x_1} = \frac{(3x_1x_2 - 4x_1 + x_2)(x_1-1)A - 3x_1(x_1x_2 - 2x_1 + x_2)}{8x_1(x_1-1)(x_1-x_2)((A-1)x_1 - A)},$$

$$c_{x_1} = \frac{A(x_2-1)^2}{4x_2(x_1-1)(x_1-x_2)((A-1)x_1-A)} + \frac{(x_1x_2+x_1-2)A-(A-1)(2x_2^2+x_1x_2+x_1-4x_2)x_1}{8x_2(x_2-1)(x_1-x_2)((A-1)x_1-A)}.$$

Thus, in the general case, the function $E(x)$ has four finite poles at points $x = 0$, $x = 1$, $x = x_1$, and $x = x_2$. We assume that all these poles are distinct. In this case, the Laurent series expansion of the function $E(x)$ in the vicinity of point $x = \infty$ has the form

$$E(x)\big|_{x=\infty} = \frac{b_\infty}{x^2} + O\left(\frac{1}{x^3}\right), \quad b_\infty = -\frac{3}{16} + \frac{A(x_1-x_2)}{(x_1-1)((A-1)x_1-A)}.$$

The following statement is valid.

Theorem 3. *Under the condition*

$$\sqrt{1+4b_\infty} \notin \mathbb{Q} \tag{11}$$

Eq. (10) *has no liouvillian solutions.* □

Proof. Taking into account the fact that for the coefficient b_∞ the condition

$$b_\infty = b_0 + b_1 + b_{x_1} + b_{x_2} + c_1 + x_1x_{x_1} + x_2c_{x_2} = \gamma,$$

holds true, we can conclude that differential equation (10) cannot have liouvillian solutions of type 3, since the necessary conditions for the existence of such solutions are not satisfied, in view of condition (11) and Theorem 2. The situation with liouvillian solutions of type 1 is similar: the constant d, which is determined by (5) in the process of searching for these, in this case would be an irrational number, whereas it should be a nonnegative integer in accordance with the algorithm. Thus, the only possible liouvillian solutions that differential equation (10) can have are solutions of type 2. To find them, we apply the Kovacic algorithm step by step for solutions of type 2.

Step 1. We obtain the following sets consisting of integers:

$$E_1 = \{-2, 2, 6\}, \quad E_0 = \{1, 2, 3\}, \quad E_{x_1} = \{1, 2, 3\}, \quad E_{x_2} = \{1, 2, 3\}, \quad \text{and} \quad E_\infty = \{2\}.$$

Step 2. Now we should consider all possible collections $s = (e_\infty, e_1, e_0, e_{x_1}, e_{x_2})$ of elements from the sets E_∞, E_1, E_0, E_{x_1}, and E_{x_2} with one of the numbers in each collection necessarily being odd.

We calculate the quantity d for each collection s by formula (6). According to the algorithm, the number d should be a nonnegative integer. Analyzing all possible collections of elements from the sets E_∞, E_1, E_0, E_{x_1}, and E_{x_2}, we assure ourselves that the only possible situation when the number d is an integer is the case $d = 0$, which is true for the three collections of numbers:

$$s_1 = (2, -2, 2, 1, 1),$$
$$s_2 = (2, -2, 1, 1, 2), \quad \text{and}$$
$$s_3 = (2, -2, 1, 2, 1).$$

Step 3. Now we test each of the above three collections. We start with the collection s_1. In accordance with the algorithm, we construct the function θ by formula (7) using the number collection s_1 to obtain

$$\theta = \frac{1}{2(x-x_1)} + \frac{1}{2(x-x_2)} + \frac{1}{x} - \frac{1}{x-1}.$$

The polynomial of degree $d = 0$ ($P \equiv 1$) should identically satisfy differential equation (8). Upon substituting $P \equiv 1$ and the explicit expressions for $h(x) = E(x)$ and θ in it, equation (8) becomes identically equal to zero in the following cases:

(a) $x_1 = x_2 = 1$, A is an arbitrary number;

(b) $x_1 = x_2 = -1$, A is an arbitrary number;

(c) $x_1 = x_2 = \dfrac{A}{A-1}$, A is an arbitrary number;

(d) $x_1 = 0$, $A = 0$, x_2 is an arbitrary number; and

(e) $x_1 = 1$, $A = 0$, x_2 is an arbitrary number.

We note that none of the above situations can be the case, since all finite poles of the function $E(x)$ are distinct by assumption. Thus, we do not manage to find a solution of type 2 of differential equation (10) for the collection of numbers s_1.

The nonexistence of liouvillian solutions of type 2 for the collections s_2 and s_3 can be proven similarly. Hence, in the case when all finite poles of the function $E(x)$ are distinct and condition (11) is satisfied, differential equation (10) has no liouvillian solutions. The theorem is proven.

Condition (11) would be valid for almost all values of the parameters of the problem. Thus, differential equation (10) has no liouvillian solutions for almost all values of the parameters of the problem. This result allows for the conclusion that, under condition (11), differential equation (1) also has no liouvillian solutions. However, we note that differential equation (10) can have liouvillian solutions in the case when condition (11) is not satisfied. Some cases of existence of liouvillian solutions of differential equation (10) are given in [3].

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