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RELEVANT GENERALIZATION STARTS HERE (and Here = 2)

Abstract. There is a productive and suggestive approach in philosophical logic based on the idea of generalized truth values. This idea, which stems essentially from the pioneering works by J.M. Dunn, N. Belnap, and which has recently been developed further by Y. Shramko and H. Wansing, is closely connected to the power-setting formation on the base of some initial truth values. Having a set of generalized truth values, one can introduce fundamental logical notions, more specifically, the ones of logical operations and logical entailment. This can be done in two different ways. According to the first one, advanced by M. Dunn, N. Belnap, Y. Shramko and H. Wansing, one defines on the given set of generalized truth values a specific ordering relation (or even several such relations) called the logical order(s), and then interprets logical connectives as well as the entailment relation(s) via this ordering(s). In particular, the negation connective is determined then by the inversion of the logical order. But there is also another method grounded on the notion of a quasi-field of sets, considered by Białynicki-Birula and Rasiowa. The key point of this approach consists in defining an operation of quasi-complement via the very specific function g and then interpreting entailment just through the relation of set-inclusion between generalized truth values.

In this paper, we will give a constructive proof of the claim that, for any finite set V with cardinality greater or equal 2, there exists a representation of a quasi-field of sets $\langle \mathcal{P}(V), \cup, \cap, - \rangle$ isomorphic to de Morgan lattice. In particular, it means that we offer a special procedure, which allows to make our negation de Morgan and our logic relevant.

Keywords: relevant logic, first-degree entailment, generalization, information.



1. Power-setting as generalization

The problem to be discussed in this paper is closely connected to the so called generalization procedure applied to some initial set of truth values, producing the new set of what has been dubbed in [8] as “generalized truth values”. To make our consideration self-contained we shall briefly delineate this procedure which can be traced back to the ideas elaborated in doctoral dissertation by J. Michael Dunn [5] (see also [6]) and much celebrated papers by Nuel D. Belnap [2, 3] of “how a computer should think” with the help of “a useful four-valued logic”. The machinery of generalized truth values can be effectively employed to provide an adequate semantic foundation for relevance logic, more concretely, for the famous system of first-degree entailment (**FDE**), which is a certain fragment of Anderson and Belnap’s systems **R** and **E**, see [1]. Recently Yaroslav Shramko and Heinrich Wansing in a series of papers, see, e.g., [8, 9], have developed this machinery further to obtain generalized truth values of any “degree”.

The first step of the procedure is to generalize some initial set of truth values (V) by taking the power-set of V and to consider the outcome of this operation the set of *generalized truth values*. If $V = \mathbf{2}$, where $\mathbf{2} = \{t, f\}$ is the set of two classical values, then the power-set of V is the following four-element set known as **4**: $\mathbf{4} = \mathcal{P}(\mathbf{2}) = \{\{t, f\}, \{t\}, \{f\}, \emptyset\}$.

On the second stage we can consider a possibility of defining a logical system on the basis of the set of generalized truth values. To do so, we have to define on this set a valuational function and introduce logical operations and the relation of logical entailment. One way of introducing these notions was proposed by Dunn in [5]. Let us take on $\mathcal{P}(V)$ the set-theoretic operations of union (\cup) and intersection (\cap). Then $\langle \mathcal{P}(V), \cup, \cap \rangle$ will be a ring of subsets of V . Dunn introduces the concept of polarity on $\mathcal{P}(V)$ as an ordered pair $X = \langle x_1, x_2 \rangle$ for $x_1, x_2 \in \mathcal{P}(V)$. Next, for any two polarities X and Y it is possible to define the generalized set-theoretical operations together with a special ordering relation:

$$\begin{aligned} X \cap Y &= \langle x_1 \cap y_1, x_2 \cup y_2 \rangle; \\ X \cup Y &= \langle x_1 \cup y_1, x_2 \cap y_2 \rangle; \\ \sim X &= \langle x_2, x_1 \rangle; \\ X \leq Y &\iff x_1 \subseteq y_1 \text{ and } y_2 \subseteq x_2. \end{aligned}$$



The structure $\langle (\mathcal{P}(V))^2, \cup, \cap, \sim \rangle$ is called a *field of polarities*. We have the following theorem:

THEOREM (Polarities Theorem, Dunn [5]). *Every de Morgan lattice is isomorphic to a field of polarities.*

Now a (generalized) valuation function v_i can be defined as a map from the set of propositional variables Var of the given propositional language to $\mathcal{P}(V)$. To extend the valuational function to arbitrary propositional formula, consider it as an isomorphism between the syntactical structure $\langle Var, \vee, \wedge, \neg \rangle$ and the field of polarities $\langle \mathcal{P}(V), \cup, \cap, \sim \rangle$. Due to Polarities Theorem, the corresponding valuational structure will always be de Morgan lattice. That is, we have the following definitions:

DEFINITION 1. For arbitrary formulas A and B of L ,

$$\begin{aligned} v_i(A \wedge B) &= v_i(A) \cap v_i(B); \\ v_i(A \vee B) &= v_i(A) \cup v_i(B); \\ v_i(\neg A) &= \sim v_i(A). \end{aligned}$$

DEFINITION 2. For arbitrary formulas A and B of L ,

$$A \models B \iff \forall v_i(v_i(A) \leq v_i(B)).$$

As it is very well known, the entailment relation defined by Definition 2 is axiomatized by system **FDE**. Dunn suggested to consider any polarity X as a pair of sets of situations in which X can be true or false. Because these situations may be inconsistent as well as incomplete the corresponding sets need not be disjoint or exhaustive. If we now consider a set of situations as a singleton $\{a\}$ the field of polarities may be depicted as on Figure 1.

The same result can also be achieved with some different presuppositions. Belnap [2, 3], by pursuing some key ideas from [6], developed his useful four-valued logic on the base of computer metaphor which later was labeled as “Belnap’s computer”. Look upon a computer receiving data from different sources. Incoming information that can be “told” to computer concerning a particular topic (proposition) may be inconsistent or incomplete. Thus, Belnap’s computer can find itself in one of the following four situations with regard to some proposition A :

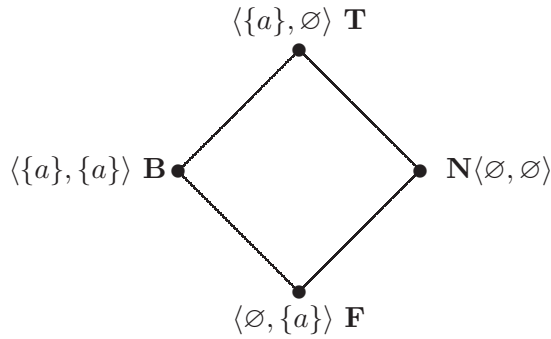


Figure 1. The field of polarities as singletons

1. A may be “told true (only)” – \mathbf{T} ;
2. A may be “told false (only)” – \mathbf{F} ;
3. A may be “told both true and false” – \mathbf{B} ;
4. A may be “told neither true nor false” – \mathbf{N} .

Interpret in a natural way Belnap’s four values as the elements of the set $\mathbf{4} = \mathcal{P}(\mathbf{2})$ and define logical ordering (t-ordering) as follows $x \leq_t y$ iff $x^t \subseteq y^t$ and $y^f \subseteq x^f$, where $x^t = \{z \in x \mid \mathbf{t} = z\}$ and $x^f = \{z \in x \mid \mathbf{f} = z\}$. The resulting structure $\langle \mathbf{4}, \leq_t \rangle$ known as “logical lattice L4” may be illustrated as on Figure 2.

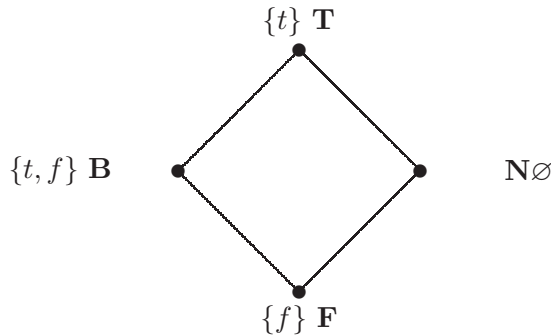


Figure 2. The lattice L4

Let \sqcap_t and \sqcup_t be the lattice meet and join on $L4$ and \sim_t be the operation of inversion of the ordering relation \leq_t , and let v_i be a map from Var into $L4$. Then we have the following definitions:

DEFINITION 3. For arbitrary formulas A and B of L ,

$$\begin{aligned} v_i(A \wedge B) &= v_i(A) \sqcap_t v_i(B); \\ v_i(A \vee B) &= v_i(A) \sqcup_t v_i(B); \\ v_i(\neg A) &= \sim_t v_i(A). \end{aligned}$$

DEFINITION 4. For arbitrary formulas A and B of L ,

$$A \models B \iff \forall v_i(v_i(A) \leq_t v_i(B)).$$

Again, the relation introduced by Definition 4 is adequately axiomatized by **FDE**. By comparing two pictures presented on figures 1 and 2, one can easily verify that they depict the same algebraic structure with **T** and **F** being the lattice top and bottom respectively. Due to the evident resemblance of Dunn’s and Belnap’s approaches, we will unite them in what follows under the label *DB-generalization*. Moreover, it might be appropriate to use the term *DBSW-generalization* since it were Shramko and Wansing [8, 9], who developed this approach further. Namely, they continued the generalization procedure and undertook power-setting of the set **4**, which gave rise to the algebraic structure of trilattice *SIX-TEEN*₃ with three ordering relations. Moreover, not limited to “secondary generalization” they go beyond it and consider unlimited power-setting applied to the set **4**. It leads to a variety of multilattices. On any stage of the repeated generalization the resulting logical system brings into game **FDE**, which allows Shramko and Wansing to acclaim that first-degree entailment is everywhere [9].

But there is also another way of defining the key logical notions on the basis of generalized truth values, which is due to Andrzej Białynicki-Birula and Helena Rasiowa [4]. We will label this latter approach as *BR-generalization* which goes the following way. Take a ring of subsets of V : $\langle \mathcal{P}(V), \cup, \cap \rangle$. Define a function g on V (where V is non-empty) satisfying the only condition of “period two”: $g(g(x)) = x$. Now if we add an operation of the “quasi-complement” defined as $-X = \mathcal{P}(V)/g(X)$, where $\mathcal{P}(V)$ is closed under this operation, the resulting structure $\langle \mathcal{P}(V), \cup, \cap, - \rangle$ is called “a quasi-field of sets”. Hereinafter we



will understand $g(X)$ (a result of application of function g to a set X) as follows: $g(X) = \{y \mid \forall x \in X, g(x) = y\}$.

THEOREM (Quasi-Fields of Sets Theorem, Białynicki-Birula and Rasiowa, [4]). *Every de Morgan lattice is isomorphic to a quasi-field of sets.*

The BR-generalization can be accomplished then as follows. First, one orders the set $\mathcal{P}(V)$ by the set-theoretic inclusion. Second, one considers the propositional language $L_{\mathbf{FDF}}$ with connectives \wedge, \vee, \neg and the algebraic structure $\langle \mathcal{P}(V), \cup, \cap, -, v_i \rangle$, where $v_i: \text{Var} \rightarrow \mathcal{P}(V)$, as a model for our logic. We have straightforward definitions:

DEFINITION 5. For arbitrary formulas A and B of L ,

$$\begin{aligned} v_i(A \wedge B) &= v_i(A) \cap v_i(B); \\ v_i(A \vee B) &= v_i(A) \cup v_i(B); \\ v_i(\neg A) &= -v_i(A). \end{aligned}$$

DEFINITION 6. For arbitrary formulas A and B of L ,

$$A \models B \iff \forall v_i(v_i(A) \subseteq v_i(B)).$$

By Quasi-Fields of Sets Theorem it must be evident that so defined semantical logic is adequate to the system **FDE**. For consistency, the proof is routine. Regarding completeness it is not difficult to give a standard Henkin-style proof.

In so doing first construct a canonical models via prime theories, closed under the FDE-consequence relation and conjunction introduction and satisfying the primeness condition: if $A \vee B \in \alpha$, then $A \in \alpha$ or $B \in \alpha$. The canonical valuation is defined by virtue of prime theories:

$$\begin{aligned} t \in v_i^c(p) &\iff p \in \alpha; \\ f \in v_i^c(p) &\iff \neg p \notin \alpha. \end{aligned}$$

Now canonical valuation lemma is at hand. Of certain interest there is only the case with $f \in v_i^c(B \wedge C)$, because for the present valuational structure, we have $f \in x \cap y$ iff $f \in x$ and $f \in y$. At the first glance it may look strange. Keeping in mind inductive assumption and canonical valuation definition, the desideratum is the equation $\neg(B \wedge C) \notin \alpha \iff$

$\neg B \notin \alpha$ and $\neg C \notin \alpha$. By evident transformation it is equivalent to $\neg(B \wedge C) \in \alpha \Leftrightarrow \neg B \in \alpha$ or $\neg C \in \alpha$, that brings us back to the standard for relevant logic reasoning pattern.

We apply then an appropriate version of Lindenbaum's Lemma (for the proof consult [7]):

LEMMA 1. *For any A and B in $L_{\mathbf{FDF}}$, if $A \not\vdash B$ then there exists a prime theory α such that $A \in \alpha$ and $B \notin \alpha$.*

And finally, we have:

THEOREM 1 (Completeness). *For any A and B in $L_{\mathbf{FDF}}$, if $A \models B$, then $A \vdash B$.*

PROOF. Let (1) $A \models B$ and assume that (2) $A \not\vdash B$. (1) means that $\forall v_i(v_i(A) \subseteq v_i(B))$, that is $\forall v_i(\forall x(x \in v_i(A) \Rightarrow x \in v_i(B)))$. Hence, $t \in v_i^c(A) \Rightarrow t \in v_i^c(B)$ hic est $A \in \alpha \Rightarrow B \in \alpha$. Applying Lindenbaum's Lemma to (2) we get $A \in \alpha$ and $B \notin \alpha$. Then we have $B \in \alpha$ and $B \notin \alpha$, that ends the proof. \square

The aforesaid allows us to specify in the first approximation the BR-generalization procedure as a sequence of the following steps:

1. Take some set of values V .
2. Construct the power-set of V .
3. Pick up the appropriate function g on V to get de Morgan quasi-complement.
4. Define a valuation function v_i as a map from the set of propositional variables into $\mathcal{P}(V)$.
5. To extend it to arbitrary formulas and to introduce an entailment relation accept Definitions 5 and 6.
6. Enjoy an adequate semantics for **FDE**.

In particular, concerning our motivating example with the sets **2** and $\mathcal{P}(\mathbf{2})$, it means that we order the latter set by the set-inclusion relation with **B** at the top and **N** at the bottom. Figure 3 visualizes the resulting structure and highlights its difference from DBSW-generalization. Due to N. Belnap this structure is known as approximation lattice A4. In order to get now a quasi-field of sets, we should define quasi-complement via



appropriate function g on the set $\mathbf{2}$. It turns out that there are only two possible candidates for this role. One of them, identical transformation taken as function g makes the corresponding quasi-complement Boolean inversion, while the other, Boolean complement on the set $\mathbf{2}$, is just what is needed, because the corresponding quasi-complement has desiderata De Morgan properties. It turns \mathbf{B} to \mathbf{N} and vice versa and unalter \mathbf{T} and \mathbf{F} . In other words this quasi-complement behaves exactly as the famous Routley's star-function!

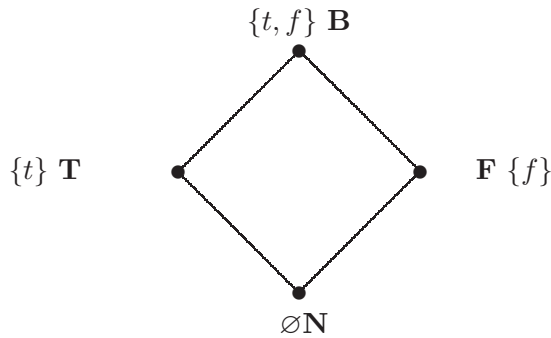


Figure 3. The lattice A4

However, in a general case step 3 in the BR-generalization procedure as described above remains unspecified. This poses the following natural question: can we *for any* $\mathcal{P}(V)$ pick up the appropriate function g , and in this way make our negation de Morgan, and thus, our logic relevant?

2. How to make our negation De Morgan and our logic relevant

Our aim is to provide the constructive proof for the following statement:

For any finite set V with cardinality greater or equal 2, there exists a representation of a quasi-field of sets $\langle \mathcal{P}(V), \cup, \cap, - \rangle$ isomorphic to de Morgan lattice.

The main problem is to delineate exactly this representation, more concretely, to specify uniquely for any $\mathcal{P}(V)$ the function g which allows to obtain the corresponding operation of quasi-complement. Let us demonstrate the way of constructing this representation.

Let V_n be an arbitrary denumerable set with cardinality $n \geq 2$. Consider two cases.

(1) n is an even number. Define mapping φ on V_n :

$$\forall k, 1 \leq k \leq n/2, \varphi(v_{2k}) = v_{2k-1} \text{ and } \varphi(v_{2k-1}) = v_{2k}.$$

In actual fact φ works as follows:

$$\begin{aligned} v_1 &\longleftrightarrow v_2 \\ v_3 &\longleftrightarrow v_4 \\ &\dots\dots\dots \\ v_{n-1} &\longleftrightarrow v_n \end{aligned}$$

It is evident that φ is the function g , because $\forall v : \varphi(\varphi(v)) = v$. Let us label this variant of g -function as g_e .

(2) n is an odd number. Define mapping ϕ on V_n :

$$\forall k, 1 \leq k \leq n - 1/2, \phi(v_{2k}) = v_{2k-1}, \phi(v_{2k-1}) = v_{2k}, \text{ and } \phi(v_n) = v_n.$$

Now ϕ works slightly in a different way:

$$\begin{aligned} v_1 &\longleftrightarrow v_2 \\ v_3 &\longleftrightarrow v_4 \\ &\dots\dots\dots \\ v_{n-2} &\longleftrightarrow v_{n-1} \\ v_n &\longleftrightarrow v_n \end{aligned}$$

Not less evident is that ϕ also satisfies the condition of period two and we label this odd g -function as g_o .

Our next task is to show that the quasi-complement determined either by g_e or by g_o :

(I) possesses the properties of de Morgan complement, that is

- (a) $--a = a$;
- (b) $a \leq b \Rightarrow -b \leq -a$;

(II) and does not possess the properties of Boolean complement, that is

- (c) $a \cap -a \not\leq -b$;
- (d) $a \not\leq b \cup -b$.



„ \Rightarrow (a)” (a.1.) n is an even number. Let $g_e(v_i) = v_j$ and abbreviate V_n/v_i as V_n^{-i} and V_n/v_j as V_n^{-j} .

(a.1.1.) a is singleton, that is $a = \{v_i\}$, where $v_i \in V_n$: $--a = V_n/g_e(V_n/g_e(a)) = V_n/g_e(V_n/v_j) = V_n/g_e(V_n^{-i}) = \{v_i\} = a$.

(a.1.2.) $a = \emptyset$: $--a = V_n/g_e(V_n/g_e(a)) = V_n/g_e(V_n/\emptyset) = V_n/g_e(V_n) = V_n/V_n = \emptyset = a$.

(a.1.3.) a is not a singleton and $a \neq \emptyset$: Then $\exists v_k, \dots, v_m \in V_n$ and $a = \{v_k\} \cup \dots \cup \{v_m\}$.

By evident properties of $/$ and g_e we may establish the following.

$--a = V_n/g_e(V_n/g_e(a)) = V_n/g_e(V_n(g_e(\{v_k\} \cup \dots \cup \{v_m\}))) = V_n/g_e(V_n/g_e(\{v_k\}) \cap \dots \cap V_n/g_e(V_n/g_e(\{v_m\}))) = V_n/g_e(V_n/g_e(\{v_k\})) \cup \dots \cup V_n/g_e(V_n/g_e(\{v_m\}))$.

By (a.1.1.) every member of this union satisfies condition of period two, hence, $--a = V_n/g_e(V_n/g_e(\{v_k\})) \cup \dots \cup V_n/g_e(V_n/g_e(\{v_m\})) = \{v_k\} \cup \dots \cup \{v_m\} = a$.

(a.2.) n is an odd number. Let $\forall i, 1 \leq i \leq n-1, g_o(v_i) = g_e(v_i)$ and $g_o(v_n) = v_n$ and abbreviate additionally V_n/v_n as V_n^{-n} .

The proof will be almost the same except for the case (a.2.1) when $a = \{v_n\}$. Consider only this sub-case. $--a = V_n/g_o(V_n/g_o(a)) = V_n/g_o(V_n/v_n) = V_n/g_o(V_n^{-n}) = V_n/V_n^{-n} = \{v_n\} = a$.

„ \Rightarrow (b)” Hereinafter the distinction between odd and even cases is not crucial, that is why we continue the proof in a more general way.

Let $a \leq b$, that is $b = a \cup c$, where $c \neq \emptyset$.

$$-b = V_n/g(b) = V_n/g(a \cup c) = V_n/(g(a) \cup g(c)).$$

$$-a = V_n/g(a).$$

By the properties of $/$, $V_n/(g(a) \cup g(c)) \subseteq V_n/g(a)$, that is $-b \leq -a$.

„ \Rightarrow (c)” We need to show that $\exists a \exists b (a \cap -a \not\leq b)$. Let $a = \{v_i\}$ and $b = \{v_k\}$, where $i \neq k$ and $1 \leq i \leq n-1$.

$$-a = V_n/g(a).$$

$$a \cap -a = a \cap V_n/g(a) = \{v_i\} \cap V_n^{-j} = \{v_i\} = a.$$

Hence, $-a \cap a \cap b = \emptyset$, that is $-a \cap a \cap b \neq a$, hence $a \cap -a \not\leq -b$.



„ \Rightarrow (d)“ The proof is similar. Let $a = \{v_i\}$ and $b = \{v_k\}$, where $i \neq k$ and $1 \leq i \leq n-1$ and $g(v_k) \neq v_i$. Then $b \cup -b \neq V_n$ and moreover $a \not\subseteq b \cup -b$. Hence, $a \not\subseteq b \cup -b$.

The question now arises of whether we could take the set V to be one-element or even empty. The answer is straightforward. If V is a singleton, there is the only way to define appropriate function g as an identical transformation: for $x \in V, g(x) = x$. Then the corresponding quasi-complement will be Boolean one that returns us to classical logic. If V is empty, then $\mathcal{P}(V)$ is a singleton and the only possible complement on V makes this logic inconsistent.

Thus, now we know how to make our negation de Morgan and our logic first-degree entailment.

3. What does it mean?

First of all, we can formulate the BR-generalization procedure with full strength (in full generality).

1. Take any set of values V containing at least two elements.
2. Construct the power-set of V .
3. Depending on cardinality of V , use one of the two ways to define function g :
 - if V contains even number of elements, define g as d_e ,
 - if V contains odd number of elements, define g as d_0 .
4. In a natural way introduce de Morgan quasi-complement.
5. Define a valuation function v_i as a map from the set of propositional variables into $\mathcal{P}(V)$.
6. To extend it for arbitrary formulas and to introduce entailment relation accept Definitions 5 and 6.
7. Enjoy an adequate semantics for **FDE**.

Second, as a corollary we can place on record that Belnap's approximation lattice A4 is enough to provide an adequate semantics for **FDE**. Logical lattice L4 is in a sense superfluous!



Finally, we have good reasons for and perfect freedom to acclaim that first-degree entailment is indeed everywhere and under every angle, however you slice it!

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