Syntax and semantics of simple paracomplete logics

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Abstract. For an arbitrary fixed element $\beta$ in $\{1, 2, 3, \ldots, \omega\}$ both a sequent calculus and a natural deduction calculus which axiomatise simple paracomplete logic $I_{2,\beta}$ are built. Additionally, a valuation semantic which is adequate to logic $I_{2,\beta}$ is constructed. For an arbitrary fixed element $\gamma$ in $\{1, 2, 3, \ldots\}$ a cortege semantic which is adequate to logic $I_{2,\gamma}$ is described. A number of results obtainable with the axiomatisations and semantics in question are formulated.

Keywords: paracomplete logic, paraconsistent logic, cortege semantics, valuation semantics, sequent calculus, natural deduction calculus

We study logics $I_{2,1}, I_{2,2}, I_{2,3}, \ldots I_{2,\omega}$ presented in [8]. These logics are paracomplete counterparts of paraconsistent logics $I_{1,1}, I_{1,2}, I_{1,3}, \ldots I_{1,\omega}$ from [7]. In the paper, (a) simple paracomplete logics $I_{2,1}, I_{2,2}, I_{2,3}, \ldots I_{2,\omega}$ are defined (see [8]); these logics form (in the order indicated above) a strictly decreasing (in terms of the set-theoretic inclusion) sequence of logics, (b) for any $j$ in $\{0, 1, 2, 3, \ldots, \omega\}$ both a sequent calculus $Gl_{2,j}$ (see [10]) and a natural deduction calculus $Nl_{2,j}$ which axiomatise logic $I_{2,j}$ are formulated, (c) for any $j$ in $\{1, 2, 3, \ldots, \omega\}$, we propose a valuation semantics for logic $I_{2,j}$ (see [9]), (d) for any $j$ in $\{1, 2, 3, \ldots\}$, we propose a cortege semantics for logic $I_{2,j}$ (see [9]). Below there are some results obtained with the semantics and calculi in question.

The language $L$ of each logic in the paper is a standard propositional language with the following alphabet: $\{\& , \lor , \rightarrow , \forall , \exists\}$.

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\(\lor, \neg, (, )p_1, p_2, p_3, \ldots\). As it is expected, \&, \lor, \supset\) are binary logical connectives in \(L\), \neg\) is a unary logical connective in \(L\), brackets (, ) are technical symbols in \(L\) and \(p_1, p_2, p_3, \ldots\) are propositional variables in \(L\). A definition of \(L\)-formula is as usual. Below, we say ‘formula’ instead of ‘\(L\)-formula’ only and adopt the convention on omitting brackets as in [4]. A formula is said to be quasi-elemental iff no logical connective in \(L\) other than \(\neg\) occurs in it. A length of a formula \(A\) is, traditionally, said to be the number of all occurrences of the logical connectives in \(L\) in \(A\). We denote the rule of modus ponens in \(L\) by \(MP\) and the rule of substitution of a formula into a formula instead of a propositional variable in \(L\) by \(Sub\). A logic is said to be a non-empty set of formulas closed under \(MP\) and \(Sub\). A theory for logic \(L\) is said to be a set of formulas including logic \(L\) and closed under \(MP\). It is understood that the set of all formulas is both a logic and a theory for any logic. The set of all formulas is said to be a trivial theory. A complete theory for logic \(L\) is said to be a theory \(T\) for logic \(L\) such that, for some formula \(A\), \(A \in T\) or \(\neg A \in T\). A paracomplete theory for logic \(L\) is said to be a theory \(T\) for logic \(L\) such that \(T\) is not a complete theory and any complete theory for logic \(L\), which includes \(T\), is a trivial theory. A paracomplete logic is said to be a logic \(L\) such that there exists a paracomplete theory for logic \(L\). Simple paracomplete logic is said to be a paracomplete logic \(L\) such that for any paracomplete theory \(T\) for logic \(L\) holds true: there exists a quasi-elemental formula \(A\) such that neither \(A\), nor \(\neg A\) belongs to \(T\).

Let us agree that anywhere in the paper: \(\alpha\) is an arbitrary element in \(\{0, 1, 2, 3, \ldots\omega\}\), \(\beta\) is an arbitrary element in \(\{1, 2, 3, \ldots\omega\}\), \(\gamma\) is an arbitrary element in \(\{1, 2, 3, \ldots\}\). We define calculus \(HI_{2, \alpha}\). This calculus is Hilbert-type calculi, the language of \(HI_{2,\alpha}\) is \(L\). \(HI_{2,\alpha}\) has \(MP\) as the only rule of inference. The notion of a derivation in \(HI_{2,\alpha}\) (of a proof in \(HI_{2,\alpha}\), in particular) is defined as usual; and for \(HI_{2,\alpha}\), both notion of a formula derivable from the set of formulas in this calculus and a notion of a formula provable in this calculus are defined as usual. Now we only need to define the set of axioms of \(HI_{2,\alpha}\).

A formula belongs to the set of axioms of calculus \(HI_{2,\alpha}\) iff it is one of the following forms (hereafter, \(A, B, C\) denote formulas):
Let us establish connections between logics $I_2, I_2, I_2, I_2, \ldots I_{2,\omega}$ and logic $I_{2,0}$ (that is, the classical propositional logic in $L$).

Let $\varphi$ be a mapping of the set of all formulas into itself satisfying the following conditions: (1) $\varphi(p)$ is not a quasi-elemental formula, for any propositional variable $p$ in $L$, (2) for any propositional variable $p$ in $L$, formulas $p \supset \varphi(p)$ and $\varphi(p) \supset p$ belong to logic $I_{2,0}$, (3) $\varphi(B \circ C) = \varphi(B) \circ \varphi(C)$, for any formulas $B, C$ and for any binary logical connective $\circ$ in $L$, (4) $\varphi(\neg B) = \neg \varphi(B)$, for any formula $B$.

Following these conditions, theorem 3 is shown.

**Theorem 3.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$ and for any formula $A$: $A \in I_{2,0}$ iff $\varphi(A) \in I_{2,j}$.

Let now $\psi$ be such a mapping the set of all formulas into itself satisfying the following conditions: (1) $\psi(p) = p$, for any propositional variable $p$ in $L$, (2) $\psi(B \circ C) = \psi(B) \circ \psi(C)$, for any formulas $B, C$ and for any binary logical connective $\circ$ in $L$, (3) $\psi(\neg B) = \psi(B) \supset \neg(p_1 \supset p_1)$, for any formula $B$.

Following these conditions, theorem 4 is shown.

**Theorem 4.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$ and for any formula $A$: $A \in I_{2,0}$ iff $\varphi(A) \in I_{2,j}$.

Let us now show a method to build up a sequent calculus $GI_{2,\beta}$ which axiomatises logic $I_{2,\beta}$. Calculus $GI_{2,\beta}$ (see [10]) is a Gentzen-
type sequent calculus. Sequents are of the form $\Gamma \rightarrow \Delta$ (hereafter, $\Gamma$, $\Delta$, $\Sigma$ and $\Theta$ denote finite sequences of formulas). The set of basic sequents of $\text{GI}_{2,\beta}$ is the set of all sequents of the form $A \rightarrow A$. The only rules of $\text{GI}_{2,\beta}$ are the rules R1-R15, R16($\beta$), R17 listed below.

\[
\begin{align*}
R1: & \quad \frac{\Gamma, A, B, \Delta \rightarrow \Theta}{\Gamma, B, A, \Delta \rightarrow \Theta} \\
R2: & \quad \frac{\Gamma \rightarrow \Theta, A, A}{\Gamma \rightarrow \Theta} \\
R3: & \quad \frac{\Gamma \rightarrow \Theta, A, A}{\Delta \rightarrow \Theta} \\
R4: & \quad \frac{\Gamma \rightarrow \Theta}{\Delta, A \rightarrow \Theta} \\
R5: & \quad \frac{\Gamma \rightarrow \Theta}{A, \Gamma \rightarrow \Theta} \\
R6: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A} \\
R7: & \quad \frac{\Gamma \rightarrow \Delta, A, B, \Sigma \rightarrow \Theta}{A \rightarrow \Theta, B, \Sigma \rightarrow \Theta} \\
R8: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A} \\
R9: & \quad \frac{\Gamma \rightarrow \Theta}{\Delta, A \rightarrow \Theta} \\
R10: & \quad \frac{\Gamma \rightarrow \Theta}{A \rightarrow \Theta, A \cdot B} \\
R11: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A \cdot B} \\
R12: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A \vee B} \\
R13: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, B \vee A} \\
R14: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, A \vee B} \\
R15: & \quad \frac{\Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta} \\
R16: & \quad \frac{E, \Gamma \rightarrow \Theta}{\Gamma \rightarrow \Theta, \neg E} \\
R17: & \quad \frac{\Gamma \rightarrow \Delta, A, \Sigma \rightarrow \Theta}{\Gamma, \Sigma \rightarrow \Delta, \Theta}
\end{align*}
\]

where $E$ is a formula which is not a quasi-elemental formula of a length less than $\beta$.

A derivation in calculus $\text{GI}_{2,\beta}$ is defined in a standard sequent calculus fashion. The definition of a sequent provable in $\text{GI}_{2,\beta}$ is as usual. The cut-elimination theorem is shown (by Gentzen’s method presented in [3]) to be valid in $\text{GI}_{2,\beta}$.

The following theorem 5 is shown.

**Theorem 5.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$ and for any formula $A$: $A \in I_{2,j}$ iff a sequent $\rightarrow A$ is provable in $\text{GI}_{2,j}$.

Let us now show a method to build up a Fitch-style natural deduction calculus $\text{NI}_{2,\beta}$ which axiomatises logic $\text{I}_{2,\beta}$.

The set of $\text{NI}_{2,\beta}$-rules is as follows, where $[A]C$ denotes a derivation of a formula $C$ from a formula $A$.

\[
\begin{align*}
& \frac{\text{CK}C_1}{C} \quad \&_{e11} \\
& \frac{\text{CK}C_1}{C_1} \quad \&_{e12} \\
& \frac{C_1 C_1}{\text{CK}C_1} \quad \&_{i1a}
\end{align*}
\]
A derivation in $\text{NI}_{2,\beta}$ is defined in a standard natural deduction calculus fashion.

The following theorem 6 is shown.

**Theorem 6.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$ and for any formula $A$ : $A \in \text{I}_{2,j}$ iff $A$ is provable in $\text{NI}_{2,j}$.

The proof search procedures which were proposed to the classical and a variety of non-classical logics are applicable [1, 2].

Let us construct $\text{I}_{2,\beta}$-valuation semantics for $\text{I}_{2,\beta}$. By $Q_{\beta}$ we denote the set of all quasi-elemental formulas of a length less or equal to $\beta$. By $I_{2,\beta}$-valuation we mean any mapping $v$ set $Q_{\beta}$ into the set $\{0, 1\}$ such that, for any quasi-elemental formula $e$ of a length less than $\beta$, if $v(e) = 1$, then $v(\neg e) = 0$. Let Form denote the set of all formulas and let $\text{Val}_{2,\beta}$ denote the set of all $I_{2,\beta}$-valuations. It can be shown there exists a unique mapping (denoted by $\xi_{2,\beta}$) satisfying the following six conditions: (1) $\xi_{2,\beta}$ is a mapping a Cartesian product $\text{Form} \times \text{Val}_{2,\beta}$ into the set $\{1, 0\}$, (2) for any quasi-elemental formula $Y$ in $Q_{\beta}$ and any $I_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(Y, v) = v(Y)$, (3) for any formulas $A, B$ and any $I_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A \& B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ and $\xi_{2,\beta}(B) = 1$, (4) for any formulas $A, B$ and any $I_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A \lor B, v) = 1$ iff $\xi_{2,\beta}(A) = 1$ or $\xi_{2,\beta}(B) = 1$, (5) for any formulas $A, B$ and any $I_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(A \supset B, v) = 1$ iff $\xi_{2,\beta}(A) = 0$ or $\xi_{2,\beta}(B) = 1$, (6) for any formula $A$ which is not a quasi-elemental formula of a length less than $\beta$, and for any $I_{2,\beta}$-valuation $v$: $\xi_{2,\beta}(\neg A, v) = 1$ iff $\xi_{2,\beta}(A, v) = 0$. A formula $A$ is said to be $I_{2,\beta}$-valid iff for any $I_{2,\beta}$-valuation $v$, $\xi_{2,\beta}(A, v) = 1$.

The following theorems 7 and 8 are shown.

**Theorem 7.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$, for any formula $A$, for any set $\Gamma$ of formulas: formula $A$ is derivable from $\Gamma$ in $\text{HI}_{2,j}$ iff for
any $I_{2,j}$-valuation $v$, if for any formula $B$ in $\Gamma$, $\xi_{2,j}(B, v) = 1$, then $\xi_{2,j}(A, v) = 1$.

**Theorem 8.** For any $j$ in $\{1, 2, 3, \ldots, \omega\}$ and for any formula $A$, $A \in I_{2,j}$ iff formula $A$ is $I_{2,j}$-valid.

It should be noted that the proposed $I_{2,j}$-valuation semantics is consistent to the requirements, which, in our point of view, N.A. Vasiliev considers to be necessary in [11]: (1) no proposition cannot be true and false at once, (2) in general case, a value of the proposition that is a negation of a proposition $P$, is not determined by the value of $P$.

Let us construct $I_{2,\gamma}$-cortege semantics for $I_{2,\gamma}$. By $I_{2,\gamma}$-cortege we mean an ordered $\gamma + 1$-tuplet of elements of the set $\{1, 0\}$ such that for any two neighboring members of this ordered $\gamma + 1$-tuplet, at least one of them is 0. By a designated $I_{2,\gamma}$-cortege we mean $I_{2,\gamma}$-cortege, where the first member is 1. By $S_{2,\gamma}$ we denote the set of all $I_{2,\gamma}$-corteges and by $D_{2,\gamma}$ we denote the set of all designated $I_{2,\gamma}$-corteges. By a normal $I_{2,\gamma}$-cortege we mean $I_{2,\gamma}$-cortege such that any two neighboring members of this $I_{2,\gamma}$-cortege are different. By a single $I_{2,\gamma}$-cortege we mean a normal $I_{2,\gamma}$-cortege such that the first member of it is 1. By a zero $I_{2,\gamma}$-cortege we mean a normal $I_{2,\gamma}$-cortege such that the first member of it is 0.

It is clear that there exists a unique single $I_{2,\gamma}$-cortege (denoted by $1_\gamma$) and there exists a unique zero $I_{2,\gamma}$-cortege (denoted by $0_\gamma$). It can be shown that there exists a unique binary operation on $S_{2,\gamma}$ (denoted by $\&_{2,\gamma}$) satisfying the following condition, for any $X, Y$ in $S_{2,\gamma}$: if the first member of $I_{2,\gamma}$-cortege $X$ is 1 and the first member of $I_{2,\gamma}$-cortege $Y$ is 1 then $X \&_{2,\gamma} Y$ is $1_\gamma$; otherwise, $X \&_{2,\gamma} Y$ is $0_\gamma$. It can be shown that there exists a unique binary operation on $S_{2,\gamma}$ (denoted by $\lor_{2,\gamma}$) satisfying the following condition, for any $X$ and $Y$ in $S_{2,\gamma}$: if the first member of $I_{2,\gamma}$-cortege $X$ is 1 or the first member of $I_{2,\gamma}$-cortege $Y$ is 1 then $X \lor_{2,\gamma} Y$ is $1_\gamma$; otherwise, $X \lor_{2,\gamma} Y$ is $0_\gamma$. It can be shown that there exists a unique binary operation on $S_{2,\gamma}$ (denoted by $\supset_{2,\gamma}$) satisfying the following condition, for any $X$ and $Y$ in $S_{2,\gamma}$: if the first member of $I_{2,\gamma}$-cortege $X$ is 0 or the first member of $I_{2,\gamma}$-cortege $Y$ is 1 then $X \supset_{2,\gamma} Y$ is $1_\gamma$; otherwise, $X \supset_{2,\gamma} Y$ is $0_\gamma$. It can be shown that there exists a unique unary
operation on $S_{2,\gamma}$ (denoted by $\neg_{2,\gamma}$) satisfying the following condition, for any $I_{2,\gamma}$-cortege $<x_1, x_2, \ldots, x_\gamma, x_{\gamma+1}>$: if $x_{\gamma+1}$ is 1 then $\neg_{2,\gamma}(<x_1, x_2, \ldots, x_\gamma, x_{\gamma+1}>)=<x_2, \ldots, x_\gamma, x_{\gamma+1},0>$ and if, if $x_{\gamma+1}$ is 0, then $\neg_{2,\gamma}(<x_1, x_2, \ldots, x_\gamma, x_{\gamma+1}>)=<x_2, \ldots, x_\gamma, x_{\gamma+1},1>$.

It is clear that $<S_{2,\gamma},D_{2,\gamma},\&_{2,\gamma},\lor_{2,\gamma},\exists_{2,\gamma},\neg_{2,\gamma},\top>$ is a logical matrix. This logical matrix (denoted by $M_{2,\gamma}$) is said to be $I_{2,\gamma}$-matrix. $M_{2,\gamma}$-valuation is said to be a mapping the set of all propositional variables in $I$ into $S_{2,\gamma}$. The set of all $M_{2,\gamma}$-valuations is denoted by $\text{Val}M_{2,\gamma}$. It can be shown that there exists a unique mapping (denoted by $\xi_{M_{2,\gamma}}$) satisfying the following conditions: (1) $\xi_{M_{2,\gamma}}$ is a mapping a Cartesian product $\text{Form} \times \text{Val}M_{2,\gamma}$ into the set $S_{2,\gamma}$, (2) for any propositional variable $p$ in $L$ and for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(p,w)=w(p)$, (3) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(A\&B,w)=\xi_{M_{2,\gamma}}(A,w)\&_{2,\gamma}\xi_{M_{2,\gamma}}(B,w)$, (4) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(A\lor B,w)=\xi_{M_{2,\gamma}}(A,w)\lor_{2,\gamma}\xi_{M_{2,\gamma}}(B,w)$, (5) for any formulas $A$, $B$ and for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(A\exists B,w)=\xi_{M_{2,\gamma}}(A,w)\exists_{2,\gamma}\xi_{M_{2,\gamma}}(B,w)$, (6) for any formula $A$ and for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(\neg A,w)=\neg_{2,\gamma}\xi_{M_{2,\gamma}}(A,w)$.

A formula $A$ is said to be $M_{2,\gamma}$-valid iff for any $M_{2,\gamma}$-valuation $w$, $\xi_{M_{2,\gamma}}(A,w)\in D_{2,\gamma}$.

The following theorems 9–11 are shown.

**Theorem 9.** For any $j$ in $\{1,2,3,\ldots\}$, for any formula $A$ and for any set $\Gamma$ of formulas, formula $A$ is derivable from $\Gamma$ in $H_{2,j}$ iff for any $M_{2,j}$-valuation $w$, if for any formula $B$ from $\Gamma$, $\xi_{M_{2,j}}(B,w)\in D_{1,j}$ then $\xi_{M_{2,j}}(A,w)\in D_{2,j}$.

**Theorem 10.** For any $j$ in $\{1,2,3,\ldots\}$ and for any formula $A$, $A\in I_{2,j}$ iff $A$ is $M_{2,j}$-valid.

**Theorem 11.** For any $j$ in $\{1,2,3,\ldots\}$ and for any formula $A$, $A$ is $M_{2,j}$-valid iff for any $M_{2,j}$-valuation $w$, $\xi_{M_{1,j}}(A,w)\in I_{j}$.

The following theorems 12–19 are shown with the help of the axiomatisations and semantics presented in the paper.
Theorem 12. Logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $I_{2,\omega}$ are simple paracomplete logics.

Theorem 13. For any $j$ and $k$ in \{1, 2, 3, $\ldots$ $\omega$\}, if $j \neq k$ then $I_{2,j} \neq I_{2,k}$.

Theorem 14. For any $j$ in \{1, 2, 3, $\ldots$ $\omega$\}, the positive fragment of logic $I_{2,j}$ is equal to the positive fragment of logic $I_{2,0}$.

Theorem 15. For any $j$ in \{1, 2, 3, $\ldots$ $\omega$\}, logic $I_{2,j}$ is decidable.

Theorem 16. For any $j$ in \{1, 2, 3, $\ldots$ \}, logic $I_{2,j}$ is finitely-valued.

Theorem 17. Logic $I_{2,\omega}$ is not finitely-valued.

Theorem 18. Logic $I_{2,\omega}$ is equal to the intersection of logics $I_{2,1}$, $I_{2,2}$, $I_{2,3}$, $\ldots$.

Theorem 19. There is a continuum of logics which include $I_{2,\omega}$ and are included in $I_{2,1}$.

References


