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Shock Wave Acceleration in a Magnetic Field

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Abstract—An exact solution of MHD equations with plane waves describing the solid-body motion of an ideally conducting gas in a given uniform gravitational field is derived. The motion is due to a piston producing a shock wave propagating throughout the initial equilibrium state with a decreasing density. The solution involves an arbitrary function of the Lagrangian variable, whose choice influences the flow pattern.

Keywords: gravitational field, magnetic field, shock waves, acceleration.

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Shock wave acceleration at the expense of a decrease in the initial density can occur in the stellar and planetary atmospheres exposed to local heating or ionization. Within the framework of gas dynamics this effect was detected by L.I. Sedov [1] in solving the problem of strong blast in a variable-density medium in the absence of counterpressure. On the other hand, when an initial constant pressure is taken into account, the density decrease automatically leads to an increase in the speed of sound and, therefore, in the shock wave velocity, thus producing the preconditions for the loss of the medium inertia, the instability, and the development of different dynamic processes.

A very simple example of the exact solution of the problem of a piston that starts to move at a constant velocity within a pressureless gas and produces an accelerating shock wave, when a certain law of decrease in the initial equilibrium density is fulfilled, is presented in [2, problem 25.37]. In a more realistic situation the effects of the counterpressure and the electromagnetic and gravitational fields must be taken into account, together with the relativistic effects. The exact solutions of this problem within the framework of the special and general relativity but without regard for the counterpressure were given in [3], while the solutions with the counterpressure in the special theory without gravitation were announced in [4], those with account of the frozen-in transverse magnetic field and the counterpressure but without gravitation within the framework of the Newtonian mechanics are presented in [5, 6], and the solutions in the linear formulation are given in [7, 8]. The general overview of the earlier studies can be found in [9]. An investigation of a class of self-similar problems without regard for magnetic fields with a power-law decrease in the initial density is given in [10].

In [5, 6] the "blow-up" effect was found to exist, when a shock wave goes to infinity for a finite time; the effect is due to the unlimited growth of the speed of sound ahead of the shock wave. Within the framework of the relativity theory the state with an infinite temperature is attained for a finite time and at a finite distance from the beginning of the motion [4].

The solution is constructed by the inverse method [11]. If the solutions ahead of and behind the shock wave involve, at least, two arbitrary functions of one variable, then, together with the law of motion of the shock wave, they are determined by three conditions at the discontinuity. In this study, we consider the case of the solid-body motion of a medium with a frozen-in magnetic field which makes it possible to construct a solution with one more arbitrary function of the Lagrangian variable, whose behavior has a considerable effect on the motion.

1. EQUATIONS AND RELATIONS AT THE DISCONTINUITY

We will consider a class of solutions of the one-dimensional problem with plane waves and a frozen-in transverse magnetic field within the framework of the Newtonian mechanics. The process is adiabatic and the gas is perfect. The Lagrangian coordinate ξ is used.

Let $x(\xi,\ t)$ be the law of the medium motion, $\xi = x(\xi,\ 0)$, $v = x_t$ is the velocity, $\rho = \rho_0(\xi)/x_\xi$ is the density, $p = f(\xi)\rho^\gamma$ is the pressure, γ is the constant adiabatic exponent, $H = H_0(\xi)/x_\xi$ is the magnetic field, $q = H^2/(8\pi)$ is the magnetic pressure, and the constant g means the gravitational field. In what follows it is assumed that $1 < \gamma < 2$. The subscripts t and ξ refer to partial derivatives and zero means the initial state of the magnetic field.

Then the equation of motion takes the form [12]:

$$\rho_0 v_t + (p + q)_\xi + \rho_0 g = 0. \tag{1.1}$$

The absence in the nature of magnetic currents leads to the continuity of the function $H_0(\xi)$ across the discontinuity, while the function $f(\xi)$ related with the entropy distribution increases. Moreover, at the shock wave the conditions of the continuity of the law of motion and the conservation of the mass, momentum, and energy fluxes are fulfilled.

A continuous mass variable $m(\xi)$ can conveniently be introduced by the equation $m_\xi = \rho_0$. Then the conditions at the discontinuity surface $t = t_s(m)$ take the form:

$$\begin{gathered} [x]_0^1 = 0, \qquad \left[v - (p + q)t_s'\right]_0^1 = 0, \\ \left[\frac{v^2}{2} + \frac{p x_m}{\gamma - 1} + q x_m - (p + q) v t_s'\right]_0^1 = 0, \end{gathered} \tag{1.2}$$

where the brackets denote the difference between the quantities in the states 1 (behind the shock wave) and 0 (ahead of it). The gas motion is produced by a piston with a fixed coordinate m_0 and the given law of motion $x_p(t) = x(t,\ 0)$.

Ahead of the shock wave the gas is assumed to be in equilibrium

$$p_0 + q_0 = g(m_\infty - m) + q_0(m_\infty) = B_0 - gm, \tag{1.3}$$

where m_∞ is the total gas mass and $q_0(m)$ and $x_0(m)$ are arbitrary functions. At $m = m_\infty$, where $p_0 = 0$, the magnetic field is assumed to be bounded.

2. A SOLUTION WITHOUT GRAVITATION

To understand the role played by gravitation we will consider the solution of Eqs. (1.1) and (1.2) at $g = 0$ which possesses the blow-up effect.

Ahead of the shock wave $p_0 + q_0 = B_0$. Behind the shock the motion with homogeneous deformation takes place

$$x = v(\xi)t, \qquad \rho = \frac{\rho_0(\xi)}{v't}, \qquad p = \frac{C_1}{t^\gamma}, \qquad q = \frac{C_2}{t^2}. \tag{2.1}$$

Then at the discontinuity we have

$$\begin{gathered} v t_s = \xi, \qquad \rho_0 v = \left(\frac{C_1}{t_s^\gamma} + \frac{C_2}{t^2} - B_0\right)t_s', \\ \rho_0 \frac{v^2}{2} + \left(\frac{C_1}{(\gamma - 1)t_s^{\gamma - 1}} + \frac{C_2}{t_s}\right)v' = \frac{B_0 - (2 - \gamma)C_2 v'^2}{\gamma - 1} + \left(\frac{C_1}{t_s^\gamma} + \frac{C_2}{t^2}\right)v t_s'. \end{gathered} \tag{2.2}$$

After ρ_0 and v have been eliminated, we arrive at a quadratic equation for $v' = F(t_s)$ which gives the differential equation with separating variables

$$\frac{1}{v}\frac{dv}{dt_s} = \frac{F(t_s)}{1 - t_s F(t_s)}, \tag{2.3}$$

$$F(t) = -\frac{\gamma - 1}{2C_2(2 - \gamma)}\Bigg[\frac{(\gamma + 1)C_1}{2(\gamma - 1)t^{\gamma-1}} + \frac{3C_2}{2t} + \frac{B_0 t}{2}$$
$$- \left(\left(\frac{(\gamma + 1)C_1}{2(\gamma - 1)t^{\gamma-1}} + \frac{3C_2}{2t} + \frac{B_0 t}{2}\right)^2 + \frac{2C_2(2 - \gamma)}{\gamma - 1}\left(\frac{(\gamma + 1)B_0}{\gamma - 1} + \frac{C_1}{t^\gamma} + \frac{C_2}{t^2}\right)\right)^{1/2}\Bigg].$$

The root of the equation $1 - tF(t) = 0$ can easily be obtained in the case in which $v' = 1/t_s$. From Eqs. (2.2) there follows

$$B_0 - \frac{C_1}{t_*^\gamma} - \frac{C_2}{t_*^2} = 0.$$

In this case, for a finite time t_* the shock wave goes at infinity at the speed of sound which, in turn, increases without bound.

The blow-up regime can also occur without magnetic field, when $C_2 = 0$ and $p = B_0$ is constant. In this case, Eq. (2.3) is fully integrable

$$v = \frac{\xi}{t_s} = v_0 \frac{p_1 - p_0 t_0^\gamma}{p_1 - p_0 t_s^\gamma}\left(\frac{t_s}{t_0}\right)^{(\gamma-1)/2}, \qquad t_* = \left(\frac{1}{p_0}\right)^{1/\gamma},$$

where $v_0 > 0$ and $t_0 > 0$ are the piston velocity at the initial moment and the moment itself, so that the Lagrangian variable is measured from the point $v_0 t_0 > 0$.

In the vicinity of $\xi = \infty$ the initial density

$$\rho_0 = \frac{(p_1 - p_0 t_s^\gamma)t_s'}{v t_s^\gamma} \sim (p_1 - p_0 t_s^\gamma)^4 \sim \frac{1}{\xi^4}.$$

Thus, the total mass is constant. As the motion is produced, the shock wave velocity, together with the initial speed of sound $a_0 = (\gamma p_0/\rho_0)^{1/2}$, increase without bound, as ξ^2.

The work done by the piston

$$A = \frac{p_1 v_0}{\gamma - 1}(t_0^{1-\gamma} - t_*^{1-\gamma}) \tag{2.4}$$

is also finite.

Formula (2.4) shows that the work done by the piston may be small but important as a catalyzer of the shock acceleration development, since in the initial state ahead of it there is an infinite energy “sea”, even at a small value of p_0. Clearly that the problem solution is model but physically it can be realized as a mechanism of the local loss of stability of the hot-gas equilibrium.

When gravitation is taken into account (Eq. (1.1)), a solution of the type (2.1) is generally impossible in the case of the initial equilibrium state (1.3). An analog can be furnished by the law of motion of the form

$$x = -1/2gt^2 + v(\xi)t, \tag{2.5}$$

which shows that behind the shock wave the gas particles start to move toward the gravitation field source with time. In this case, the entire flow pattern considerably changes.

3. GRAVITATION FIELD INFLUENCE

The law of motion (2.5) leads to a fairly complicated first-order equation whose solution is qualitatively investigated. However, a simpler exact solution with a single arbitrary function of the Lagrangian variable m can also be constructed assuming that behind the shock wave we have the solid-body motion $x = at^2/2 + u(m)$ and p and q are functions of m.

Then from the equation of motion (1.1) it follows that

$$p + q = B_1 - (a + g)m, \qquad p = \frac{f(m)}{u'^{\gamma}}, \qquad q = \frac{q_0(m)}{u'^2}, \tag{3.1}$$

where $B_1 = \text{const}$. With account for Eq. (1.3), there are four arbitrary functions of the mass m, namely, x_0, u, q_0, and t_s, of which three are determined by the conditions at discontinuity (1.2) and one remains arbitrary.

The law of momentum flux conservation is immediately integrated and yields

$$t_s = \frac{C}{B_1 - B_0 - am}, \tag{3.2}$$

where $C = \text{const}$. If we assume that $B_1 = (a + g)m_\infty + q_0(m_\infty)$, then the pressure $p(m_\infty) = 0$, together with the initial pressure p_0 (on-design regime). In this case, the magnetic field is equalized. At smaller B_1 the shock wave does not reach the gas layer edge even for infinite time and is actually stopped. At greater B_1 the arbitrary discontinuity breaks down at the edge of the layer, whose thickness is finite. Then the constant $C = t_0 g m_\infty$, where t_0 is the moment of the beginning of the piston motion. Choosing the measurement units as $g = 1$, $m_\infty = 1$, and $t_0 = 1$ we have $t_s = 1/(1 - m)$.

The condition of the energy flux continuity can be transformed to the conservation of the generalized enthalpy

$$\left[\frac{(D - v)^2}{2} + \frac{\gamma p}{(\gamma - 1)\rho} + \frac{2q}{\rho}\right]_0^1 = 0, \tag{3.3}$$

where $D = dx_0/dt = x_0'/t_s'$ is the discontinuity velocity.

Equation (3.3) is linear with respect to q_0; for this reason, after some algebra and with account for integral (3.2) in can be reduced to the equality

$$q_0 = \frac{1}{2 - \gamma}\left(\frac{D}{t_s} - a\right)\left(\frac{D}{t_s} - \frac{\gamma + 1}{2}a - \gamma - \gamma q_0(1) t_s\right) \tag{3.4}$$

with the restriction $q_0 \geq 0$.

Hence there follows the asymptotics of the shock wave velocity, as $t \to \infty$

$$D = 2\gamma q_0(1) t^2. \tag{3.5}$$

After the function $D(t)$ has been specified, together with its asymptotics, all the other functions are determined using integral (3.2), Eq. (3.4), and the continuity of the law of motion $x_0 = at_s^2/2 + u$.

4. EXAMPLES OF FLOWS

We will consider several examples.

(a) Let a magnetic field be absent. Then we have

$$D = \left(\frac{\gamma + 1}{2}a + \gamma\right)t,$$

$$x_0 = \left(\frac{\gamma + 1}{2}a + \gamma\right)\frac{1}{2(1 - m)^2} = \xi.$$

Here, the additive constant is omitted. Hence follows that the initial density $\rho_0 = m_\xi \sim \xi^{-3/2}$.

In this case, even at an arbitrarily small piston acceleration a the shock wave goes to infinity at the acceleration γ in the units of g.

(b) Let the initial magnetic field be constant, $q_0 = q_0(1)$. Then the velocity D is determined by the greater solution of the quadratic equation (3.4). Using the leading term of asymptotics (3.5) $D \approx \gamma q_0 t^2$ gives the following result

$$x_0 \approx \frac{\gamma q_0 t_s^3}{3} = \frac{\gamma q_0}{3(1-m)^3}.$$

The initial density $\rho_0 \approx (\gamma q_0/81)^{1/3}\xi^{-4/3}$.

(c) At large t the case of a "shivery" shock wave can also be considered, when $D = 2q_0(1)t^2 + A\sin(\omega/t)$. In this case, the magnetic pressure is determined by Eq. (3.4) with the replacement $t_s = (1-m)^{-1}$.

Summary. With reference to simple examples pertaining to a class of exact solutions of the MHD equations corresponding to the special motion of a conducting gas behind a shock wave, it is shown that, despite the decelerating influence of the gravitational field, under certain conditions determined by a decrease in the initial gas velocity even a very small piston acceleration can produce the shock wave motion at a finite acceleration. The presence of a magnetic field only enhances this effect leading to an unbounded growth of the discontinuity acceleration.

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