ISSN 1547-4771, Physics of Particles and Nuclei Letters, 2014, Vol. 11, No. 7, pp. 933–937.

Once More on Parastatistics[1]

V. N. Tolstoy

Lomonosov Moscow State University, Skobeltsyn Institute of Nuclear Physics, Moscow, 119991 Russian Federation
e-mail: vntolstoy@yandex.ru

Abstract—Equivalence between algebraic structures generated by parastatistics triple relations of Green (1953) and Greenberg–Messiah (1965), and certain orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras is found explicitly. Moreover, it is shown that such superalgebras give more complex para-Fermi and para-Bose systems then ones of Green–Greenberg–Messiah.

DOI: 10.1134/S1547477114070449

1. INTRODUCTION

The usual creation and annihilation operators of identical particles, fermions $a_i^{\pm}$ $(i = 1, ..., m)$ and bosons $b_j^{\pm}$ $(j = 1, ..., n)$, satisfy the canonical commutation relations:

$$\{a_i^{\zeta}, a_j^{\eta}\} = \frac{1}{2}|\eta - \zeta|\delta_{ij}, \quad [b_i^{\zeta}, b_j^{\eta}] = \frac{1}{2}(\eta - \zeta)\delta_{ij}. \tag{1.1}$$

Here and elsewhere the Greek letters $\zeta, \eta \in \{+, -\}$ if they are upper indexes, and they are interpreted as $+1$ and -1 in the algebraic expressions of the type $\eta - \zeta$.

From the relations (1.1) it follows the so-called "*symmetrization postulate*" (*SP*): *States of more than one identical particle must be antisymmetric (fermions) or symmetric (bosons) under permutations.*

In 1953 Green [1] proposed to refuse *SP* and he introduced algebras with the triple relations:

$$[[a_i^{\zeta}, a_j^{\eta}], a_k^{\xi}] = |\xi - \eta|\delta_{jk}a_i^{\zeta} - |\xi - \zeta|\delta_{ik}a_j^{\eta} \quad \text{(parafermions)}, \tag{1.2}$$

$$[\{b_i^{\zeta}, b_j^{\eta}\}, b_k^{\xi}] = (\xi - \eta)\delta_{jk}b_i^{\zeta} + (\xi - \zeta)\delta_{ik}b_j^{\eta} \quad \text{(parabosons)}. \tag{1.3}$$

The usual fermions and bosons satisfy these relations but also another solutions exist.

In 1962 Kamefuchi and Takahashi [2] (also see [3]) shown that the parafermionic algebra is isomorphic to the orthogonal Lie algebra $\mathfrak{o}(2m + 1) := \mathfrak{o}(2m + 1, \mathbb{C})$. Later in 1980 Ganchev and Palev [4] proved that the parabosonic algebra is isomorphic to the orthosymplectic $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1|2n)$.

In 1965 Greenberg and Messiah [5] considered parasystem consisting simultaneously from parafermions and parabosons and they defined the relative commutation rules between parafermions and parabosons. There are two types of such relations:

$$\begin{gathered}
[[a_i^{\zeta}, a_j^{\eta}], b_k^{\xi}] = 0, \quad [\{b_i^{\zeta}, b_j^{\eta}\}, a_k^{\xi}] = 0, \\
[[a_i^{\zeta}, b_j^{\eta}], a_k^{\xi}] = -|\xi - \zeta|\delta_{ik}b_j^{\eta}, \\
\{[a_i^{\zeta}, b_j^{\eta}], b_k^{\xi}\} = (\xi - \eta)\delta_{jk}a_i^{\zeta},
\end{gathered} \tag{1.4}$$

$$\begin{gathered}
[[a_i^{\zeta}, a_j^{\eta}], b_k^{\xi}] = 0, \quad [\{b_i^{\zeta}, b_j^{\eta}\}, a_k^{\xi}] = 0, \\
\{\{a_i^{\zeta}, b_j^{\eta}\}, a_k^{\xi}\} = |\xi - \zeta|\delta_{ik}b_j^{\eta}, \\
[\{a_i^{\zeta}, b_j^{\eta}\}, b_k^{\xi}] = (\xi - \eta)\delta_{jk}a_i^{\zeta},
\end{gathered} \tag{1.5}$$

where $i, j, k = 1, 2, ..., m$ for the symbols a's and $i, j, k = 1, 2, ..., n$ for the symbols b's. The first case (1.4) was called as *the relative para-Fermi set* and the second case (1.5) was called as *the relative para-Boson set*.[2]

In 1982 Palev [6] shown that the case (1.4) with (1.2) and (1.3) is isomorphic to the orthosymplectic $\mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m + 1|2n)$. No any similar solution for the second case (1.5) was known up to now.

Here we show that the case (1.5) with (1.2) and (1.3) is isomorphic to the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra osp(1, 2m|2n, 0). Moreover it will demonstrate that the more general mixed parasystem, which simultaneously involves the relative para-Fermi and relative para-Bose sets, contains two sorts of parafermions and one sort of parabosons and it is isomorphic to the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$. All previous cases are particular (degenerated) variants of this general case.

The paper is organized as follows. Section 2 provides a definition and general structure of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebras and also a matrix realization and

[1] The article is published in the original.

[2] The names *the relative para-Fermi and para-Boson set* are directly related to type of the Lie bracket (commutator or anticommutator) given between parafermion and paraboson elements.

a Cartan–Weyl basis of the general linear $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. In Section 3 we describe the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ and show that a subset of its defining triple relations in the terms of short root vectors coincides with the relative para-Bose set.

2. SUPERALGEBRA $\mathfrak{gl}(m_1, m_2|n_1, n_2)$

At first we remind a general definition of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra [7, 8].

The $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded LSA $\mathfrak{g}$, as a linear space, is a direct sum of four graded components

$$\mathfrak{g} = \bigoplus_{\mathbf{a} = (a_1, a_2)} \mathfrak{g}_{\mathbf{a}} = \mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)} \oplus \mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)} \quad (2.1)$$

with a bilinear operation $[\![\cdot, \cdot]\!]$ satisfying the identities (grading, symmetry, Jacobi):

$$\begin{aligned} \deg([\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!]) &= \deg(x_{\mathbf{a}}) + \deg(x_{\mathbf{b}}) = \mathbf{a} + \mathbf{b} \\ &= (a_1 + b_1, a_2 + b_2), \end{aligned} \quad (2.2)$$

$$[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!] = -(-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, x_{\mathbf{a}}]\!], \quad (2.3)$$

$$[\![x_{\mathbf{a}}, [\![y_{\mathbf{b}}, z]\!]]\!] = [\![[\![x_{\mathbf{a}}, y_{\mathbf{b}}]\!], z]\!] + (-1)^{\mathbf{ab}}[\![y_{\mathbf{b}}, [\![x_{\mathbf{a}}, z]\!]]\!], \quad (2.4)$$

where the vector $(a_1 + b_1, a_2 + b_2)$ is defined mod (2, 2) and $\mathbf{ab} = a_1b_1 + a_2b_2$. Here in (2.2)–(2.4) $x_{\mathbf{a}} \in \mathfrak{g}_{\mathbf{a}}$, $x_{\mathbf{b}} \in \mathfrak{g}_{\mathbf{b}}$, and the element $z \in \mathfrak{g}$ is not necessarily homogeneous. From (2.2) it is follows that $\mathfrak{g}_{(0,0)}$ is a Lie subalgebra in $\mathfrak{g}$, and the subspaces $\mathfrak{g}_{(1,1)}$, $\mathfrak{g}_{(1,0)}$ and $\mathfrak{g}_{(0,1)}$ are $\mathfrak{g}_{(0,0)}$-modules. It should be noted that $\mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$ is a Lie subalgebra in $\mathfrak{g}$ and the subspace $\mathfrak{g}_{(1,0)} \oplus \mathfrak{g}_{(0,1)}$ is a $\mathfrak{g}_{(0,0)} \oplus \mathfrak{g}_{(1,1)}$-module, and moreover $\{\mathfrak{g}_{(1,1)}, \mathfrak{g}_{(1,0)}\} \subset \mathfrak{g}_{(0,1)}$ and vice versa $\{\mathfrak{g}_{(1,1)}, \mathfrak{g}_{(0,1)}\} \subset \mathfrak{g}_{(1,0)}$. From (2.2) and (2.3) it is follows that the general Lie bracket $[\![\cdot, \cdot]\!]$ for homogeneous elements posses two value: commutator $[\cdot, \cdot]$ and anticommutator $\{\cdot, \cdot\}$ as well as in a case of usual $\mathbb{Z}_2$-graded Lie superalgebras [9].

Now we construct a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded matrix superalgebras $\mathfrak{gl}(m_1, m_2|n_1, n_2)$.

Let an arbitrary $(m_1 + m_2 + n_1 + n_2) \times (m_1 + m_2 + n_1 + n_2)$-matrix M be presented in the following block form:[3]

$$M = \begin{pmatrix} A_{(0,0)} & A_{(1,1)} & A_{(1,0)} & A_{(0,1)} \\ B_{(1,1)} & B_{(0,0)} & B_{(0,1)} & B_{(1,0)} \\ C_{(1,0)} & C_{(0,1)} & C_{(0,0)} & C_{(1,1)} \\ D_{(0,1)} & D_{(1,0)} & D_{(1,1)} & D_{(0,0)} \end{pmatrix}, \quad (2.5)$$

where the diagonal block matrices $A_{(0,0)}$, $B_{(0,0)}$, $C_{(0,0)}$, $D_{(0,0)}$ have the dimensions $m_1 \times m_1$, $m_2 \times m_2$, $n_1 \times n_1$ and $n_2 \times n_2$ correspondingly, the dimensions of the non-diagonal block matrices $A_{(1,1)}$, $A_{(1,0)}$, $A_{(0,1)}$, etc. are easy determined by the dimensions of these diagonal block matrices. The matrix M can be split into the sum of four matrices:

$$\begin{aligned} M &= M_{(0,0)} + M_{(1,1)} + M_{(1,0)} + M_{(0,1)} \\ &= \begin{pmatrix} A_{(0,0)} & 0 & 0 & 0 \\ 0 & B_{(0,0)} & 0 & 0 \\ 0 & 0 & C_{(0,0)} & 0 \\ 0 & 0 & 0 & D_{(0,0)} \end{pmatrix} \\ &+ \begin{pmatrix} 0 & A_{(1,1)} & 0 & 0 \\ B_{(1,1)} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{(1,1)} \\ 0 & 0 & D_{(1,1)} & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & A_{(1,0)} & 0 \\ 0 & 0 & 0 & B_{(1,0)} \\ C_{(1,0)} & 0 & 0 & 0 \\ 0 & D_{(1,0)} & 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 & 0 & A_{(0,1)} \\ 0 & 0 & B_{(0,1)} & 0 \\ 0 & C_{(0,1)} & 0 & 0 \\ D_{(0,1)} & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (2.6)$$

Let us define the general commutator $[\![\cdot, \cdot]\!]$ on a space of all such matrices by the following way:

$$\begin{aligned} [\![M_{(a_1,a_2)}, M'_{(b_1,b_2)}]\!] &:= M_{(a_1,a_2)}M'_{(b_1,b_2)} \\ &- (-1)^{a_1b_1 + a_2b_2} M'_{(b_1,b_2)}M_{(a_1,a_2)} \end{aligned} \quad (2.7)$$

for the homogeneous components $M_{(a_1,a_2)}$ and $M_{(b_1,b_2)}$. For arbitrary matrices M and M' the commutator $[\![\cdot, \cdot]\!]$ is extended by linearity. It is easy to check that

$$[\![M_{(a_1,a_2)}, M'_{(b_1,b_2)}]\!] = M''_{(a_1+a_2, b_1+b_2)}, \quad (2.8)$$

where the sum $(a_1 + a_2, b_1 + b_2)$ is defined mod (2, 2). Thus the *grading* condition (2.2) is available. The *symmetry* and *Jacobi* identities (2.3) and (2.4) are available

[3] It is evidently supposed that all such matrices in each block-row or in each block-column have the same number of rows or columns.

too. Hence we obtain a Lie superalgebra which is called $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. It should be noted that

$$
\begin{aligned}
[\![M_{\mathbf{a}}, M'_{\mathbf{b}}]\!] &= [M_{\mathbf{a}}, M'_{\mathbf{b}}] \quad \text{if } \mathbf{ab} = 0, 2,\\
[\![M_{\mathbf{a}}, M'_{\mathbf{b}}]\!] &= [M_{\mathbf{a}}, M'_{\mathbf{b}}] \quad \text{if } \mathbf{ab} = 1.
\end{aligned}
\tag{2.9}
$$

Now we consider the Cartan–Weyl basis of $\mathfrak{gl}(m_1, m_2|n_1, n_2)$ and its supercommutation ($\mathbb{Z}_2 \times \mathbb{Z}_2$-graded) relations. In accordance with the block structure of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded matrix (2.5) we introduce a $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded function (grading) $\mathbf{d}(\cdot)$ defined on the integer segment $[1, 2, ..., m_1, m_1 + 1, ..., m_1 + m_2, m_1 + m_2 + 1, ..., m_1 + m_2 + n_1, m_1 + m_2 + n_1 + 1, ..., m_1 + m_2 + n_1 + n_2]$ as follows:

$$
\mathbf{d}_i := \mathbf{d}(i) = \begin{cases}
(0, 0) \quad \text{for } i = 1, 2, \dots, m_1,\\
(1, 1) \quad \text{for } i = m_1 + 1, \dots, m_1 + m_2,\\
(1, 0)\\
\text{for } i = m_1 + m_2 + 1, \dots, m_1 + m_2 + n_1,\\
(0, 1) \quad \text{for } i = m_1 + m_2 + n_1 + 1, \dots,\\
m_1 + m_2 + n_1 + n_2.
\end{cases}
\tag{2.10}
$$

Let e_{ij} be the $(m_1 + m_2 + n_1 + n_2) \times (m_1 + m_2 + n_1 + n_2)$ matrix (2.5) with 1 is in the (i, j)-th place and other entries 0. The matrices e_{ij} $(i, j = 1, 2, ..., m_1 + m_2 + n_1 + n_2)$ are homogeneous, moreover, the grading $\deg(e_{ij})$ is determined by

$$
\deg(e_{ij}) = \mathbf{d}_{ij} := \mathbf{d}_i + \mathbf{d}_j \quad (\mathrm{mod}(2, 2)),
\tag{2.11}
$$

and the supercommutator for such matrices is given as follows

$$
[\![e_{ij}, e_{kl}]\!] := e_{ij}e_{kl} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}} e_{kl}e_{ij}.
\tag{2.12}
$$

It is easy to check that

$$
[\![e_{ij}, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}} \delta_{il}e_{kj}.
\tag{2.13}
$$

The elements e_{ij} $(i, j = 1, 2, ..., m_1 + m_2 + n_1 + n_2)$ with the relations (2.13) generates the Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. The elements $h_i := e_{ii}$ $(i, j = 1, 2, ..., m_1 + m_2 + n_1 + n_2)$ compose a basis in the Cartan subalgebra $\mathfrak{h}(m_1 + m_2|n_1 + n_2) \subset \mathfrak{gl}(m_1, m_2|n_1, n_2)$.

The Lie superalgebra $\mathfrak{gl}(m_1, m_2|n_1, n_2$ play a special role among all finite dimensional $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebras. Namely, a general Ado's theorem is valid. It states: *Any finite dimensional Lie $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra can be realized in terms of a subalgebra of* $\mathfrak{gl}(m_1, m_2|n_1, n_2)$. This theorem was proved by Scheunert [10] for all finite dimensional graded generalized Lie algebras including our cases.

As an illustration of the Ado's theorem, in the next section we give realization of the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ in terms of the superalgebra $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$ and, moreover, we present a Cartan–Weyl basis of the orthosymplectic superalgebra and its explicit commutation relations and we also show that a subset of the short root vectors of the Cartan–Weyl basis generates this superalgebra and describe the parastatistics with the relative para-Fermi and para-Bose sets simultaneously.

3. ORTHOSYMPLECTIC SUPERALGEBRA $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ AND ITS RELATION WITH PARASTATISTICS

We start with an explicit description of embedding of the orthosymplectic Lie superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ in the general linear Lie superalgebra $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$. For this propose the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded integer segment $\mathbb{S}_N^{(\mathbf{d})} := [1, 2, ..., 2N + 1]$, where $N = m_1 + m_2 + n$, with the grading $\mathrm{d}(\cdot)$ given by

$$
\mathbf{d}_i := \mathbf{d}(i) = \begin{cases}
(0, 0) \quad \text{for } i = 1, 2, \dots, 2m_1,\\
(1, 1) \quad \text{for } i = 2m_1 + 1, \dots, 2m_1 + 2m_2,\\
(1, 0) \quad \text{for } i = 2m_1 + 2m_2 + 1, \dots,\\
2m_1 + 2m_2 + 2n,
\end{cases}
\tag{3.1}
$$

is reindexed by the following way $\tilde{\mathbb{S}}_N^{(\mathbf{d})} := [0, \pm 1, \pm 2, ..., N]$ with the grading $\mathbf{d}(\cdot)$ given by

$$
\mathbf{d}_i := \mathbf{d}(i)
$$

$$
= \begin{cases}
(0, 0) \quad \text{for } i = 0, \pm 1, \pm 2, \dots, \pm m_1,\\
(1, 1) \quad \text{for } i = \pm(m_1 + 1), \dots, \pm(m_1 + m_2),\\
(1, 0) \quad \text{for } i = \pm(m_1 + m_2 + 1), \dots,\\
\pm(m_1 + m_2 + n),
\end{cases}
\tag{3.2}
$$

Rows and columns of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded $(2N + 1) \times (2N + 1)$-matrices are enumerated by the indices $0, 1, -1, 2, -2, ..., ..., N, -N$ $(N = m_1 + m_2 + n)$. Let $e_{ij}(i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})})$ be the standard (unit) basis of $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$ with the given indexing and the canonical supercommutation relations:

$$
[\![e_{ij}, e_{kl}]\!] = \delta_{jk}e_{il} - (-1)^{\mathbf{d}_{ij}\mathbf{d}_{kl}} \delta_{il}e_{kj},
\tag{3.3}
$$

where $\mathbf{d}_{ij} = \mathbf{d}_i + \mathbf{d}_j$ and the grading $\mathbf{d}(\cdot)$ is given by (3.2).

The orthosymplectic ($\mathbb{Z}_2 \times \mathbb{Z}_2$-graded) Lie superalgebra $\mathfrak{osp}(2m_1 + 1, 2m_2|2n, 0)$ is embedded in $\mathfrak{gl}(2m_1 + 1, 2m_2|2n, 0)$ as a linear span of the elements

$$
x_{ij} := e_{i,-j} - (-1)^{\mathbf{d}_i\mathbf{d}_j + \mathbf{d}_{ij}^2} \phi_i \phi_j e_{j,-i} \quad (i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})}),
\tag{3.4}
$$

where the index function ϕ_i is given as follows

$$\phi_i := \begin{cases} 1 & \text{if } i = 0, \pm 1, \pm 2, \ldots, \pm(m_1+m_2), \\ 1 & \text{if } i = m_1+m_2+1, \ldots, m_1+m_2+n, \\ -1 & \text{if } i = -m_1-m_2-1, \ldots, -m_1-m_2-n. \end{cases} \tag{3.5}$$

It is easy to verify that the elements (3.4) satisfy the following supercommutation relations

$$\begin{aligned} [\![x_{ij}, x_{kl}]\!] &= \delta_{j,-k}x_{il} - \delta_{j,-l}(-1)^{\mathbf{d}_k\mathbf{d}_l + \mathbf{d}_{kl}^2}\phi_k\phi_l x_{ik} \\ &- \delta_{i,-k}(-1)^{\mathbf{d}_i\mathbf{d}_j + \mathbf{d}_{ij}^2}\phi_i\phi_j x_{jl} - \delta_{i,-l}(-1)^{\mathbf{d}_{ij}\mathbf{d}_{ik}}x_{kj}. \end{aligned} \tag{3.6}$$

Not all elements (3.4) are linearly independent because they satisfy the relations

$$x_{ij} = -(-1)^{\mathbf{d}_i\mathbf{d}_j + \mathbf{d}_{ij}^2}\phi_i\phi_j x_{ji} \quad (i, j \in \tilde{\mathbb{S}}_N^{(\mathbf{d})}), \tag{3.7}$$

and what is more

$$x_{ii} = 0 \quad \text{for } i = 0, \pm 1, \pm 2, \ldots, \pm(m_1+m_2). \tag{3.8}$$

From the general supercommutation relations (3.6) it follows at once that the short root vectors x_{0i} ($i = \pm 1, \pm 2, \ldots, \pm(m_1 + m_2 + n)$) satisfy the following triple relations:

$$[\![[\![x_{0i}, x_{0j}]\!], x_{0k}]\!] = -\delta_{j,-k}\phi_j x_{0i} + \delta_{i,-k}(-1)^{\mathbf{d}_i\mathbf{d}_j}\phi_i x_{0j}. \tag{3.9}$$

Conversely, let the abstract $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded generators x_{0i} ($i = \pm 1, \pm 2, \ldots, \pm(m_1 + m_2 + n)$) with the grading $\deg(x_{0i}) = \mathbf{d}_{0i} \equiv \mathbf{d}_0 + \mathbf{d}_i = \mathbf{d}_i$, where $\mathbf{d}_i$ is given by (3.2), satisfy the relations (3.9), where the index function ϕ_i is determined by (3.5), then it is not difficult to check that these relations generate for the superalgebra $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$.

The defining relations (3.9) can be rewritten in detailed in accordance with the explicit grading of their generators using the following notations:[4]

$$\begin{aligned} a_i^{-\zeta} &:= \zeta\sqrt{2}x_{0,\zeta i} \quad (\deg(a_i^{\zeta}) = (0,0)) \\ &\qquad \text{for } i = 1, 2, \ldots, m_1, \\ \tilde{a}_i^{-\zeta} &:= \zeta\sqrt{2}x_{0,\zeta(m_1+i)} \quad (\deg(\tilde{a}_i^{\zeta}) = (1,1)) \\ &\qquad \text{for } i = 1, 2, \ldots, m_2, \\ b_i^{-\zeta} &:= \zeta\sqrt{2}x_{0,\zeta(m+i)} \quad (\deg(b_i^{\zeta}) = (1,0)) \\ &\qquad \text{for } i = 1, 2, \ldots, n, \end{aligned} \tag{3.10}$$

where $m := m_1 + m_2$. Substituting (3.10) in (3.9) we obtain the different types of defining triple relations.

1. Parafermion relations:

(a1) the defining relations of $\mathfrak{o}(2m_1+1)$:

$$\begin{aligned} [[a_i^{\zeta}, a_j^{\eta}], a_k^{\xi}] &= |\xi-\eta|\delta_{jk}a_i^{\zeta} - |\xi-\zeta|\delta_{ik}a_j^{\eta} \\ &\text{for } i, j, k = 1, 2, \ldots, m_1; \end{aligned} \tag{3.11}$$

(a2) the defining relations of $\mathfrak{o}(2m_2+1)$:

$$\begin{aligned} [[\tilde{a}_i^{\zeta}, \tilde{a}_j^{\eta}], \tilde{a}_k^{\xi}] &= |\xi-\eta|\delta_{jk}\tilde{a}_i^{\zeta} - |\xi-\zeta|\delta_{ik}\tilde{a}_j^{\eta} \\ &\text{for } i, j, k = 1, 2, \ldots, m_2; \end{aligned} \tag{3.12}$$

(a3) the mixed parafermion relations:

$$[[a_i^{\zeta}, a_j^{\eta}], \tilde{a}_k^{\xi}] = 0, \quad [[\tilde{a}_i^{\zeta}, \tilde{a}_j^{\eta}], a_k^{\xi}] = 0, \tag{3.13}$$

$$\begin{aligned} [[a_i^{\zeta}, \tilde{a}_j^{\eta}], a_k^{\xi}] &= -|\xi-\zeta|\delta_{ik}\tilde{a}_j^{\eta}, \\ [[a_i^{\zeta}, \tilde{a}_j^{\eta}], \tilde{a}_k^{\xi}] &= |\xi-\eta|\delta_{jk}a_i^{\zeta}, \end{aligned} \tag{3.14}$$

where $i, j, k = 1, \ldots, m_1$ for the symbols a's and $i, j, k = 1, \ldots, m_2$ for the symbols $\tilde{a}$'s.

2. Paraboson relations:

(b1) the defining relations of $\mathfrak{osp}(1|2n)$:

$$\begin{aligned} [\{b_i^{\zeta}, b_j^{\eta}\}, b_k^{\xi}] &= (\xi-\eta)\delta_{jk}b_i^{\zeta} + (\xi-\zeta)\delta_{ik}b_j^{\eta} \\ &\text{for } i, j, k = 1, 2, \ldots, n. \end{aligned} \tag{3.15}$$

2. Mixed parafermion and paraboson relations:

(ab1) the relative para-Fermi set:

$$\begin{aligned} &[[a_i^{\zeta}, a_j^{\eta}], b_k^{\xi}] = 0, \quad [\{b_i^{\zeta}, b_j^{\eta}\}, a_k^{\xi}] = 0, \\ &[[a_i^{\zeta}, b_j^{\eta}], a_k^{\xi}] = -|\xi-\zeta|\tilde{b}_{ik}b_j^{\eta}, \\ &\{[a_i^{\zeta}, b_j^{\eta}], b_k^{\xi}\} = (\xi-\eta)\delta_{jk}a_i^{\zeta}, \end{aligned} \tag{3.16}$$

where $i, j, k = 1, 2, \ldots, m_1$ for the symbols a's and $i, j, k = 1, \ldots, n$ for the symbols b's;

(ab2) the relative para-Bose set:

$$\begin{aligned} &[[\tilde{a}_i^{\zeta}, \tilde{a}_j^{\eta}], b_k^{\xi}] = 0, \quad [\{b_i^{\zeta}, b_j^{\eta}\}, \tilde{a}_k^{\xi}] = 0, \\ &\{\{\tilde{a}_i^{\zeta}, b_j^{\eta}\}, \tilde{a}_k^{\xi}\} = |\xi-\zeta|\delta_{ik}b_j^{\eta}, \\ &[\{\tilde{a}_i^{\zeta}, b_j^{\eta}\}, b_k^{\xi}] = (\xi-\eta)\delta_{jk}\tilde{a}_i^{\zeta}, \end{aligned} \tag{3.17}$$

where $i, j, k = 1, 2, \ldots, m_2$ for the symbols $\tilde{a}$'s and $i, j, k = 1, \ldots, n$ for the symbols b's;

(ab3) the relations with distinct grading elements:

$$\begin{aligned} \{[a_i^{\zeta}, \tilde{a}_j^{\eta}], b_k^{\xi}\} &= [\{\tilde{a}_j^{\eta}, b_k^{\xi}\}, a_i^{\zeta}] \\ &= \{[b_k^{\xi}, a_i^{\zeta}], \tilde{a}_j^{\eta}\} = 0. \end{aligned} \tag{3.18}$$

The result connected with the relation (3.9) can be reformulated the following way. *If we have two sorts of the parafermions* a_i^{ζ} ($i = 1, 2, \ldots, m_1$) and $\tilde{a}_i^{\zeta}$ ($i = 1, 2, \ldots, m_2$) *with the triple relations (3.11)–(3.14) and one sort of the parabosons* b_i^{ζ} ($i = 1, 2, \ldots, n$) *with the triple relations* (3.15) *which together satisfy the relative para-Fermi set* (3.16) *and relative para-Bose set* (3.17), *and they obey also the triple relations of the form* (3.18) *then this parasystem generate the orthosymplectic* $\mathbb{Z}_2 \times \mathbb{Z}_2$-*graded Lie superalgebra* $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$.

[4] Here anywhere $\zeta, \eta, \xi \in \{+, -\}$.

We consider two particular cases which are degenerations of $\mathfrak{osp}(2m_1+1, 2m_2|2n, 0)$.

• If the parasystem consist of only one sort of the parafermions a_i^{ξ} ($i = 1, 2, ..., m_1$) and one sort of the parabosons b_i^{ξ} ($i = 1, 2, ..., n$) then we have the parasystem with the relative Fermi set and it generates the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(2m_1+1|2n) = \mathfrak{osp}(2m_1+1, 0|2n, 0)$.

• If the parasystem contains one sort of the parafermions $\tilde{a}_i^{\xi}$ ($i = 1, 2, ..., m_2$) and one sort of the parabosons b_i^{ξ} ($i = 1, 2, ..., n$) then we have the case of a parasystem with the relative Bose set (see the relations (1.2), (1.3) and (1.5)) and it generates the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m_2|2n, 0)$.

Thus we shown that the para-Fermi and para-Boose triple relations (1.2), (1.3) together with the relative para-Bose set (1.5) generate the orthosymplectic $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie superalgebra $\mathfrak{osp}(1, 2m|2n, 0)$. Moreover, it was shown that the superalgebras $\mathfrak{osp}(2m_{1+1}, 2m_2|2n, 0)$ give more complex para-Fermi and para-Bose system which contain the relative para-Fermi and para-Bose sets simultaneously.

It should be noted that, probably, for the first time the $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded structure of the relative para-Bose set (1.5) was observable in [11, 12] (also see [13]).

It should be also noted that the obtained relation between the parastatistics and the orthosymplectic superalgebras allows to apply all mathematical power of the representation theory of the superalgebras for a detailed description of the parastatistics, e.g. their Fock spaces etc. (for example, see [14–16]).

ACKNOWLEDGMENTS

The author would like to thank the Organizers for the kind invitation to speak at the International Workshop Supersymmetries and Quantum Symmetries, Dubna, July 29–August 3, 2013. The paper was supported by the RFBR Grant no. 11-01-00980-a and the Grant no. 12-09-0064 of the Academic Fund Program of the National Research University Higher School of Economics.

REFERENCES

1. H. S. Green, "A Generalized method of field quantization," Phys. Rev. **9**, 270–273 (1953).
2. S. Kamefuchi and Y. Takahashi, "A generalization of field quantization and statistics," Nucl. Phys. **36**, 177–206 (1962).
3. C. Ryan and E. C. G. Sudarshan, "Representations of parafermi rings," Nucl. Phys. **47**, 207–211 (1963).
4. A. C. Ganchev and T. D. Palev, "A Lie superalgebraic interpretation of the para-Bose statistics," J. Math. Phys. **21**, 797–799 (1980).
5. O. W. Greenberg and A. M. L. Messiah, "Selection rules for parafields and the absence of para-particles in nature," Phys. Rev. B **138**, 1155–1167 (1965).
6. T. D. Palev, "Para-Bose and para-Fermi operators as generators of orthosymplectic Lie superalgebras," J. Math. Phys. **23**, 1100–1102 (1982).
7. V. Rittenberg and D. Wyler, "Sequences of $\mathbb{Z}_2 \times \mathbb{Z}_2$-graded Lie algebras and superalgebras," J. Math. Phys. **19** (10), 2193–2200 (1978).
8. V. Rittenberg and D. Wyler, "Generalized superalgebras," Nucl. Phys. B **139** (10), 189–202 (1978).
9. V. G. Kac, "Lie superalgebras," Adv. Math. **26,** 8–96 (1977).
10. M. Scheunert, "Generalized Lie algebras," J. Math. Phys. **20** (4), 712–720 (1979).
11. W. Yang and S. Jing, "A new kind of graded Lie algebra and parastatistical supersymmetry," Science in China, Series A **44** (9), 1167–1173 (2001); arXiv:math-ph/0212004v1.
12. W. Yang and S. Jing, "Graded Lie algebra generating of parastatistical algebraic structure," Commun. Theor. Phys. **36**, 647–650 (2001); arXiv:math-ph/0212009v1.
13. K. Kanakoglou, C. Daskaloyannis, and A. Herrera-Aguilar, "Mixed paraparticles, colors, braidings and a new class of realizations for Lie superalgebras," arXiv:0912.1070v1 [math-ph], 1–19 (2009).
14. S. Lievens, N. I. Stoilova, and J. Van der Jeugt, "The paraboson Fock space and unitary irreducible representations of the Lie superalgebra $\mathfrak{osp}(1|2n)$," Commun. Math. Phys. **281**, 805–826 (2008); arXiv:0706.4196v2 [hep-th].
15. N. I. Stoilova and J. Van der Jeugt, "The parafermion Fock space and explicit $\mathfrak{so}(2n+1)$ representations," J. Phys. A **41** (075202), 1–13 (2008); arXiv:0712.1485v1 [hep-th].
16. N. I. Stoilova, "The parastatistics Fock space and explicit Lie superalgebra representations," arXiv:1311.4042v1 [math-ph], 1–14 (2013).