rate of \( \frac{dM}{dt} \approx 5 \times 10^{12} \) g/sec would be sufficient for the wind, averaged over the parallel, to have a regular meridional component of \(-1\%)\) with respect to the mass of matter transferred of the random component. The expenditure of energy on the transfer of matter toward the equator comprises no more than

\[
E \approx \frac{v_s}{L^2} \sqrt{\frac{S}{2}} \approx \frac{v_s}{L} \left( \frac{4}{3} L 0.1 n \right) \sqrt{\frac{S}{2}} \approx 4 \times 10^{-19} \text{ J}
\]

(where \( v_s \) is the turbulent viscosity), which is at least an order of magnitude less than the kinetic energy of the entire atmosphere.\(^5\)\(^6\) i.e., the value of \( E \) is fully acceptable. Thus, the scheme presented does not contradict the known physical parameters of the atmosphere.

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\(^1\)One of the first such studies is Ref. 5.

\(^2\)For all seasons.

\(^3\)It should be noted that there is also a noticeable pressure decrease at latitudes of 80 and 85°.


Translated by Edward U. Oldham

### Periodic motions of an axisymmetric rigid body in the gravitational field of a sphere

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The existence of periodic solutions to the problem of the translational–rotational motion of a rigid body possessing an axis of dynamic symmetry in the gravitational field of a sphere is studied by the small-parameter method of Poincaré. The motion of the bodies is described by equations in Delaunay–Andoyer canonical osculating elements. The analytical conditions for the existence of spatially periodic motions are obtained and they are studied in the report.

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The translational–rotational motion of a rigid body possessing an axis of dynamic symmetry in the gravitational field of a sphere is studied in the present report.

The first particular solutions of this problem were found by Kondurant\(^{1-4}\) and Duboshin.\(^{5,6}\) The authors based their studies on the Lagrangian equations of motion in generalized coordinates, the different forms of which were obtained by Duboshin\(^7\) in a general formulation of the n-body problem. A new method for studying particular solutions of an analogous problem, based on the use of equations of motion in osculating elements, was proposed in Refs. 8 and 9. But only cases of plane translational–rotational motions of rigid bodies were analyzed in these reports.

In the present report we study the three-dimensional translational–rotational motion of two rigid bodies \(M_b\) and \(M_r\), of which \(M_b\) is a homogeneous sphere and \(M_r\) is a rigid body possessing an axis of dynamic symmetry. The small-
parameter method of Poincaré is applied to the canonical system of differential equations in Delaunay–Andoyer osculating elements with which the motion of the bodies is described, and the problem of the existence of periodic solutions is solved.

1. STATEMENT OF THE PROBLEM, EQUATIONS OF MOTION

Let us consider the translational–rotational motion of two rigid bodies: a sphere $M_2$ and an axisymmetric body $M_1$. We designate the respective masses of the bodies as $m_2$ and $m_1$ and the principal central moments of inertia of body $M_1$ as $A = B = C$.

To describe the motion of the bodies we introduce into the analysis the following coordinate systems: OXYZ is a relative coordinate system with the origin at the center $O$ of the sphere and with axes which retain a constant orientation in space; $O_1XY$ is a Cartesian coordinate system with the origin at the center of mass $O_1$ of body $M_1$, the axes of which are parallel to the corresponding axes of coordinate system OXYZ; $O_1xyz$ is a moving coordinate system whose axes are directed along the principal central axes of inertia of body $M_1$.

The rotational motion of body $M_1$ about its center of mass is described by the Andoyer canonical osculating elements

$$L_1, G_1, H_1, l_1, g_1, h_1,$$

for which we introduce the intermediate plane P passing through the center $O_1$ normal to the vector $G_1$ of the angular momentum of the rotational motion of body $M_1$. The orientation of the intermediate plane to the axes $O_1XY$ is determined by the angles $\rho$ and $h_1$: $\rho$ is the angle measured between the $O_1Z$ axis and the vector $G_1$ (the inclination of the intermediate plane); $h_1$ is the angle between the $O_1X$ axis and the line of intersection of the planes $P$ and $O_1XY$ (the longitude of the ascending node of the intermediate plane); the position of the axes $O_1xyz$ of body $M_1$ relative to the vector $G_1$ and the intermediate plane are determined by the angles $\theta$, $g_1$, and $l_1$: $\theta$ is the angle measured between the $O_1Z$ axis and the vector $G_1$ (the inclination of the equator of the satellite to the intermediate plane); $g_1$ is the angle measured from the line of intersection of the planes $O_1XY$ and $P$ to the line of intersection of the planes $O_1xy$ and $P$ (the longitude of the ascending node of the equator in the intermediate plane, measured from the ascending node of the intermediate plane); $l_1$ is the angle measured from the line of intersection of the planes $P$ and $O_1xy$ to the axis $O_1X$ (the angle of proper rotation of body $M_1$ measured from the intermediate plane $P$).

The variables $L_1$ and $H_1$ are introduced through the relations

$$L_1 = G_1 \cos \theta, \quad H_1 = G_1 \cos \rho,$$

where $G_1$ is the magnitude of the angular momentum of the rotational motion of body $M_1$. Thus, $L_1$ is the projection of the vector $G_1$ onto the axis of dynamic symmetry of body $M_1$ while $H_1$ is the projection of the vector $G_1$ onto the fixed axis $O_1Z$.

The relative translational motion of bodies $M_2$ and $M_1$ is described by the Delaunay canonical elements:

$$L = m_2 \mu, \quad G = m_2 \mu (1 - e^2),$$

$$H = H_2 \cos i, \quad h = (h_2 \cos i),$$

where $\mu$ is the mean anomaly, $G$ is the angular distance to the pericenter, $H$ is the longitude of the ascending node, $h$ is the eccentricity of the orbit and $i$ is its inclination to the principal plane OXY,

$$m = m_2 m_1 / (m_2 + m_1), \quad \mu = f (m_2 + m_1),$$

and $f$ is the gravitational constant.

Taking as the unperturbed motion of bodies $M_2$ and $M_1$ their Keplerian translational motion and free rotational motion of body $M_1$ about its center of mass, we describe their translational–rotational motion by the equations (1)–(2) of perturbed motion in Delaunay–Andoyer canonical elements, the derivation of which is based on the theory of the Hamilton–Jacobi method and the method of variation of arbitrary constants. We have

$$d(L, G, H, L_1, G_1, H_1) = \frac{\partial F}{\partial (l, g, h, l_1, g_1, h_1)},$$

$$d(L, G, H, L_1, G_1, H_1) = \frac{\partial F}{\partial (L, G, H, L_1, G_1, H_1)},$$

The characteristic problem of the function is connected with the perturbing function $R$ by the equation

$$F = \frac{1}{2} m^2 \left( \frac{G_1}{2A} - \frac{G_1}{2A} \right) L_1^2 + R,$$

where the first three terms correspond to the unperturbed motion.

The perturbing function of the problem represents an expansion by associated Legendre polynomials

$$R(r, \delta) = \sum_{n=1}^N A_n P_n (\cos \delta),$$

where $r$ is the distance between the centers of inertia $O_1$ of the bodies; $\delta$ is the angle between the radius vector of the center of mass of body $M_1$ and its axis of symmetry $O_1Z$; $A_n$ are numerical coefficients characterizing the dynamic structure of body $M_1$.

We represent $R$ as a function of the Delaunay–Andoyer variables with the help of the equations of unperturbed motion:

$$r = \frac{G_1}{m^2 (1 + \varepsilon \cos \nu)}, \quad \cos \delta = \gamma_1 \cos(\nu + \gamma_2 \sin \nu + \gamma_3),$$

where $\varepsilon = \sqrt{G_1^2 - L_1^2 / L_2}$ is the true anomaly of the Keplerian motion, defined by the equation

$$l = \frac{G_1}{L_2} \int_0^\nu \frac{dv}{(1 + \varepsilon \cos v)^{\frac{3}{2}}},$$

while the quantities $\gamma_1$ and $\gamma_2$ are represented as functions of the elements (1)–(2) by the following equations:

$$\gamma_1 = \frac{\gamma_2}{G_1^2} \left[ L_1 \sin(h_1 - h) \cos g_1 + G_1 \cos(h_1 - h) \sin g_1 \right].$$

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From Eqs. (6) and (7) it follows that the perturbed function assigned by the expansion (5) depends on the elements h and h1 only through their difference, while it depends on the element g through the sum v + g and does not depend on the element l1 at all.

The first five integrals of the problem under consideration are known: the energy integral, three area integrals, and the first integral characterizing the constancy of the projection of the angular velocity of rotation of the body onto its axis of dynamic symmetry. The enumerated integrals are written in the Delaunay–Andoyer canonical variables as follows:

\[
\frac{\mu m^3}{2L^2} G_1^3 \left( \frac{1}{2} \frac{1}{C} - \frac{1}{A} \right) L_1^3 + R = e_i, \tag{8}
\]

\[
\sqrt{G_1^3 - H_1^3} \cos h + \sqrt{G_1^3 - H_1^3} \cos h_1 = c_i, \tag{9}
\]

\[
\sqrt{G_1^3 - H_1^3} \sin h + \sqrt{G_1^3 - H_1^3} \sin h_1 = c_i, \tag{10}
\]

\[
H + H_1 = c_i, \tag{11}
\]

\[
L_1 = c_i, \tag{12}
\]

where the arbitrary constants of integration are designated as c_i.

Using the integral (12) we eliminate from the further analysis the variable l1 (i.e., the rotation of body M1 about the axis of symmetry), which is determined from the known elements

\[
L, G, H, G_1, H_1, i, g, h, g_1, h_1 \tag{13}
\]

by the following quadrature:

\[
l - l_1 = \int \frac{\partial F}{\partial L_1} dt, \tag{14}
\]

where \(l_1 = g_i(t_0)\).

Then the motion of the bodies is described by a system of 10-th order differential equations

\[
d(L, G, H, G_1, H_1) = \frac{\partial F}{\partial (L, G, H, G_1, H_1)}, \tag{15}
\]

where

\[
F = \frac{\mu m^3}{2L^2} \left( \frac{1}{2} \frac{1}{C} - \frac{1}{A} \right) L_1^3 + R. \tag{16}
\]

Equations (14) admit of an energy integral and three area integrals.

We make the assumption that the ellipsoid of inertia of body M_0 is close to a sphere and the dimensions of body M_0 are small in comparison with the distance between bodies M_0 and M_i. Then the characteristic function of the problem can be represented in the form necessary for the application of the Poincaré small-parameter method. The dynamic flattening of body M_i can be chosen as the small parameter \(\nu\), for example, i.e., \(\nu = (C - A)/A\).

With this assumption we will have

\[
F = F_0 + \nu F_1 + O(\nu^2), \tag{17}
\]

where

\[
F_0 = \frac{\mu m^3}{2L^2} \left( \frac{1}{2} \frac{1}{C} - \frac{1}{A} \right) L_1^3 + R, \tag{18}
\]

is the unperturbed part of the function F, while the part \(\nu F_1\) of the characteristic function, representing a term of the first order, for a broad class of bodies can be identified with the principal term of the expansion of the perturbing function R:

\[
\nu F_1 = \nu m_0 A (1 - 3 \cos^2 \delta)/2r^2, \tag{19}
\]

with all the remaining terms in the expansion for F being terms of a higher order.

With \(\nu = 0\) the solution of Eqs. (14) corresponds to the unperturbed motion in which the center of mass of body M_i describes a Keplerian elliptical orbit while the rotation of body M_i about the center of mass takes place by Euler's laws. In this case because of the properties of the Keplerian translational motion and the Eulerian rotational motion periodic solutions exist as soon as the average orbital velocity \(v\) and the velocity of precession \(\psi\) are connected by the relation \(q = q_n\), where \(q\) and \(q_n\) are integers. Such motion, which we take as the generating motion, is periodic with a period \(T = 2\pi q_n\), where \(q\) and \(q_n\) are integers.

The study of the existence of periodic solutions of Eqs. (14) with small values of \(\nu\) is connected with cumbersome transformations. To eliminate this drawback we reduce the order of Eq. (14), using the area integrals.

Up to now the principal plane OXY has been chosen arbitrarily. Now let us choose it in such a way that it coincides with the Laplace plane, for which \(c_2 = c_3 = 0\) and \(c_1 = c\), and then the area integrals (9)-(11) are converted to the following form:

\[
h - h_1 = \pi, \tag{20}
\]

\[
\sqrt{G_1^3 - H_1^3} = \sqrt{G_1^3 - H_1^3}, \tag{21}
\]

Equations (19)-(21) have an origin in common with the analogous equations of the three-body problem and are convenient for reducing the order of the system of equations (14).

If the constant Laplace plane is chosen as the principal plane then the right sides of Eqs. (14) will not contain the variables \(h\) and \(h_1\). The variables \(H\) and \(H_1\) are eliminated from the equations using the formulas:

\[
H = \frac{c}{2} + \frac{1}{2c} (G_1^3 - G_1^3), \quad H_1 = \frac{c}{2} - \frac{1}{2c} (G_1^3 - G_1^3), \tag{22}
\]

which result directly from Eqs. (20) and (21).

For the variables \(G\) and \(G_1\) contained in F we introduce the new designations \(G = \Gamma\) and \(G_1 = \Gamma_1\). Then the equations of motion of the problem, retaining the canon-
ical form, are written in the following form:

\[
\begin{align*}
\frac{dL}{dt} &= -\frac{\partial F}{\partial l}, \\
\frac{d\Gamma}{dt} &= -\frac{\partial F}{\partial \Gamma}, \\
\frac{d\Gamma}{dt} &= -\frac{\partial F}{\partial \Gamma}, \\
\frac{d\Gamma}{dt} &= -\frac{\partial F}{\partial \Gamma},
\end{align*}
\]  (23)

where

\[
F = \frac{\mu m^3}{2L^2} + \frac{\Gamma^2}{2A} + vF_i(L, \Gamma, l, g, g_i) + O(v). \tag{24}
\]

Equations (23) admit of only one first integral:

\[F = c_i. \tag{25}\]

Later we will need an explicit expression for the function \(vF_i\). Omitting the intermediate calculations, we present the final result:

\[
vF_i = k \sum R_{n, \lambda, \mu}(L, \Gamma, l, g) \cos (k_l + k_g + k_g_i), \tag{26}\n\]

where

\[k = j m \Delta v / a,\]

the summation is carried out over the indices \(k_1 (0, 1, 2, \ldots, \infty), k_2 (0, \pm 1, \pm 2), \) and \(k_3 (0, \pm 1, \pm 2)\), and the coefficients of the expansion are determined by the following equations:

\[
\begin{align*}
R_{n,\lambda,\mu} &= \left(1 - \frac{3}{2} \sin^2 \theta \right) \left(1 - \frac{3}{2} \sin^2 j \right) C_{n, \lambda}^2, \\
R_{n,\lambda,\mu} &= \frac{3}{4} \sin^2 j \left(1 - \frac{3}{2} \sin^2 \theta \right) (C_{n, \lambda}^{21} + \epsilon S_{n, \lambda}^{21}), \\
R_{n,\lambda,\mu} &= \frac{3}{8} \sin^2 \theta \sin J C_{n, \lambda}^{12}, \\
R_{n,\lambda,\mu} &= \frac{3}{8} \sin 2 \theta \sin J C_{n, \lambda}^{11}, \\
R_{n,\lambda,\mu} &= \frac{3}{8} \sin 2 \theta \sin J (\cos J - \epsilon \theta) (C_{n, \lambda}^{21} + \epsilon S_{n, \lambda}^{21}), \\
R_{n,\lambda,\mu} &= \frac{3}{16} \sin^2 \theta (1 - \epsilon \cos J)^j (C_{n, \lambda}^{21} + \epsilon S_{n, \lambda}^{21})^j,
\end{align*}
\]  (27)

where for brevity of notation we introduce the designations \(\varepsilon = \pm 1\) and \(\sigma = \pm 2\). The coefficients \(C_{k_1}^{11}, C_{k_1}^{21}, S_{k_1}^{21}\) represent well-known series\(^\text{13}\) expanded by increasing powers of the orbital eccentricity \(e = \sqrt{1 - \frac{L}{L_0}}\), while the quantities \(\epsilon\) and \(J\) are defined by the equations

\[
\cos \theta = L / \Gamma, \cos J = (c^2 - \Gamma^2 - \Gamma_0) / 2\Gamma_0. \tag{28}\n\]

2. POINCARÉ PERIODIC SOLUTIONS

For brevity in deriving the conditions for the existence of Poincaré periodic solutions we introduce new designations of the variables:

\[
\begin{align*}
x_i &= L_i, \quad x_2 = \Gamma, \quad x_3 = l, \\
y_i &= \lambda, \quad y_2 = \theta, \quad y_3 = g_i.
\end{align*}
\]  (29)

Then Eqs. (23) are written in the form

\[
\begin{align*}
\frac{dx_i}{dt} &= \frac{\partial F}{\partial y_i}, \\
\frac{dy_i}{dt} &= -\frac{\partial F}{\partial x_i} \quad (i=1, 2, 3),
\end{align*}
\]  (30)

where

\[
\begin{align*}
F &= F_0 + F_1 + O(v), \\
F_0 &= \frac{\mu m^3}{2A}, \\
vF_i &= k \sum R_{n, \lambda, \mu}(x_1, x_2, x_3) \cos (k_1 y + k_2 \theta + k_3 g_i). \tag{31}
\end{align*}
\]

With \(v = 0\) we obtain from (30) the generating system of equations, the general solution of which has the form

\[
\begin{align*}
x_1 &= a_1, \\
y_1 &= a_2, \quad y_2 = a_3 + \Omega \tau, \\
y_3 &= a_4 + \Omega \tau + \omega_1 \tau + \omega_2 \tau + \omega_3 \tau.
\end{align*}
\]  (32)

Here \(a_1, a_2, a_3, a_4, \Omega, \omega_1, \omega_2, \omega_3\) are arbitrary constants of integration and

\[
\begin{align*}
\frac{\partial F_1}{\partial x_1} &= \mu m^3, \\
\frac{\partial F_1}{\partial x_2} &= 0, \\
\frac{\partial F_1}{\partial x_3} &= a_3.
\end{align*}
\]  (33)

This solution will be periodic with a period \(T_0\) if the conditions \(x_1(T_0) = x_1(0) = 0, y_1(T_0) - y_1(0) = n_1 \theta T_0 = 2k_1 \pi\) are satisfied, where \(k_1\) are integers such that \(n_1 \theta\) are commensurate.

Now let us examine the solution of Eqs. (30) with the initial conditions

\[
\begin{align*}
x_1 &= a_1 + \beta_1, \quad y_1 &= a_2 + \omega_1 + \gamma_1, \quad \gamma_1 = a_3, \quad \gamma_1 = a_4 + \omega_1 \tau + \omega_2 \tau + \omega_3 \tau.
\end{align*}
\]  (34)

which we represent in the form

\[
\begin{align*}
x_1 &= a_1 + \beta_1 + \gamma_1, \\
y_1 &= a_2 + \omega_1 + \gamma_1 + \gamma_2.
\end{align*}
\]  (35)

We write the equations of motion in the new variables \(\xi_1\) and \(\eta_1\). We will have

\[
\begin{align*}
\frac{d\xi_1}{dt} &= \frac{\partial K}{\partial \eta_1}, \\
\frac{d\eta_1}{dt} &= -\frac{\partial K}{\partial \xi_1} \quad (i=1, 2, 3),
\end{align*}
\]  (36)

where

\[
K = \xi_1 n_1 + \xi_2 n_2 + \sum_{\ell=1}^{\infty} \left[ \frac{\partial F}{\partial a_1} \xi_1 + \frac{\partial F}{\partial a_2} \xi_2 \right] + O(\xi_3^2). \tag{37}
\]

The necessary and sufficient conditions for the existence of periodic solutions of the system (38) will be

\[
\xi_1(T_0) = 0, \quad \eta_1(T_0) = 0 \quad (i=1, 2, 3). \tag{39}
\]

Retaining terms of the first order, we write Eq. (38) in the following form:

\[
\begin{align*}
\frac{d\xi_1}{dt} &= \frac{\partial F}{\partial a_1}, \\
\frac{d\eta_1}{dt} &= -\eta_1 - \frac{\partial F}{\partial a_2}.
\end{align*}
\]  (40)

Expanding \(F(\beta_1 + \gamma_i n_1 \theta + \omega_1 + \gamma_i)\) in a power-law series with respect to \(\beta_1, \gamma_i, \) and \(\xi_1\) from Eqs. (41) we obtain explicit expressions for the conditions (40):

\[
\begin{align*}
\frac{\xi_1(T_0) \gamma_1}{vT_0} &= \frac{\partial [F_1]}{\partial \theta},
\end{align*}
\]
in the case of the commensurability

\[ N_n^{(\alpha)} = 2n^{(\alpha)} \tag{A} \]

where \( N \) is an odd integer, and

\[
[F_1] = \frac{\hbar m A}{a} \left( R_{n,3}^{(3)} + R_{n,3,4}^{(3)} \cos 2g_i + R_{n,4,5}^{(3)} \cos g_i \right) + R_{n,3,4}^{(3)} \cos (2g_i - g_i^*) + R_{n,3}^{(3)} \cos (2g_i^* - g_i) + R_{n,5}^{(3)} \cos 2(g_i + g_i^*)
\]

in the case of the commensurability

\[ N_n^{(\alpha)} = n^{(\alpha)} \tag{B} \]

where \( N \) is any integer. The index means that the coefficients are calculated with the generating values of the variables.

Substituting the expression for \([F_1]\) into the conditions of existence (46) and solving them, we obtain the generating values of the angular values of the angular quantities:

\[
l_i = g_i = 0, \quad g_i^* = 0, \quad \pi \frac{2}{\pi}, \quad \frac{3}{2} \pi
\]

for the case of the commensurability (A) and

\[
l_i = g_i = 0, \quad g_i^* = \pi
\]

for the case of the commensurability (B).

Using the solutions found, we convert the condition (47) to the following form:

\[
\varphi (\Gamma^i, \Gamma^i, L_i, L^i, c) = 0
\]

where \( c \) and \( L^i \) are arbitrary constants of integration, while the quantities \( L_0 \) and \( \Gamma^i \) are chosen in such a way that the condition of commensurability

\[
q - \frac{\mu m^4}{L_0} = q_i \frac{\Gamma^i}{A}
\]

is satisfied, where \( q \) and \( q_i \) are commensurability indices corresponding to the cases (A) and (B).

Consequently, with given \( q, q_i, \Gamma^i, L_0 \) and arbitrary chosen \( m, A, L^i \), and \( c \) we find the generating value of the variable \( \Gamma \) from the condition (51):

\[
\Gamma = \Gamma_i (m, A, L^i, c)
\]

Then from the equations

\[
\cos \theta = \frac{L^i}{\Gamma^i}, \quad \cos J = (c^2 - \Gamma^i - \Gamma^i^*) / 2 \Gamma^i, \quad \varepsilon = \sqrt{1 - \frac{\Gamma^i}{L^i}}
\]

we find the generating values of the quantities \( \theta, J, \) and \( \varepsilon \).

We note that in the actual motion the quantities \( \Gamma_0, L_0 \), and \( L^i \) must satisfy the conditions

\[
|L^i| \ll |\Gamma^i|, \quad |\Gamma_i| \ll |L_0|, \quad \left| \frac{c^2 - \Gamma^i - \Gamma^i^*}{2 \Gamma^i} \right| \ll 1
\]

Following Poincaré's arguments, one can show that the last condition (48) for the existence of periodic solutions is satisfied.
CONCLUSION

The analytical conditions for the existence of periodic solutions were established by the Poincaré small-parameter method, which was applied to the equations of motion in Delaunay–Andoyer osculating canonical elements.

From the conditions of existence it follows that the generating solutions correspond to motions of body \( M_1 \) along an elliptical orbit whose eccentricity is determined from the algebraic equation (51) as a function of the values of \( \theta \) and \( \varphi \). In this case at the pericenter of the orbit the body \( M_1 \) occupies one of four possible positions determined by the solutions (49) and (50).

In the future we propose to make a numerical study of Eq. (51) for different cases of commensurability from (A) and (B) and to generalize the results of the work to the case of the translational–rotational motion of a rigid body \( M_1 \) possessing an arbitrary dynamic structure.

The results of the work have practical importance for

the study of resonance translational–rotational motions of celestial bodies.


Translated by Edward U. Oldham

Solution of the Lane–Emden problem in series

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A solution of the Lane–Emden differential equation is obtained in series form. The question of the convergence of the series obtained is briefly discussed.

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1. As is well known, the Lane–Emden equation (LEE) of index \( n \)

\[(\rho\dot{\rho})' = -\frac{n}{x^2} \rho^4 \]

with the boundary conditions

\[x = 0, \quad \theta = 1, \quad \phi = 0 \]

has an analytical solution only for three values of \( n \):

\[n = 0, \quad \theta(x) = 1, \quad x^2 \]

\[n = 1, \quad \theta(x) = \frac{\sin x}{x} \]

\[n = 5, \quad \theta(x) = (1 + \frac{1}{3} x^2)^{-5} \]

For an arbitrary value of \( n \) the LEE must be solved numerically, such as by the Runge–Kutta method. Because of the fundamental role of the LEE in the theory of the internal structure of self-gravitating bodies (planets, stars) it makes sense to attempt a solution of the LEE in series. Functional series are one of the powerful methods of mathematical analysis and are no less (and sometimes more) convenient than the elementary functions. Here it is pertinent to note that modern computers, for example, often use series in the calculation of the majority of "elementary" functions; even such a simple one as \( 2^n \). In the present note we will demonstrate the possibility of obtaining solutions of the LEE in series form.

2. Let us transform (1), introducing the substitution

\[z = x^2 \]

\[2 \left( \frac{d\theta(z)}{dz} + 2z \frac{d^2\theta(z)}{dz^2} \right) = -z \]

boundary condition:

\[z = 0, \quad \theta(z) = 1, \quad d\theta(z)/dz = 0 \]

We seek the solution \( \theta(z) \) in series form:

\[\theta(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \]

From the formula for raising a series to a power we have

\[\theta^z(z) = \left[ 1 + \sum_{k=1}^{\infty} c_k z^k \right]^n = 1 + \sum_{k=1}^{\infty} c_{nk} z^k \]

\[c_{nk} = \frac{1}{n} \sum_{k=1}^{\infty} \left( n - m + 1 \right) c_m c_{n-m, k} \]

Substituting (8) and (9) into (6) and equating the coefficients with the same powers of \( z \), we obtain the relation between the coefficients \( c \) and \( a \):}