Asymptotic Methods in Probability and Statistics with Applications

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Birkhäuser
Boston • Basel • Berlin
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Library of Congress Cataloging-in-Publication Data
Asymptotic methods in probability and statistics with applications / edited by
N. Balakrishnan, I.A. Ibragimov, V.B. Nevzorov.
p. cm. (Statistics for industry and technology)
Includes bibliographical references and index.
ISBN 0-8176-4214-5 (acid-free paper)
1. Asymptotic distribution (Probability theory) I. Balakrishnan, N., 1956--
II. Ibragimov, I.A. (I'I'dar Abdulovich) III. Nevzorov, Valery B., 1946--
IV. Series.
QA273.6 .A78 2001
519.2—dc21
2001025400
CIP

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ISBN 3-7643-4214-5 SPIN 10785725

Production managed by Louise Farkas; manufacturing supervised by Jacqui Ashri.
Typeset by the editors in $\LaTeX$.
Cover design by Vernon Press, Boston, MA.
Printed and bound by Hamilton Printing Co., Rensselaer, NY.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1
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Characterization and Stability Problems for Finite Quadratic Forms

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Abstract: Sufficient conditions are given under which the distribution of a finite quadratic form in independent identically distributed symmetric random variables defines uniquely the underlying distribution. Moreover, a stability theorem for quadratic forms is proved.

Keywords and phrases: Quadratic forms, characterization problem, stability problem

3.1 Introduction

Let $Z_1, \ldots, Z_n$ be independent identically distributed (i.i.d.) standard normal random variables and $a_1, \ldots, a_n$ be real numbers with $a_1^2 + \ldots + a_n^2 \neq 0$. Suppose that $X_1, \ldots, X_n$ are i.i.d. random variables such that

$$a_1 Z_1 + \ldots + a_n Z_n \overset{d}{=} a_1 X_1 + \ldots + a_n X_n,$$

where $\overset{d}{=}$ denotes the equality in distribution. Then, by Cramér's decomposition theorem for the normal law [see Linnik and Ostrovski (1972, Theorem 3.1.4)], the $X_i$ are standard normal too.

Lukacs and Laha (1964, Theorem 9.1.1) considered a more general problem. Namely, let $X_1, \ldots, X_n$ be i.i.d. random variables such that their linear combination $L = a_1 X_1 + \ldots + a_n X_n$ has analytic characteristic function and

$$a_1^s + \ldots + a_n^s \neq 0 \text{ for all } s = 1, 2, \ldots$$

Then the distribution of $X_1$ is uniquely determined by that of $L$.  

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The aim of this Chapter is to obtain a similar characterization property for quadratic forms in i.i.d. random variables $Z_1, \ldots, Z_n$. Furthermore, we state a stability property of such quadratic forms.

3.2 Notations and Main Results

Consider a symmetric matrix $A = (a_{ij})_{i,j=1}^n$. Let

$$Q(x_1, \ldots, x_n) = \sum_{i,j=1}^n a_{ij}x_ix_j$$

be a quadratic form in variables $x_1, \ldots, x_n$. Assume that $Q$ is non-degenerate in the sense that $A$ is not a zero matrix. Suppose $Z_1, \ldots, Z_n$ are i.i.d. random variables with a symmetric distribution $F$.

We say that a pair $(Q, F)$ has a characterization property (CP) iff for a sequence of i.i.d. symmetric random variables $X_1, \ldots, X_n$, the equality

$$Q(Z_1, \ldots, Z_n) \overset{d}{=} Q(X_1, \ldots, X_n)$$

(3.1)

implies

$$Z_1 \overset{d}{=} X_1.$$

**Remark 3.2.1** We require in the definition of CP that the random variables $X_1, \ldots, X_n$ are symmetric. Otherwise the problem does not have solution even in the case $n = 1$ and $Q(x_1) = x_1^2$. Equation (3.1) holds for $X_1 = Z_1$ as well as for $X_1 = |Z_1|$.

**Remark 3.2.2** With a symmetric distribution $F$ an answer is trivial in the one dimensional case, i.e. any pair $(Q, F)$ has CP. Therefore we assume that $n \geq 2$ everywhere below.

In this Chapter, sufficient conditions are given under which the pair $(Q, F)$ has CP. The solution of the problem depends also on the coefficients of the matrix $A$, where the following possibilities occur:

1. $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$.

2. $a_{ii} \neq 0$ for some $i = 1, 2, \ldots, n$.
   2.1. $a_{11}^{2k+1} + a_{22}^{2k+1} + \ldots + a_{nn}^{2k+1} \neq 0$ for all $k = 0, 1, 2, \ldots$
   2.2. $a_{11} + a_{22} + \ldots + a_{nn} = 0$.
      2.2.1. $a_{ij} = 0$ for all $i \neq j$. 

2.2.2. \( a_{ij} \neq 0 \) for some \( i \neq j \).
2.3. \( a_{11}^{2k+1} + a_{22}^{2k+1} + \ldots + a_{nn}^{2k+1} = 0 \) for some \( k = 1, 2, \ldots \)

Here, we consider cases 1, 2.1 and 2.2.1.

Define now a class \( \mathcal{F} \) of probability distributions so that \( F \in \mathcal{F} \) iff the following two conditions are satisfied:

(C1) \( F \) has moments \( \alpha_k = \int_{-\infty}^{\infty} x^k dF(x) \) of all orders \( k \).

(C2) \( F \) is uniquely specified by \( \alpha_1, \alpha_2, \ldots \)

The following examples demonstrate when probability distribution \( F \in \mathcal{F} \).

**Example 3.2.1** If \( F \) has an analytic characteristic function, then \( F \in \mathcal{F} \).

Remember [see Lukacs (1970, §7.2)] that a characteristic function is analytic iff

(i) the condition (C1) is satisfied and

(ii) \( \limsup_{n \to \infty} \alpha_{2n}^{1/(2n)}/(2n) < \infty \).

The latter condition leads to (C2); see Lukacs and Laha (1964, Ch. 9).

We say that a probability distribution \( F \) satisfies *Cramér's condition* CC iff

\[
\int_{-\infty}^{\infty} \exp\{h|x|\} dF(x) < \infty \quad \text{for some} \quad h > 0.
\]

**Example 3.2.2** Let \( F \) satisfies CC, then \( F \in \mathcal{F} \).

It follows from the fact that \( F \) satisfies CC iff its characteristic function is analytic [see Lukacs (1970, §7.2)].

**Example 3.2.3** If the moments \( \{\alpha_k\} \) of \( F \) satisfy Carleman condition, i.e.

\[
\sum_{n=1}^{\infty} \alpha_{2n}^{-1/(2n)} = \infty,
\]  

then \( F \in \mathcal{F} \).

In fact, the condition (3.2) yields the uniqueness of the moment problem for \( \mathcal{F} \); see, for example, Shohat and Tamarkin (1970, Theorem 1.10).

Note that Carleman condition is weaker than CC. Other examples of probability distributions belonging to \( \mathcal{F} \) as well as detailed discussion concerning the moment problem and other related topics; see Akhiezer (1965), Feller (1971, Sec. VII.3) and Stoyanov (1987, Sec. 8.12 and 11).

**Theorem 3.2.1** Let \( F \in \mathcal{F} \) and the matrix \( A \) be such that \( a_{ii} = 0 \) for all \( i = 1, 2, \ldots, n \). Then, \((Q, F)\) has CP.
Example 3.2.4 Let $Z_1, Z_2, Z_3$ be i.i.d. standard normal random variables and $X_1, X_2, X_3$ be i.i.d. symmetric random variables such that

$$Z_1 Z_2 - Z_2 Z_3 \overset{d}{=} X_1 X_2 - X_2 X_3,$$

then by Theorem 3.2.1 the random variables $X_1, X_2, X_3$ are standard normal.

Theorem 3.2.2 Let $F \in \mathcal{F}$ and the matrix $A$ be such that for all $k = 0, 1, 2, \ldots$

$$a_{11}^{2k+1} + a_{22}^{2k+1} + \ldots + a_{nn}^{2k+1} \neq 0.$$

Then, $(Q, F)$ has CP.

Example 3.2.5 Let $Z_1, Z_2$ be i.i.d. random variables with distribution $F$ and density function

$$p(x) = (1/4) \exp\{-|x|^{1/2}\}, \quad x \in (-\infty, \infty)$$

Then, $F \in \mathcal{F}$; see Stoyanov (1987, p. 98).

Let $X_1, X_2$ be i.i.d. symmetric random variables such that

$$2Z_1^2 + 4Z_1 Z_2 - Z_2^2 \overset{d}{=} 2X_1^2 + 4X_1 X_2 - X_2^2.$$

Then by Theorem 3.2.2, the random variables $X_1$ and $X_2$ have the density function defined in (3.3) too.

Theorem 3.2.3 Let $a_{ii} \neq 0$ for some $i = 1, 2, \ldots, n$, but $a_{11} + a_{22} + \ldots + a_{nn} = 0$ and $a_{ij} = 0$ for all $i \neq j$. Then for any $F$, the pair $(Q, F)$ does not have CP.

Example 3.2.6 Let $Z$ be a random variable with symmetric distribution $F$ independent of the random variable $\zeta$ with $P(\zeta = 1) = P(\zeta = -1) = 1/2$ and let $c > 0$ be a real constant. Put

$$X = \zeta (Z^2 + c)^{1/2}.$$

Suppose now that both $Z, Z_1, Z_2, \ldots, Z_n$ are i.i.d. and $X, X_1, X_2, \ldots, X_n$ are i.i.d. too. Under the conditions of Theorem 3.2.3 varying the constant $c$, we find a family of symmetric distributions of $X_1$ such that (3.1) holds. In particular, if

$$Z_1^2 - Z_2^2 \overset{d}{=} X_1^2 - X_2^2,$$

then the distributions of $X_1$ and $Z_1$ may differ.

Example 3.2.6 proves Theorem 3.2.3. The proofs of Theorems 3.2.1 and 3.2.2 are given in Section 3.4. They are based on the following:

a) If $F \in \mathcal{F}$, then $X_1$ has moments $EX_1^k$ of all orders $k$.

b) Under the given conditions, we have

$$EX_1^k = EZ_1^k \quad \text{for all} \quad k = 1, 2, \ldots.$$
Moreover, we also prove also a stability theorem.

**Theorem 3.2.4** Suppose that the pair \((Q, F)\) has CP. Let \(X_{N,1}, \ldots, X_{N,n}\) for \(N = 1, 2, \ldots\) be a series of i.i.d. symmetric random variables and

\[
Q(X_{N,1}, \ldots, X_{N,n}) \xrightarrow{d} Q(Z_1, \ldots, Z_n) \quad \text{as} \quad N \to \infty,
\]

where \(\xrightarrow{d}\) denotes the convergence in distribution. Then,

\[
X_{N,1} \xrightarrow{d} Z_1 \quad \text{as} \quad N \to \infty.
\]

Theorem 3.2.4 will be proved in Section 3.4 using the tightness of the converging sequences of quadratic forms.

---

### 3.3 Auxiliary Results

At first, we give some simple relations for a quadratic form which enable us to remove undesirable elements to get inequalities between tail probabilities of \(X_1\) and \(Q(X_1, \ldots, X_n)\). Denote

\[
\text{tr}A = a_{11} + \ldots + a_{nn} \quad \text{and} \quad M = \max_{i,j} |a_{i,j}|.
\]

**Lemma 3.3.1** We have

\[
a_{11}x_1^2 + \ldots + a_{nn}x_n^2 = 2^{-n} \sum_{\varepsilon(1,n)} Q(\varepsilon_1 x_1, \ldots, \varepsilon_n x_n) \tag{3.4}
\]

and

\[
a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2 = 2^{2-n} \sum_{\varepsilon(3,n)} Q(x_1, x_2, \varepsilon_3 x_3, \ldots, \varepsilon_n x_n), \tag{3.5}
\]

where \(\sum_{\varepsilon(i,n)}\) for \(i \leq n\) denotes the summation over all vectors \(\varepsilon(i,n) = (\varepsilon_i, \ldots, \varepsilon_n)\) with \(\varepsilon_j \in \{-1, 1\}\).

**Lemma 3.3.2** Assume that \(\text{tr}A = 0\) and put

\[
Q^*(x_1, \ldots, x_n) = a_{11}x_1^2 + \ldots + a_{nn}x_n^2 + 2a_{12}x_1x_2.
\]

Then

\[
2a_{12}(x_1x_2 + x_nx_1 + \ldots + x_2x_3)
\]

\[
= Q^*(x_1, x_2, \ldots, x_n) + Q^*(x_n, x_1, \ldots, x_{n-1}) + \ldots + Q^*(x_2, x_3, \ldots, x_n, x_1).
\]
Lemma 3.3.3 Let $X_1, \ldots, X_n$ be i.i.d. symmetric random variables. Then for any permutation $(i_1, \ldots, i_n)$ of indices $(1, \ldots, n)$ and any vector $(\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_j \in \{-1, 1\}$, we have

$$Q(X_1, \ldots, X_n) \overset{d}{=} Q(\varepsilon_1 X_{i_1}, \ldots, \varepsilon_n X_{i_n}).$$

Proofs of Lemmas 3.3.1–3.3.3 are obvious.

Lemma 3.3.4 If $a_{11} \neq 0$, then

$$|Q(x_1, \ldots, x_n)| \geq 0.75 |a_{11}| x_1^2 - c_1(A)(x_2^2 + \ldots + x_n^2),$$

with $c_1(A) = \frac{4}{|a_{11}|} \sum_{j=2}^{n} a_{1j}^2 + \max_{2 \leq i \leq n} \left\{ \sum_{j=2}^{n} |a_{ij}| \right\}$.

Proof. Since

$$Q(x_1, \ldots, x_n) = a_{11} x_1^2 + 2x_1 (a_{12} x_2 + \ldots + a_{1n} x_n) + Q(0, x_2, \ldots, x_n),$$

we find

$$|Q(x_1, \ldots, x_n)| \geq |a_{11}| x_1^2 - |2x_1 (a_{12} x_2 + \ldots + a_{1n} x_n)| - |Q(0, x_2, \ldots, x_n)|.$$

Using in the second term of the right hand side

$$2|ab| \leq a^2 + b^2$$

with $a = \frac{1}{2} \sqrt{|a_{11}|} x_1$ and $b = \frac{2}{\sqrt{|a_{11}|}} (a_{12} x_2 + \ldots + a_{1n} x_n)$,

and $2|a_{1i}a_{1j} x_i x_j| \leq a_{1i}^2 x_i^2 + a_{1j}^2 x_j^2$, we obtain

$$|2x_1 (a_{12} x_2 + \ldots + a_{1n} x_n)| \leq \frac{1}{4} |a_{11}| x_1^2 - \frac{4}{|a_{11}|} \sum_{j=2}^{n} a_{1j}^2 (x_2^2 + \ldots + x_n^2).$$

The inequality $2|x_i x_j| \leq x_i^2 + x_j^2$ leads to

$$|Q(0, x_2, \ldots, x_n)| \leq \max_{2 \leq i \leq n} \left\{ \sum_{j=2}^{n} |a_{ij}| \right\} (x_2^2 + \ldots + x_n^2)$$

which completes the proof of Lemma 3.3.4.

We now prove inequalities between tail probabilities of both $X_1$ and $Q_X = Q(X_1, \ldots, X_n)$.

Lemma 3.3.5 Let $X_1, \ldots, X_n$ be i.i.d. symmetric random variables. For any positive $u$, we have

$$P\{|Q_X| \geq u\} \leq n P\{X_1^2 \geq u/(M n^2)\}.$$
PROOF. The obvious inequality
\[ |Q(x_1, x_2, \ldots, x_n)| \leq n M (x_1^2 + \ldots + x_n^2) \]  \hspace{1cm} (3.6)
leads to the statement.

Denote by \( m = \text{med} X_1^2 \) a median of \( X_1^2 \), i.e.
\[ P\{X_1^2 \geq m\} \geq 1/2 \quad \text{and} \quad P\{X_1^2 \leq m\} \geq 1/2. \]

**Lemma 3.3.6** There are positive constants \( c_1 \) and \( c_2 \) depending only on the elements of matrix \( A \) such that

(a) if \( a_{ii} = 0 \) for all \( i = 1, 2, \ldots, n \), then for any \( u \geq 0 \) we have
\[ P^2\{|X_1| \geq \sqrt{u}\} \leq 2^{n-2} P\{|Q_X| \geq c_1 u\}; \]  \hspace{1cm} (3.7)

(b) if \( a_{ii} \neq 0 \) for some \( i = 1, 2, \ldots, n \), then for any \( u \geq 0 \) we have
\[ P\{X_1^2 \geq u + c_2 (n-1) m\} \leq 2^{n-1} P\{|Q_X| \geq c_1 u\}. \]  \hspace{1cm} (3.8)

PROOF. Case a: Since \( A \) is not a zero matrix, there exists \( a_{ij} \neq 0 \) with \( i \neq j \). Without loss of generality, we may assume \( a_{12} \neq 0 \). Then using (3.5), we get
\[ 2|a_{12} X_1 X_2| \leq 2^{2-n} \sum_{\varepsilon \in \{+1, -1\}^n} |Q(X_1, X_2, \varepsilon^3 X_3, \ldots, \varepsilon^n X_n)|. \]

Therefore, by Lemma 3.3.3, we have for any positive \( u \)
\[ P^2\{|X_1| \geq \sqrt{u}\} \leq P\{2|a_{12} X_1 X_2| \geq 2|a_{12}| u\} \leq 2^{n-2} P\{|Q_X| \geq 2|a_{12}| u\}. \]

Case b: Without loss of generality, we assume \( a_{11} \neq 0 \). Put
\[ \alpha = 4 c_1(A) (n-1) m / (3|a_{11}|), \]
where \( c_1(A) \) is defined in Lemma 3.3.4 and \( m \) is the median of \( X_1^2 \).

For any \( u \geq 0 \), we find now
\[ P\{X_1^2 \geq u + \alpha\} \leq 2^{n-1} P\{X_1^2 \geq u + \alpha, X_2^2 \leq m, \ldots, X_n^2 \leq m\} \]
\[ \leq 2^{n-1} P\{|Q_X| \geq 0.75 |a_{11}| (u + \alpha) - c_1(A) (n-1) m\} \]
\[ \leq 2^{n-1} P\{|Q_X| \geq 0.75 |a_{11}| u\} \]

and Lemma 3.3.6 is proved.

Using the last two lemmas, we find the following statement which is of its own interest.

**Lemma 3.3.7** Random variables \( X_1 \) and \( |Q_X|^{1/2} \) satisfy or do not satisfy CC simultaneously.
PROOF. It follows from Lemma 3.3.5 and the equality for any $h > 0$
\[ E \exp\{h |X|\} = 1 + h \int_0^\infty \exp\{hu\} P\{|X| \geq u\} du \]  
(3.9)
that $|Q_X|^{1/2}$ satisfies CC if $X_1$ satisfies CC.
Suppose now $E \exp\{h_0 |Q_X|^{1/2}\} < \infty$ and $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$. By 
(3.9), (3.7) and the Markov inequality, we find
\[ E \exp\{h \, X_1\} \leq 1 + 2^{(n-2)/2} h \int_0^\infty \exp\{hu\} \left( \frac{E \exp\{h_0 |Q_X|^{1/2}\}}{\exp\{h_0 c_1^{1/2} u\}} \right)^{1/2} du < \infty. \]
Hence, CC holds with some $0 < h < h_0 c_1^{1/2}$.
Let now $a_{ii} \neq 0$ for some $i = 1, 2, \ldots, n$. Then by (3.9), (3.8) and $P\{|X_1| \geq u + c_2 (n - 1)m\} \leq P\{|X_1|^2 \geq u^2 + c_2 (n - 1)m\}$, we find
\[ E \exp\{h |X_1|\} \leq c_3 + c_4 \int_0^\infty e^{hu} P\{|X_1|^{1/2} \geq c_1^{1/2} u\} du \]
with some finite constants $c_3$ and $c_4$. Hence, $X_1$ satisfies CC.

Lemma 3.3.8 Random variables $X_1$ and $Q_X$ have moments of all orders simultaneously.

PROOF. Let $X_1$ have moments of all orders, then by (3.6) $E|Q_X|^k < \infty$ for $k = 1, 2, \ldots$, too. If $Q_X$ has moments of all orders, then the existence of the absolute moments of all orders of $X_1$ follows now from the equality
\[ E|X_1|^k = k \int_0^\infty u^{k-1} P\{|X_1| \geq u\} du \quad \text{for any integer} \quad k \geq 1, \]
Lemma 3.3.6 and Markov inequality in the same way as in the second part of the proof of Lemma 3.3.7.

Lemma 3.3.9 Let $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$. Then, $E Q_X^{2k}$ is an increasing function of $\beta_{2k} = E X_1^{2k}$ for all $k = 1, 2, \ldots$

PROOF. We have
\[ E Q_X^{2k} = B \beta_{2k}^2 + C \beta_{2k} + D \]  
(3.10)
for all $k = 1, 2, \ldots$, where $B, C$ and $D$ depend on the matrix $A$ and $\beta_{2k-2l}$ with $l = 1, 2, \ldots, k-1$. It is enough to prove that $B > 0$ and $C \geq 0$.
We obtain
\[ E Q_X^{2k} = 2^{2k} E \sum_{i' < j'} a_{i'j'} X_{i'j'} \ldots a_{i'2k} X_{i'2k} X_{j'2k}, \]  
(3.11)
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where \( \sum_{i' < j'} \) denotes the summation over all \( 2k \) pairs:

\[
1 \leq i_1 < j_1 \leq n, \ldots, 1 \leq i_{2k} < j_{2k} \leq n.
\]

It is clear from (3.11) that \( B \) in the representation (3.10) equals

\[
B = 2^{2k} \sum_{1 \leq i < j \leq n} a_{ij}^{2k} > 0.
\]

In order to prove that \( C \geq 0 \), we introduce notation for a finite set \( M = \{m_1, \ldots, m_l\} \) of integers \( m_1, \ldots, m_l \). Let \( \#(M) \) be the number of elements in \( M \) and \( \#^*(M) \) be the number of different elements in \( M \). For example, if \( M = \{3, 2, 2, 1\} \), then \( \#(M) = 4 \) and \( \#^*(M) = 3 \).

The coefficient \( C \) in (3.10) up to factors \( \beta_{2k-2l} \) with \( l = 1, 2, \ldots, k - 1 \) is a sum of products \( 2^{2k} a_{i_1 j_1} \cdots a_{i_{2k} j_{2k}} \) [see (3.11)] such that the set of their indices \( E = \{i_1, j_1, \ldots, i_{2k}, j_{2k}\} \) satisfies the following three conditions:

a) There is a subset \( E_1 \subset E \) with \( \#(E_1) = 2k \) and \( \#^*(E_1) = 1 \). This yields that a corresponding summand in (3.11) has a factor \( \beta_{2k} \).

b) \( \#^*(E \setminus E_1) \geq 2 \). It implies that we consider a summand with factor \( \beta_{2k} \), but not \( \beta_{2k}^2 \). Note that \( \#^*(E) = \#^*(E \setminus E_1) + \#^*(E_1) \).

c) Each value from the set \( E \setminus E_1 \) is taken by even number of elements from \( E \setminus E_1 \). Otherwise, the corresponding expectation equals to zero since the random variables \( X_j, j = 1, 2, \ldots, n \), are symmetric and independent.

It follows from the above three conditions that \( C \geq 0 \). Thus, Lemma 3.3.9 is proved.

A similar idea of monotony was used by Khakhubiya (1965).

### 3.4 Proofs of Theorems

**Proof of Theorem 3.2.1.** We get from (3.1) and Lemma 3.3.8 that \( X_1 \) has moments of all orders. Obviously, \( EX_1^{2k+1} = EZ_1^{2k+1} = 0 \) for all \( k = 0, 1, 2, \ldots \)

We now show that

\[
EX_1^{2k} = EZ_1^{2k} \quad \text{for all} \quad k = 1, 2, \ldots \tag{3.12}
\]

Comparing moments of \( Q_X = Q(X_1, \ldots, X_n) \) and \( Q_Z = Q(Z_1, \ldots, Z_n) \), we get (3.12). In fact, it follows from (3.1) that

\[
EQ_X^{2k} = EQ_Z^{2k} \quad \text{for all} \quad k = 1, 2, \ldots \tag{3.13}
\]
Since \( a_{ii} = 0 \) for all \( i = 1, 2, \ldots, n \), in \( E Q_X^{2k} \) there occur only moments \( E X_1^2 \) up to order \( 2k \). Taking \( k = 1 \) in (3.13), we get

\[
(E X_1^2)^2 \operatorname{tr}(A^2) = (E Z_1^2)^2 \operatorname{tr}(A^2).
\]

Therefore, we obtain (3.12) for \( k = 1 \).

Then taking \( k \geq 2 \) in (3.13) and using Lemma 3.3.9, we get (3.12) for \( k \geq 2 \) by induction.

Since \( F \in \mathcal{F} \), it is uniquely specified by its moments. We proved that all moments of \( X_1 \) and \( Z_1 \) coincide, respectively. Hence, the distribution of \( X_1 \) is uniquely defined by its moments too, and Theorem 3.2.1 is proved.

**Proof of Theorem 3.2.2.** Similar to the proof of Theorem 3.2.1, it is enough to show that (3.12) holds. With (3.1), we find now

\[
E Q_X^k = E Q_Z^k \quad \text{for all} \quad k = 1, 2, \ldots
\]  

(3.14)

Taking \( k = 1 \) in (3.14), we obtain

\[
E X_1^2 \operatorname{tr} A = E Z_1^2 \operatorname{tr} A,
\]

i.e. we get (3.12) for \( k = 1 \) since \( \operatorname{tr} A \neq 0 \).

The proof of (3.12) for \( k \geq 2 \) can be done by induction using (3.14) and the conditions on the elements of matrix \( A \).

**Proof of Theorem 3.2.4.** Put

\[
Q_{X,N} = Q(X_{N,1}, \ldots, X_{N,n}) \quad \text{and} \quad Q_Z = Q(Z_1, \ldots, Z_n).
\]

Since \( Q_{X,N} \xrightarrow{d} Q_Z \) as \( N \to \infty \), the sequence \( \{Q_{X,N}\} \) is relatively compact. It is known [see Prohorov (1956)] that \( \{Q_{X,N}\} \) is relatively compact if and only if \( \{Q_{X,N}\} \) is tight, i.e.

\[
\sup_N \mathbb{P}\{|Q_{X,N}| > v\} \to 0 \quad \text{as} \quad v \to +\infty.
\]  

(3.15)

In order to prove Theorem 3.2.4, it is enough to show that \( \{X_{N,1}\} \) is also tight. In fact, in this case for any infinite subset of \( \{X_{N,1}\} \) there exists a subsequence \( \{X_{N_k,1}\} \) which converges in distribution to some symmetric random variable \( V_1 \). Since \( Q \) is continuous in distribution in each argument, we have

\[
Q(X_{N_k,1}, \ldots, X_{N_k,n}) \xrightarrow{d} Q(V_1, \ldots, V_n) \quad \text{as} \quad N_k \to \infty,
\]

where \( V_1, \ldots, V_n \) are i.i.d. symmetric random variables. From the assumption of Theorem 3.2.4, we get

\[
Q(V_1, \ldots, V_n) \xrightarrow{d} Q(Z_1, \ldots, Z_n).
\]
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It yields $V_1 \overset{d}{=} Z_1$. Therefore, any limit point of $\{X_{N,1}\}$ has the same distribution as $Z_1$, which proves the statement of Theorem 3.2.4.

In order to prove the tightness of $\{X_{N,1}\}$, we consider two cases with respect to diagonal elements of $A$.

Case 1 if $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$. It follows from Case a of Lemma 3.3.6 that $\{X_{N,1}\}$ is tight when (3.15) holds.

Case 2 if $a_{ii} \neq 0$ for some $i = 1, 2, \ldots, n$. It follows from (3.8) that the sequence $\{X_{N,1}^2 - c_2(n-1)m_N\}$ with $m_N = \text{med}(X_{N,1}^2)$ is also tight when (3.15) holds. Therefore, it is enough to show that

$$\sup_N m_N \leq c < \infty$$ (3.16)

with some absolute constant $c$.

Put

$$\hat{Q}(x_1, \ldots, x_n) = \sum_{i=1}^{n} (|a_{ii}| - a_{ii}) x_i^2 + Q(x_1, \ldots, x_n).$$

The quadratic form $\hat{Q}$ differs from $Q$ only by the diagonal elements of the matrix $A$, which are $|a_{ii}|$ in $\hat{Q}$ instead of $a_{ii}$ in $Q$, $i = 1, 2, \ldots, n$. Using (3.4) of Lemma 3.3.1 and Lemma 3.3.3, we get

$$2^{-n} \leq P\left\{a_{11}X_{N,1}^2 \geq |a_{11}|m_N, \ldots, |a_{nn}|X_{N,n}^2 \geq |a_{nn}|m_N\right\}$$

$$\leq P\left\{\sum_{i=1}^{n} |a_{ii}|X_{N,i}^2 \geq m_N \sum_{i=1}^{n} |a_{ii}|\right\}$$

$$\leq P\left\{2^{-n} \sum_{\varepsilon(1, n)} |\hat{Q}(\varepsilon_1 X_{N,1}, \ldots, \varepsilon_n X_{N,n})| \geq m_N \sum_{i=1}^{n} |a_{ii}|\right\}$$

$$\leq 2^n P\{|\hat{Q}_{X,N}^2| \geq m_N \sum_{i=1}^{n} |a_{ii}|\}.$$

Comparing the last inequality with (3.15), we find (3.16). It proves the tightness of $\{X_{N,1}\}$ in this case too. Thus, Theorem 3.2.4 is proved.

Acknowledgements. This research was carried out with partial support from RFBR and INTAS under grants RFBR 96-01-01919 and INTAS-RFBR 95-0099.

References


