Optimal growth of global hydrodynamical perturbations in sub-Keplerian toroidal flows

Dmitriy A. Razdoburdin and Viacheslav V. Zhuravlev

Physical Department of MSU, Sternberg Astronomical Institute

SAI MSU

Abstract

A hypersonic rotational polytropic flow in the vicinity of a gravitating body has no global spectral modes which grow on the dynamical timescale. Nevertheless, the fundamental question remains unclear whether the accretion is possible if some sort of small perturbations is set up in the medium, i.e whether the perturbations will ever exhibit a substantial growth at the dynamical time span. In spite of the fact that a traditional modal analysis gives answer "no" for that issue, this doesn't arrive us to the conclusion that linear mechanisms are not related to the basic problem of non-magnetic angular momentum transfer in the rotating flows. On the contrary, it is also known that due to the non-orthogonality of eigenmodes in a shear background there can exist a particular initial perturbations which exhibit a substantial (optimal) growth in energy at finite time intervals. Here we study a specific sort of small inertial-acoustic perturbations being the linear combination of slow neutral eigenmodes with the corotation located beyond the outer boundary of the basic flow. We consider a geometrically thin compressible configurations with free boundaries assuming that $\delta \equiv h/\Delta << 1$, where h is the disc half-thickness and Δ is its finite radial size. We obtain a substantial transient growth of optimal perturbations at the timescale of $t_s \sim (\delta\Omega_0)^{-1}$, where Ω_0 is the typical keplerian frequency in the disc. We note that t_s is longer than the dynamical time intervals but shorter that the viscous, duffusion timescale, $t_\nu \sim \delta^{-2}\Omega_0^{-1}$. So results obtained here can be related to the problem of angular momentum transfer and various temporal phenomena in Keplerian flows including accretion discs which already have an effective turbulent viscosity.

Thin disc

In our research we make the following assumptions:

- The thin disk approximation, $\delta << 1$
- The sub-Keplerian rotation with the small fixed deviation from the pure keplerian angular velocity profile, $\Omega_K \propto r^{-3/2}$, where r is the distance from the gravitating point-mass object
- The polytropic equation of state
- The non-viscous, compressible fluid
- We assume that the basic flow has a finite radial size and the free boundary where the pressure, p, vanishes

For such model, in cylindrical coordinates the stationary configuration is described by the following equations:

$$H(x) = \delta\, x \left[\frac{x_1(1+\ln x) - x(1+\ln x_1)}{x_1 - 1 - \ln x_1}\right]^{1/2}$$
$$h(x,y) = \frac{H^2}{2x^3}\left[1 - \left(\frac{y}{H}\right)^2\right], \qquad (1)$$

where $x \equiv r/r_0$ and $y \equiv z/r_0$ are dimensionless radial and vertical coordinates, $H(x)$ is the half-thickness profile, x_1 and x_2 are the points where $H = 0$, so that $\Delta = x_2 - x_1$. Finally, $h(x,y)$ is an enthalpy distribution in the disc, $h = 0$ corresponds the free surface of the disc.

Spectral problem

First of all, we have to solve the spectral problem of the flow considered. In general, it's a difficult task, which was carried out by a number of authors in the past for a variety of background models. The main complicating factors are the presence of resonances in the flow for definite-frequency modes (i.e. lindblad and corotation resonances) and the necessity to take into account the vertical perturbed motions.

However, we concern about non-modal dynamics here, so we prefer to take the simplest part of modal spectrum as a sample to study the transient dynamics phenomenon. Specifically, we consider only the eigenmodes with no node in the vertical direction and with all resonances beyond the outer boundary of the disk, x_2.

Since the stationary configuration is axially symmetric, the eigenmodes are proportional to $\propto exp(-i\omega t + m\varphi)$, where t is time, φ - azimuthal coordinate, ω is an eigen-frequency and m is an azimuthal number. The basic equations for such modes has the following form:

$$\frac{D}{x\rho}\frac{\partial}{\partial x}\left(\frac{x\rho}{D}\frac{\partial W}{\partial x}\right) - \frac{D}{\rho\bar\omega^2}\frac{\partial}{\partial y}\left(\rho\frac{\partial W}{\partial y}\right) - \left[\frac{2m}{\bar\omega}\frac{D}{x\rho}\frac{\partial}{\partial x}\left(\frac{\Omega\rho}{D}\right) + \frac{D}{a^2} + \frac{m^2}{x^2}\right] W = 0, \qquad (2)$$

where Ω is an angular velocity of the rotating flow, ρ is the density, $W = \delta h/\rho$, δh is the fourier-amplitude of the enthaply euler perturbation, $\bar\omega = \omega - m\Omega$, $\kappa^2 = \frac{2\Omega}{x}\frac{d}{dx}(\Omega x^2)$ is the epicyclic frequency and $D = \kappa^2 - \bar\omega^2$.

We study perturbed flow with a vertical hydrostatic equilibrium, $W = W(x)$ and equation (2) can be integrated over the vertical direction:

$$\frac{D}{x\Sigma}\frac{d}{dx}\left(\frac{x\Sigma}{D}\frac{dW}{dx}\right) - \left[\frac{2m}{\bar\omega}\frac{D}{x\Sigma}\frac{d}{dx}\left(\frac{\Omega\Sigma}{D}\right) + (n+1/2)\frac{D}{h_*} + \frac{m^2}{x^2}\right] W = 0 \qquad (3)$$

where Σ is the surface density:

$$\Sigma(r) = \int_{-H}^{H} \rho dz \propto H\left(\frac{H^2}{x^3}\right)^n \quad \text{and} \quad h_* = \frac{H^2}{2x^3} \qquad (4)$$

is equatorial value of enthalpy. For our primitive eigenmodes in the modal framework, and we employ the WKBJ-method to solve (3):

$$W = C_0 S_1(x)\cos\left[S_0(x) + \varphi_0\right], \qquad (5)$$

where $S_{0,1}$ are known slowly varying functions of x. WKBJ-solution is irregular in boundary points, so we use the regular bessel solutions in the vicinity of $x_{1,2}$:

$$W = C_{1,2}\,\tilde x^{-(2n-1)/4} J_{n-1/2}(z), \qquad (6)$$

where $z \propto \tilde x^{1/2}/\delta$, $\tilde x \equiv |x - x_{1,2}|$ and n is the usual polytropic index (6) match the WKBJ-solution for discrete values of ω given by the dispersion relation:

$$\int_{x_1}^{x_2}\left((2n+1)\frac{-Dx^3}{H^2} - \frac{m^2}{x^2}\right)^{1/2} dx = \pi(n+k) \qquad (7)$$

Optimization problem

Consider the linear span S_N of a finite number of eigenmodes:

$$q = \sum_{n=1}^{N}\kappa_n\tilde q_n \qquad (8)$$

where vector

$$\tilde q_n = \{\delta v_r, \delta v_\varphi, W\}_n \times e^{im\varphi},$$

δv_r, δv_φ are the euler perturbations of radial and azimuthal velocity components and the temporal evolution is determined by the coefficients

$$\kappa_n(t) = \kappa_n^0 e^{-i\omega_n t}.$$

κ_n^0 are the arbitrary complex numbers and ω_n are the corresponding eigen-frequencies.

First, we introduce the inner product in S_N so that the norm of individual perturbations is equal to its full acoustic energy:

$$E_a = \frac{1}{2}\int\Sigma\left(\delta v_r^2 + \delta v_\varphi^2 + (n+1/2)\frac{\delta h}{h_*}\right) x dx d\varphi, \qquad (9)$$

where the vertical integration is done.

Then the growth factor for the individual perturbations is

$$g(t) = \frac{\|\kappa(t)\|^2}{\|\kappa^0\|^2} = \frac{\|e^{\Lambda t}\kappa^0\|^2}{\|\kappa^0\|^2}, \qquad (10)$$

where by definition $\kappa^0 = (\kappa_1^0, \kappa_2^0, ..., \kappa_N^0)^T$, $\kappa = (\kappa_1, \kappa_2, ..., \kappa_N)^T$ and the optimal perturbation at every moment is

$$G(t) = \max_{\kappa^0 \neq 0} g(t) \equiv \|e^{\Lambda t}\|^2, \qquad (11)$$

where $e^{\Lambda T}$ is the propagator acting on the initial perturbations, $\Lambda = diag\{-i\omega_1, -i\omega_2, ..., -i\omega_N\}$.

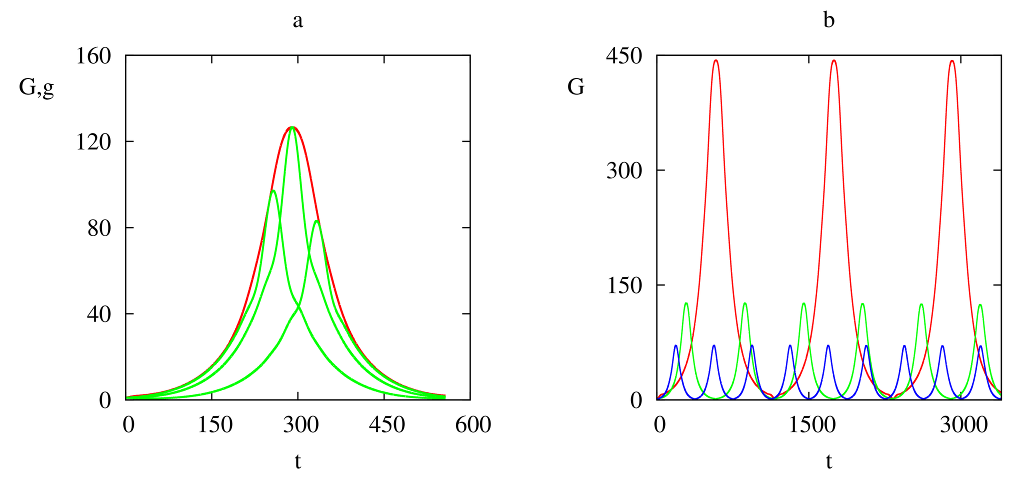
G is calculated employing the standard procedure, specifically, it equals to the first singular value of propagator matrix. It is important to note that the numerical tests show that saturation of G always takes place as we increase N. Namely, G doesn't change noticeably when N gets to $20 - 30$ of higher.

Optimal growth

In the figure we present the results for optimal growth calculation in our model.

On the plot (a) you can see the energy growth for linear combinations that attain maximum at $t = 250, 290, 390$ (green curves) and optimal growth of perturbations (red curve). The fixed parameters are $\delta = 0.002$, $m = 25$, $N = 20$, $\Delta = 1.0$, $n = 3/2$

On the plot (b) you can see the optimal growth for different values of δ. Red, green and blue curves correspond to $\delta = 0.001, 0.002$ and 0.003 accordingly. For all curves here $m = 25$, $N = 20$, $\Delta = 1.0$, $n = 3/2$



We conclude that the linear combination of neural inertial-acoustic modes we've chosen for this study is capable of substantial energy growth at the sonic timescale, t_s, which is intermediate between the the dynamical, t_d, and viscous, t_ν, timescales. Note that as the same time full energy of each separate eigenmode remains constant in time. Moreover, as we can see, the transient growth get higher as the rotation approaches the keplerian profile, i.e. as the disc becomes more and more geometrically thin.

Angular momentum flux

In a rotating disc an angular momentum density of perturbations is equal to

$$K = x < \delta\Sigma\delta v_\varphi >, \qquad (12)$$

where $<>$ means averaging over the azimuthal coordinate, $\delta\Sigma$ is the perturbation of surface density. The non-zero (radial) angular momentum flux density equals to

$$F = x\Sigma < \delta v_r \delta v_\varphi > \qquad (13)$$

These quantities obey the conservation law

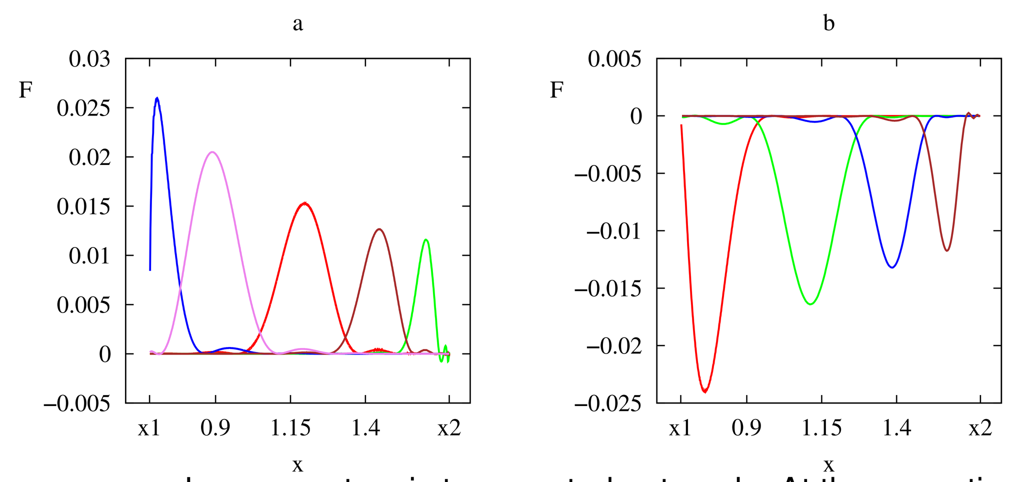
$$\frac{\partial K}{\partial t} + \frac{1}{x}\frac{\partial(xF)}{\partial x} = 0 \qquad (14)$$

F is simply related to the work done by the Reynolds force which is responsible for the energy transport from the basic flow to perturbations. In fact, the total acoustic energy of perturbations changes due to the action of the Reynolds force only:

$$\frac{dE_a}{dt} = -\int x\frac{d\Omega}{dx}\Sigma\delta v_r\delta v_\varphi x\, dx\, d\varphi \qquad (15)$$

Let us see how the profile of $F(x)$ changes with time during the transient growth and decay.

In the figure you can see the angular momentum flux for perturbations with optimization time $t = 290$. The fixed parameters are $\delta = 0.002$, $\Delta = 1.0$, $m = 25$, $n = 3/2$, $N = 20$.
On the plot (a) green, brown, red, magenta and blue curves correspond to $t = 50, 100, 150, 200, 240$.
On the plot (b) red, green, blue and brown curves correspond to $t = 290, 350, 400, 450$.



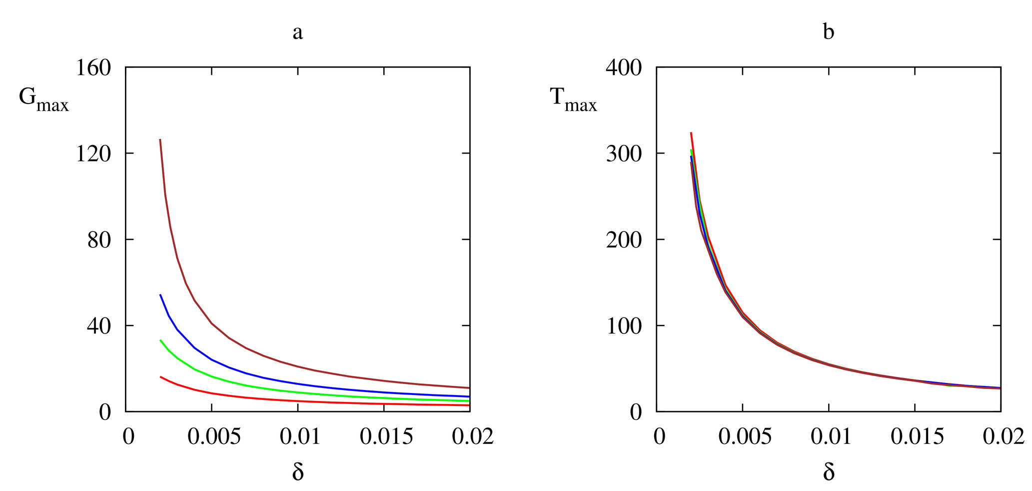
It's clear, that when the energy of perturbations grows up angular momentum is transported outwards. At the same time, the narrow region of the most intensive angular momentum flux moves from the outer edge of the disc to it's inner part. When the evolution changes to the decay epoch, we see the reverse process. Note again, that $F = 0$ at all points of (x_1, x_2) for each separate neutral eigenmode.

Parametric study

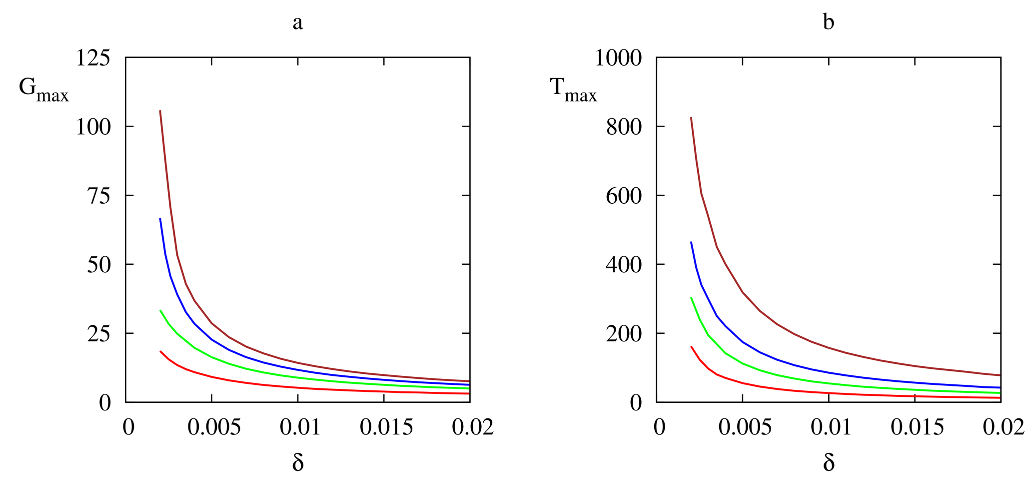
Here we'd like to present a short parametric study of optimal growth in thin polytropic disc. We have few parameters need to be fixed for the particular model. These are the polytropic index ,n, which changes in the range $(3/2, 3)$ from perfect cool gas to relativistic gas, the wave number in azimuthal direction, m, the disc half-thickness, δ, and it's radial size, Δ.

To display G behaviour, we take two characteristic quantities, G_{max}, which is the magnitude of the first maximum on $G(t)$ curve, see the figures above, and the time, T_{max}, at which an optimal growth attains G_{max}.

Let us note at first that we din't find a considerable dependence on n in the range mentioned. On the plots (a) and (b) you can see the dependence of G_{max} and T_{max} on δ. $N = 20$, $\Delta = 2.0$, $n = 3/2$. Red, green, blue and brown curves are obtained for $m = 5, 10, 15, 25$ accordingly.



On the plots (a) and (b) you can see the dependence of G_{max} and T_{max} on δ as well. But now we fix m and change to different radial size of the disc: $N = 20$, $m = 10$, $n = 3/2$, and red, green, blue and brown curves are obtained for $\Delta = 0.5, 1.0, 1.5, 2.5$ accordingly.



Evidently, the transient growth phenomenon becomes more prominent while approaching small δ, i.e. the disc rotation tends to the keplerian profile. This is opposite to sonic modal instability that emerges from resonant interaction between the basic flow and the mode of perturbations (of between two perturbation modes with total energy of opposite signs) at the corotation critical layer and is strongly suppressed close to the keplerian rotation. We also notice that the optimal growth increases as we increase m or Δ.

Comments

Our study shows the significance of the transient growth concept in application to astrophysical systems. We stress that in contrast to the majority of publications in this area we use here the full grobal approach to the problem, so we consider the perturbation dynamics in the whole sub-keplerian thin disc including the influence of the free boundary and the vorticity gradient.

We found that a linear combination of the most simple type of the eigenmodes exhibit a substantial transient growth of acoustic energy with positive angular momentum transfer, i.e. outwards from the central object. Also, the characteristic timescale of optimal transient growth in thin disk is of order of the sonic timescale, which is longer than the dynamical one but shorter that the period of viscous evolution of the configuration as a whole.

Finally, we see several directions to develop our research. That is, one has to include the vertical structure of eigenmodes into consideration, and eigenmodes with corotation resonance inside the flow. Then, it'd be important to include a small viscous corrections since this must be necessary to do on small spatial scale of perturbations. At the same time the latter would possibly allow us to consider a quasi-stationary dynamics of thin hypersonic disc under the external stochastic forcing on the time intervals shorter than the viscous timescale of the whole disc.