A study of transient dynamics of perturbations in Keplerian discs using a variational approach

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Abstract

The behaviour of hydrodynamical fluctuations in gaseous component of young protoplanetary disks is of primary importance for planetary system formation and evolution. It is also related to the major problem of the enhanced angular momentum transfer and nature of the effective viscosity in protoplanetary disks and, more generally, in Keplerian disks. Having a significant gradient of angular velocity, these disks are capable of a substantial transient effects in dynamics of perturbations. In our study we consider such effects employing a variational formulation of the optimisation problem what allows one to obtain an optimal initial perturbations that exhibit the highest possible growth at a specified time interval. In particular, we use our method to study the transient dynamics in a shearing sheet approximation. It is shown that the most rapidly growing shearing harmonic has azimuthal wavelength of order of the disc thickness. Moreover, its initial shape is always nearly identical to vortical perturbation having the same potential vorticity. We also extend our study to a global spatial scale taking into account the background vorticity gradient and the disc cylindrical geometry. It is shown that global vortices with azimuthal wavelengths more than an order of magnitude greater than the disc thickness still are able to attain the growth of dozens of times in a few Keplerian periods at the inner disk boundary. We estimate that if disc is already in a turbulent state with small effective viscosity, these large scale vortices have the most favorable conditions to be transiently amplified before they are damped. At the same time, turbulence is a natural source of the potential vorticity for this transient activity. Thus, we conclude that transiently growing vortical structures on scales above the disc thickness should provide an additional angular momentum transfer in discs and should affect their variability properties as well.

Introduction

A conventional way to study non-stationary phenomena in astrophysical discs is spectral (i.e. modal) analysis, when disc eigenfrequencies are determined by looking for the set of solutions that vary exponentially with time. The next problem is whether the modes are really excited because of spectral instabilities, turbulent motions or external forcing. Contrary to the modal framework, the non-modal approach sorts out perturbations according to the amount of energy they gain from the background during a specified time interval. Here we would like to tackle the optimal configurations of compressible perturbations and the corresponding optimal growth factors. We would also like to illustrate the potential of a relatively new technique for the optimization, which is based on the variational framework and has been successfully applied to a number of complex hydrodynamical flows. The advantage of the variational approach is that, in contrast to the usual optimization method, it does not rely on the representation in the basis of modes or on any other discretization procedure. This implies that it can be easily generalized to non-stationary background flows and even to a non-linear problem.

Variational approach

Variational approach can briefly be outlined as follows. If an evolution of a perturbation state vector $\mathbf{q}$ is controlled by an operator $\mathbf{U}$, $\mathbf{q}(t) = \mathbf{U}\mathbf{q}(0)$, the first right singular vector of $\mathbf{U}$ will show a maximal possible growth, and this growth will be equal to the square of the first singular value of $\mathbf{U}$. From the operator theory, it is known that the first singular value of any operator is the square root of the largest eigenvalue of a positive definite composite operator, which is the original times its adjoint. Thus, in order to solve an optimization problem, we have to determine the largest eigenvalue of $\mathbf{U}^\dagger \cdot \mathbf{U}$, which is equivalent to the advance of the perturbation first forward in time using the $\mathbf{U}$ and then backward in time using $\mathbf{U}^\dagger$. The direct way to converge to the largest eigenvalue of $\mathbf{U}^\dagger \cdot \mathbf{U}$ is to iterate an arbitrary initial perturbation, advancing it recurrently by the operator itself, $(\mathbf{U}^\dagger \cdot \mathbf{U})^{p\to\infty}\, \mathbf{q}(0)$, where p is a natural number. This procedure is usually called the power iteration.

If operator $\mathbf{U}$ is controlled by the basic dynamical equation, the adjoint operator $\mathbf{U}^\dagger$ would be controlled by the so-called adjoint equation, which can be found from the ordinary definition of the adjoint operator.

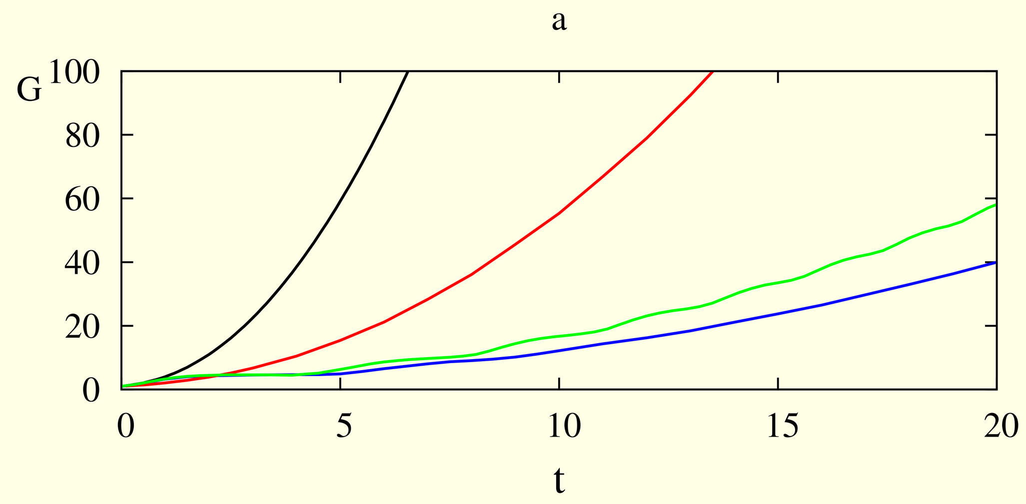
It should be noted that the existence of the largest eigenvalue of $\mathbf{U}^\dagger \cdot \mathbf{U}$ is guaranteed by the Krein–Rutman theorem of functional analysis.

Note that we do not make any assumptions about the background flow. So, the iteration scheme described above can be employed in a wide class of complex flows when the solution of the spectral problem commonly used to evaluate the transient growth can be quite an involved task.

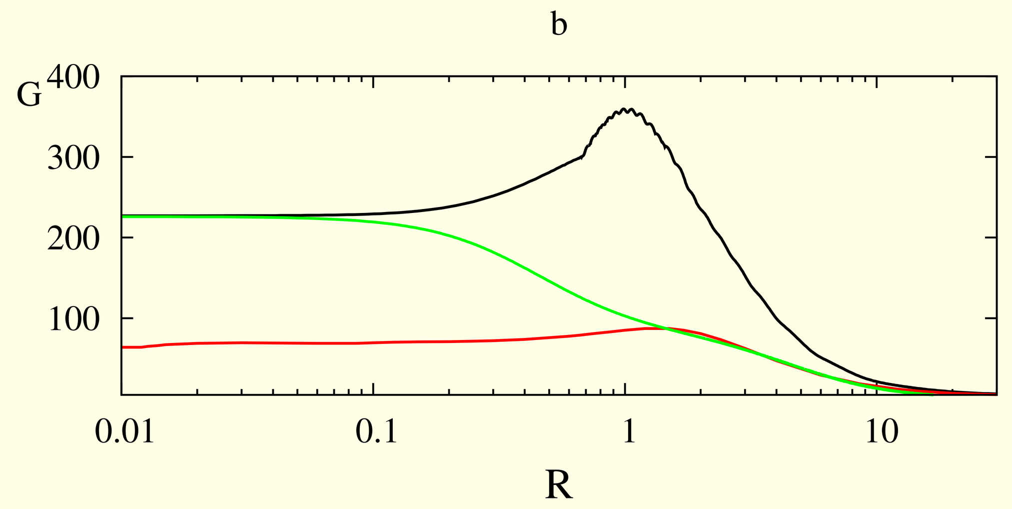
Optimal solutions

To illustrate variational approach in an astrophysical context, we would like to consider small perturbations in a Keplerian disc in a global and local case. We neglect the effects of viscosity and consider only the model case of barotropic equation of state. To measure the growth rate of perturbation we use its acoustic energy.

For a parametrization of the problem we use a parameter $\boldsymbol{R}$, which is approximately equal to the ratio of the azimuthal part of a perturbation's wavelength to the thickness of the disk for the Keplerian angular velocity profile.



In the figure (a) you can see the dependence of the maximal possible growth rate of perturbation's energy $G = \frac{||\mathbf{q}(t)||^2}{||\mathbf{q}(0)||^2}$ on time. Black and green curves show G obtained in local case for $R = 0.12$ and $R = 12$, respectively. Red and blue curves show G obtained in global case for the same values of R. Note that for perturbations with scales much larger then thickness of the disk ($R \gg 1$) the difference between global and local cases become insignificant.



In the figure (b) you can see the dependence of G on R for local (black curve) and global (red curve) cases. In both cases perturbations are optimized at $t = 10$ in units of the inverse Keplerian frequency at the inner boundary of the disk. Note that the most significant growth is demonstrated by the perturbations with $R \sim 1$. Also we estimate the transient growth magnitude analytically in the local approach (green curve in the figure) considering the slow varying vortical solutions in limits $R \gg 1$, $R \ll 1$. Particularly, in the limit of large azimuthal wavelengths and long times, the result becomes especially simple $G \approx \frac{4\Omega_0^4 q^4 t^2}{\kappa^4 R^2}$, where Ω_0 is an angular velocity in the inner radius of the disk, κ is epicyclic frequency and q is an index in the power law dependence of the angular velocity from the radius $\Omega = \Omega_0 \left(\frac{r}{r_0}\right)^{-q}$. For Keplerian disks $q = 3/2$, however, in regions with non-Keplerian rotation, namely, which is close to uniform angular momentum radial distribution, the transient growth of large scale vortices can be much stronger due to the steep dependence on Ω/κ.

Conclusions

We have studied the transient dynamics of linear perturbations in thin Keplerian discs. We show that substantial non-modal growth can exist at all spatial scales including those when the azimuthal wavelength of perturbations is much larger than the disc thickness.

It is important to note that in the case of $R \gg 1$ the magnitude of transient growth is $\propto \frac{4\Omega^4}{\kappa^4}$, which suggests that such vortices exhibit much stronger amplification in a shear flow that approaches a uniform specific angular momentum distribution.

In the local case, for $R \gg 1$, the optimal growth is highly suppressed in comparison with its incompressible ($R \ll 1$) magnitude. However, the situation changes when we extend our study to a global spatial scale, taking into account the background vorticity gradient and the disc's cylindrical geometry. Contrary to what we have in the local problem, the global vortices with the highest azimuthal wavelengths, at least in the range $R \lesssim 10$, exhibit transient growth comparable to what has been found previously in the simplified model of incompressible Keplerian flow. Thus, any kind of persistent source of the potential vorticity on the scales above the disc thickness can lead to the formation and growth of global vortices, providing an enhanced angular momentum transfer to the disc's periphery.

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