International Conference on Modern Algebra in conjunction with the 17th annual Shanks Lectures

Vanderbilt University
Nashville, Tennessee, USA

May 21-24, 2002
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**Lattice Theory and Ordered Structures**

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Universal Algebra and Model Theory

Computational Complexity of Problems in Universal Algebra
Clifford Bergman*
Iowa State University
Giora Slutzki (ISU)

Algebras have long considered algorithmic questions to be of central interest. Traditionally, the focus has been on the decidability of the problem under consideration. As early as 1953, Tarski proved that the equational theory of relation algebras is undecidable. More recently, in 1996, McKenzie did the same for the question of whether a given finite algebra is finitely based.

With the development of the theory of computational complexity, it is possible to sharpen our discussion of algorithmic questions. In this talk I will survey what is known about these questions, and propose several open problems, some of which seem to be quite difficult.

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Quasiprimitive groups and congruence lattices of finite algebras
Ferdinand Boerner
University of Potsdam, Germany

A transitive permutation group is called quasiprimitive if all of its nontrivial normal subgroups are transitive. Similar as in the O'Nan–Scott Theorem for primitive groups, quasiprimitive groups can be classified according to the properties of their minimal normal subgroups (C. Praeger, 1993).

The finite lattice representation problem is the (open) question, whether every finite lattice is isomorphic to the congruence lattice of a finite algebra. This problem is equivalent to the question, whether every finite lattice is isomorphic to the subgroup lattice of a finite group (P.P. Puffy and P. Pudlak, 1980).

Let $G$ be a transitive permutation group on a finite set $A$ and let $e \in A$. If now the interval $[G(e); G]$ in the subgroup lattice of $G$ has a special property, called the LP-property, then $G$ is quasiprimitive. On the other hand, the LP-property is relatively weak. Therefore many lattices have this property.

Using some ideas of R. Baddley (1998), we investigate the structure of groups with special intervals in their subgroup lattice. Finally, we obtain an equivalent formulation of the finite lattice representation problem.

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Algebraic methods in the Constraint Satisfaction Problem
Andrei Bulatov
Ural State University (Ekaterinburg, Russia)

The Constraint Satisfaction Problem (CSP) provides a general framework for a huge variety of combinatorial problems. The links between CSP and some areas of universal algebra established by P. Jeavons and coauthors allows us to make a breakthrough in the study of CSP. In this talk we further develop the algebraic approach to CSP, and show how some notions and results of tame congruence theory can be used.

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The Saga of the High School Identities
Stanley Burris
University of Waterloo

The High School Identities that have been so well known since Dedekind's famous 1888 monograph "sind und was sollen die Zahlen" have proved to be difficult to analyze from the universal algebra perspective. This talk will review highlights of research into this subject and discuss a number of challenging questions the subject poses.

Standard Topological Algebras and Syntactic Congruences
David M. Clark
SUNY New Paltz
Brian Davey, Miroslav Haviar, Marcel Jackson, Jane Pitkethly, Rashed Talukder

A topological quasi-variety (TQV) $\mathcal{X}$ is a category obtained from a finite algebraic structure $\mathbf{M}$ carrying the discrete topology by closing $\{\mathbf{M}\}$ under the formation of direct products, topologically closed substructures and isomorphic images. TQVs are of interest to algebraists since they arise as the duals of algebraic quasi-varieties under natural dualities. In order to make use of a natural duality, it is necessary to have a clear understanding of the structure of the members of its dual category $\mathcal{X}$. A standard topological quasi-variety (STQV) is a TQV in which such an understanding arises in a canonical fashion: $\mathcal{X}$ consists exactly of those algebraic structures having the type of $\mathbf{M}$ which carry a compatible Boolean topology and are members of the quasi-atomic theory of $\mathbf{M}$. We present a congruence condition based on the terms of a topological algebra $\mathbf{M}$ which guarantees that the TQV it generates is standard.

Brian Davey
La Trobe University
David Clark (SUNY, New Paltz)

From a review of the second edition: "Since the first edition of this text appeared in 1998 much progress has been made in the development of the theory of natural dualities. This new edition brings the theory to date and brings into focus the major unsolved problems in the area." In this talk more details from the review will be presented.

(Footnote: The second edition is a figment of the speaker's imagination, but what it might contain is not.)
Decidability of discriminator varieties with group stalks
Dejan Delic
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One of the fundamental problems in universal algebra is to provide the full characterization of locally finite varieties whose first-order theory is decidable. In order for this question to be settled, a satisfactory characterization of decidable locally finite discriminator varieties needs to be provided, in light of the decomposition theorem of R. McKenzie and M. Valeriote. In this talk, we will discuss the decidability of locally finite discriminator varieties arising from universally axiomatized locally finite classes of groups and try to shed some light on algebraic and model-theoretic aspects of this problem.

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Term-valuated Equational Theory
Klaus Denecke
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A term valuation is a special homomorphism from the term algebra into another algebra of the same type on which a partial order relation is defined. Examples are the operation symbol count of a term, the variable count, the minimum depth and the maximum depth of a term. For a given equational theory, for a given natural number \( k \) we ask for all equations from the equational theory such that both sides are terms of a value greater or equal to \( k \). The model classes of such sets of equations, called \( k \)-normalizations of the given equational theories, form complete sublattices of the lattice of all varieties of the given type. An important problem is to find a generating set of the set of all identities of the \( k \)-normalization using the generating system of the given variety.

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The spectrum function and varieties
Bradd Hart
Fields Institute for Research in Mathematical Sciences

In recently published work, Hrushovski, Laskowski and I completed the computation of all uncountable spectra for countable theories. A precursor to this was work done by Starchenko, Valeriote and I on the infinite spectra for varieties. This latter work was inspired by the beautiful decomposition result of McKenzie and Valeriote for decidable, locally finite varieties. In my talk, I will describe the possible spectra and give examples of each which are varieties.

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How finite is a three-element unary algebra?
Jennifer Hyndman*
University of Northern British Columbia
Jane Pitkeithly (La Trobe University)

Within the class of three-element unary algebras there is a tight connection between several nice finiteness properties. In 1989, Bestsenyi introduced three special three-element unary algebras, \( V \), \( L \), and \( D \).

**Theorem** Let \( M \) be a three-element unary algebra. Then the following are equivalent.

1. \( M \) does not have \( V \), \( L \) or \( D \) as a reduct;
2. the quasi-equational theory of \( M \) is finitely based;
3. \( M \) has finite rank;
4. \( M \) has enough algebraic operations;
5. \( M \) is quasi-injective.

Bestsenyi has already proven that conditions (1) and (2) are equivalent, and Lampe, McNulty and Willard have shown that (4) implies (3) in general.

Although the properties of finite rank and enough algebraic operations come from duality theory, the equivalences in this theorem are independent of dualisability. There are several three-element unary algebras that have enough algebraic operations, but are not dualisable. For those algebras that are dualisable, known results from duality theory were used to establish the equivalences.

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Generative complexity in algebra
Pawel M. Idziak*
(Jagiellonian University, Krakow, Poland)
Joel Berman (University of Illinois, Chicago, IL, USA), Ralph McKenzie (Vanderbilt University, Nashville, TN, USA), Matthew Valeriote (McMaster University, Hamilton, ON, Canada)

The generative complexity of a class of models is the function that counts, up to isomorphism, the number of at most \( k \)-generated models in the class.

The infinite counterpart of the problem of counting non-isomorphic models is widely studied in Model Theory, and is one of the fundamental topics for Shelah's classification theory and for stability theory. Note that in the infinite realm (and for a countable language) being \( k \)-generated and having \( k \) elements are the same.

However in the finite setting it is enormously hard to count models, especially algebras, with a given number of elements. There are many reasons for these difficulties. One reason that interests us is that algebras are often described by means of a set of generators. Once we know the generators of an algebra \( A \) in a given class \( \mathcal{V} \) and some of the conditions that these generators satisfy, our freedom in building the rest of the model is heavily restricted. This effect is widely used in group theory where a group is usually presented by a (finite) set of generators and a set of relations the generators must obey. The constraints put on the behavior of the generators place restrictions on the structure of the entire algebra \( A \). However, there is in general no obvious or transparent way to determine the cardinality of \( A \). This makes the counting of all \( n \)-element or at most \( n \)-element algebras difficult, if at all possible, even if we content ourselves with an asymptotic estimate.

One possible way to overcome these difficulties is to count \( k \)-generated models instead of \( k \)-element ones. This is a more tractable problem when the classes are varieties of algebras, and we believe is the proper
setting for asymptotic probabilities in algebra. Note however that these numbers are the same for purely
minimal languages.

Two main results of our work are:

- a full characterization of locally finite varieties with polynomial generative complexity as varietal
  products of an affine variety over a ring of finite representation type and several varieties that are
  matrix powers of pointed $G$-sets,

- a full characterization of finitely generated congruence modular varieties (or, in fact, omitting TCT
  type 1) with singly exponential generative complexity.

The characterization of locally finite varieties with polynomial generative complexity was obtained by
P. Idziak, R. McKenzie and M. Valeriote. The conditions in this situation are very simple and easily
stated. The conditions involved in the second characterization, obtained by J. Berman and P. Idziak, are
more complicated, although they also have a natural algebraic meaning. In both cases we know that the
bound for the number of algebras implies a very transparent and manageable structure. For example, when
specializing our results to groups we get the following:

- every finitely generated variety of groups has at most doubly exponential generative complexity,

- a finitely generated variety of groups has singly exponential generative complexity if and only if it is
  nilpotent,

- a finitely generated variety of groups has polynomial generative complexity if and only if it is Abelian,

while for commutative rings with unit our characterization reduces to:

- every finitely generated variety of rings has at most doubly exponential generative complexity,

- a finitely generated variety of commutative rings with unit has singly exponential generative complexity
  if and only if the Jacobson radical in the generating ring squares to 0,

- no nontrivial variety of rings with unit has polynomial generative complexity.

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Constraints, Complexity, and Polymorphisms
Peter Jeavons
Oxford University Computing Laboratory

Many natural combinatorial problems can be expressed as "constraint satisfaction problems", so this
class of problems has received a lot of attention within Computer Science. Although these problems are NP-
hard in general, certain restrictions on the form of the constraints can ensure tractability. By considering
the algebra of polymorphisms associated with constraint relations we can translate questions about the
computational complexity of this kind of problem into questions about the associated algebra. The talk will
provide an introduction to this approach and some results and open questions.

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Quasiequational theories of flat algebras
Jaroslav Ježek*
Department of Mathematics, Vanderbilt University
Miklós Maróti

A finite oriented graph \((V, E)\) can be made into an algebra \(A(V, E)\) with two binary operations, multiplication and meet, in this way: the underlying set is the union of \(V\) with \(E\) and \(\{0\}\); this is a flat semilattice with the least element \(0\) (all the other elements are atoms); we put \(ae = b\) if \(e\) is an edge from \(a\) to \(b\); in all other cases put \(xy = 0\). The equational theories of the algebras \(A(V, E)\) are known to be mostly (inherently) nonfinitely based. We have proved that the quasiequational theories are all finitely based. There are also more general results.

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On categorical equivalence of finite arithmetical algebras
Kalle Kaarli
University of Tartu, Estonia

C. Bergman [1] completely solved the categorical equivalence problem for finite algebras with majority term. As a special case, he proved that the categorical equivalence class of a finite algebra \(A\) which has no proper subalgebras and generates an arithmetical variety, is fully determined by the full subcategory \(Q(A)\) of \(V(A)\) whose objects are the quotient algebras of \(A\). The present work addresses the question: what is the formal structure of the categories \(Q(A)\) which appear in this situation? So far we have mainly studied algebras with all quotients congruence rigid. The latter means that every automorphism preserves all congruences. The system of automorphism groups of the quotients of \(A\) is a main example of what we call a group scheme. If all quotients of \(A\) are congruence rigid then the group scheme of \(A\) determines the category \(Q(A)\). We study which group schemes give rise to finite algebras having no proper subalgebras and generating arithmetical varieties.


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Specht type problems and related questions.
Alexei Kanel-Belov
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There was a problem, posed by W. Specht in 1950:
Specht problem Does any increasing chain of \(T\)-ideals stabilize? Or — Is any set of identities finitely based (can be presented by a finite subset)?

W. Specht had in mind the case of algebras over a field of zero characteristic. This problem was solved affirmatively by A.R. Kemer. A.I. Maltsev gave another point of view. He considered the more general case (any characteristic and over an arbitrary ring).

The author proved the next theorem:
Theorem 1 The following set of identities \(R_n\)

\[ R_n = \langle [E, T], T \rangle \eta_n([T, [T, F]]([E, T], T])^{\eta-1} [T, [T, F]] \]
is infinitely based. ([·, ·] denotes commutator, \( Q(x, y) = x^{p-1}y^{p-1}[x, y] \), \( p \) is characteristic of main field, 
\( q = y^p \). \( Q_{1} = Q(x_{1}, y_{1}) \cdots Q(x_{n}, y_{n}) \).

The finite basis problem can be considered in the local case (i.e. chain conditions on sets of identities in 
a finitely generated algebra). There was a well known problem, related to this question:

Does any increasing chain of T-ideals in a finitely generated algebra stabilize?

Is any finitely generated relatively free algebra representable? (i.e. embeddable in a matrix algebra over 
a Noetherian commutative ring)

Can any relatively free PI-algebra be approximated by finite dimensional ones? These problems where 
posed by L. Bokut', I. Lvov, A.I. Maltsev. A.R. Kemer obtained a positive answer in the homogeneous case, 
i.e. when the main field is infinite.

Theorem 2 (Below) Every relatively free PI-ring is representable, and any increasing chain of ideals of 
identities in a finitely generated ring stabilizes.

There was a problem, posed by A.I. Maltsev in 1967 (and also by P. Cohn, Tarski);

Maltsev problem Let \( f \) be an identity, \( g_i \) be a finite set of identities. The question is: is \( f \) a consequence 
of \( \{g_i\}\)? Does a general algorithm solving this question exist?

In case of groups the answer is "No" (as was shown by Yu. Kleiman). The author proved the next result.

Theorem 4 There exists such general algorithm in the case of associative rings.

The Specht-type problems are closely connected with properties of Hilbert series of algebras. C. Procesi 
posed a problem about rationality of Hilbert series of algebra of general matrices. In case of \( 2 \times 2 \) matrices 
this problem was solved by V. Drensky. He also proved rationality of Hilbert series of relatively free algebras 
in non-matrix varieties.

Theorem 5 a) The Hilbert series of relatively free algebras is rational.

b) There is a representable algebra with a transcendental Hilbert series.

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1990, 54, N4, pp.726-753.
pp.133-134.

Keywords: PI-ring, Specht problem, Hilbert series, Identities, Normal basis.

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Flatness of Semimodules over Semirings

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In the talk, there will be considered tensor product bifunctors in the context of semimodules over semirings, 
defined in such a fashion that they become left adjoints of the Hom functors (see, for example, Y. Katsov, 
Tensor products and injective envelopes of semimodules over additively regular semirings, Algebra Colloquium 
4.2 (1997) 121-131). As usual, a left semimodule \( A \) over a semiring \( R \) is said to be flat if the tensor 
multiplication by \( A \) preserves all finite limits. On the other hand, \( A \) is called L-flat if it is a filtered colimit of 
finally generated free semimodules. Using Shannon's characterization of L-flatness for algebraic categories, 
we establish that in the setting of semimodules over semirings both concepts — flatness and L-flatness — 
coincide (Gowers-Lazard's theorem for the varieties of semimodules over semirings). Then, there will be de 
scribed flat semimodules over finite Boolean algebras, in particular 0-flat semilattices, as well as commutative 
semirings over which all semimodules are flat.

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Free spectra
Keith A. Kearnes
University of Colorado

If $V$ is a variety of algebras, then $F_V(n)$ denotes the $V$-free algebra on $n$ generators. The function $f_V(n) = |F_V(n)|$ is called the free spectrum of $V$. We will discuss problems and results concerning the growth rates of free spectra of finitely generated varieties.

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Universal Algebra, Permutation Groups, and Supergraphs
Andrzej Kisielewicz
Wrocław University

A few results concerning clones and equational theories of semigroups are presented that have a strong link with permutation groups and combinatorics. Not only are some results on permutation groups and graphs applied to solve problems in universal algebra, but these problems provide a motivation for opening a new interesting area of research in permutation groups and graphs.

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Separator Algebras
Arthur Knoebel

In unitary rings, one can characterize direct products internally in five different ways: central idempotents, factor ideals, orthogonal projections, factor congruences and decomposing functions. The two objectives of this talk are: (1) to show how to define direct products internally in these five ways for a much larger class of algebras; and (2) to prove embeddability of the new in the old. A separator algebra is an algebra having a quaternary term function $q$ and two constants 0 and 1 satisfying $q(1,0,x,y) = x$ and $q(0,1,x,y) = y$. Examples are unitary rings and bounded lattices: $q(a,b,x,y) = ax + by$. Any discriminator algebra is polynomially equivalent to a separator algebra, but not conversely. Theorem: Each separator algebra $A$ has a Boolean algebra of factor elements $\text{Elem}'A$ that is isomorphic to the Boolean algebra of factor congruences, and so isomorphic to the Boolean algebras of all the other factor objects. Theorem: For any separator algebra $A$ there is a ring $B$, perhaps with additional operations, such that $A$ is a subalgebra of a reduct of $B$; $\text{Elem}'A$ is a subalgebra of $\text{Elem}'B$; and $q(e,e',x,y) = ex + e'y$ for all $e$ in $\text{Elem}'B$ and all $x,y$ in $B$. In a similar fashion, any separator algebra is embeddable in a distributive lattice or a subdirect product of discriminator algebras.

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The decidability of ordinal sums
Chris Laskowski*
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The ordinal sum of a family of semigroups indexed by a linear order is discussed in Fuchs [1]. (The universe of the ordinal sum is the disjoint union of the semigroups and the semigroup operation is extended by asserting that $x \cdot y = y \cdot x = x$ whenever $y$ is in a semigroup below the semigroup containing $x$).

We generalize this notion and define the ordinal sum of models of a theory $T$ in any finite language $L$ with no constant symbols. We prove that if $T$ is a decidable $L$-theory then the class of ordinal sums of models of $T$ is decidable. We then use this result to show that the elementary theory of MV-chains and Hajek’s BL-chains [2] are decidable.


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Very large cardinals
Richard Laver
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If $\phi(k)$ and $\psi(k)$ are properties of a cardinal $k$, say that $\phi(k)$ strongly implies $\psi(k)$ if $\phi(k)$ implies that $\psi(k)$ and that for some $k' < k$, $\psi(k')$. Then the large cardinal properties of set theory tend to be well ordered under strong implication. We will review some facts about large cardinals and their relation to determinacy, and, in the case of very large cardinals, mention some consequences of and consider the question of strong implications between these axioms.

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The variety generated by tournaments
Miklós Maróti
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We prove that every subdirectly irreducible member of the variety $T$ generated by tournaments (commutative groupoids satisfying $xy \in \{x,y\}$) is a tournament. This has many interesting consequences: The lattice of subvarieties of $T$ is distributive. Every finitely generated subvariety of $T$ is finitely based and has a finite residual bound.

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Locally Finite Varieties
Ralph McKenzie
Vanderbilt University

I will discuss solved and unsolved problems for locally finite varieties. Detailed study of the interaction between polynomial functions and congruences in finite algebras produced a tool, tame congruence theory, which has been applied very successfully to discover and prove unsuspected general results about locally finite varieties. It is a mystery why so many of these results turned out to be provable, by other means, for non-locally finite varieties.

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Which Finite Algebras are Finitely Based?
George F. McNulty
University of South Carolina

Ralph McKenzie’s resolution of Tarski’s Finite Basis Problem tells us that there is no algorithm for determining which among all finite algebras with finitely many fundamental operations have finitely based equational theories. We must settle instead for widely and easily applicable sufficient conditions and for such necessary conditions. This talk will take up sufficient conditions.

In the literature, apart from some apparently ad hoc results, there seem to be four general directions: that direction loosely based on investigation of the clone of term operations of the algebra (e.g., work of Lyndon, Murskii, and Berman); that direction leading to the conclusion that each finite group and each finite ring is finitely based (e.g., work of Oates and Powell and of L’vov and of Kruse); that direction launched by Ralph McKenzie’s proof that finite lattices with operators are finitely based which led, through work of Kirby Baker, to Ross Willard’s proof concerning finite algebras which generate residually finite congruence meet-semidistributive varieties; and finally that direction launched, again by McKenzie, centered on the role of definable principal congruences and leading to recent results of Kirby Baker and Ju Wang.

The bulk of the talk will address advances and prospects for unifying the last three directions. The advances are collaborative work with Kirby Baker and Ju Wang. The talk will conclude with a number of open problems.

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Elementary equivalence of free algebras of varieties
Alexander Pinus
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The elementary equivalence of derived structures of free algebras
A.G. Pinus (Novosibirsk, Russia)

As is known all infinitely generated free algebras of all varieties are elementary equivalent. Another situation we have for derived structures of this algebras: semigroups of endomorphisms $EndF_V(k)$, groups of automorphisms $AutF_V(k)$, lattices of subalgebras $SubF_V(k)$ and the lattices of congruences $ConF_V(k)$. Here $F_V(k)$ is $k$-generated $V$-free algebra for some variety $V$. So, this relation of elementary equivalence of derived structures of those free algebras gives some basic for some classification of infinitely generated
V-free algebras for any variety V. This relation the following equivalence relations on the class of all infinite cardinals:
\[ k \equiv V_{\text{end}} \lambda \iff \text{End}F_V(k) \equiv \text{End}F_V(\lambda), \]
\[ k \equiv V_{\text{aut}} \lambda \iff \text{Aut}F_V(k) \equiv \text{Aut}F_V(\lambda), \]
\[ k \equiv V_{\text{sub}} \lambda \iff \text{Sub}F_V(k) \equiv \text{Sub}F_V(\lambda), \]
\[ k \equiv V_{\text{con}} \lambda \iff \text{Con}F_V(k) \equiv \text{Con}F_V(\lambda). \]
S. Shelah (1974) proved that the relation \( \equiv V_{\text{end}} \) coincides with the relation \( \equiv_2 \) for any nontrivial variety V. Here \( \equiv_2 \) is relation of equivalence of cardinals in full second order logic. Let us denote the as \( \nabla \) and \( \equiv_n \) the following relations on the class of all infinite cardinals:
\[ K \nabla \lambda \iff \text{for any infinite cardinals } K \text{ and } \lambda, \]
\[ K \equiv_3 \lambda \iff \text{the cardinals } K \text{ and } \lambda \text{ are equivalent in the second order logic with the quantifiers for any permutation on this cardinals.} \]
In all known cases, the relations \( \equiv_{\text{aut}}V, \equiv_{\text{sub}}V, \equiv_{\text{con}}V \) coincide with either relation: \( \nabla, \equiv_n \) or \( \equiv_2 \). Here, we give a survey of the known results for relations \( \equiv_{\text{aut}}V, \equiv_{\text{sub}}V, \equiv_{\text{con}}V \) for some varieties V, we formulate also some natural problems which are connected with this relations and formulate the following new results.

**Theorem 1** For any nontrivial variety V of groups the lattice \( \text{Con}F_V(k)(\text{Sub}F_V(k)) \), is elementary definable in the class of all this lattices iff the cardinal k is definable in the full second order logic.

**Theorem 2** For any variety V of algebras the relations \( \equiv_{\text{sub}}V \) and \( \nabla \) are the same.

**Theorem 3** For the variety V of all semigroups (of all commutative semigroups) the relations \( \equiv_{\text{con}}V, \equiv_{\text{sub}}V \equiv_2 \) are the same.

**Theorem 4** For the variety V of all semilattices the relations \( \equiv_{\text{con}}V \) and \( \equiv_2 \) are the same.

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**A Galois connection between operations and relations and an application to Tame Congruence Theory**

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The Galois connection induced by "a function preserves a relation" can be used to describe properties of algebras by invariant relations. Several applications will be shown, e.g. the characterization of minimal algebras (in the sense of Tame Congruence Theory) by a ternary relation.

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**Is Modularity prime?**

Luís Sequeira

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The lattice \( L \) of interpretability types was defined in [4]. Maltsev-type conditions naturally correspond to Maltsev filters in this lattice, i.e., filters generated by a countable chain of finitely based varieties of finite type.

The interpretability types of congruence modular varieties, or of congruence distributive varieties, constitute Maltsev filters [1, 3].

Garcia and Taylor [2] conjectured in 1984 that the congruence-modularity filter is prime. On the other hand, the following easy example shows that the congruence-distributivity filter is not prime. Let \( \mathcal{P} \) denote the Maltsev variety, which has a single ternary fundamental operation \( p \) satisfying the identities

\[ p(x, x, z) \equiv z, \quad p(x, x, z) \equiv z \]
is a majority term for $\mathcal{P} \lor \mathcal{S}$. This term has "depth 2" (in the sense of [6]), for it has the form of a $\mathcal{P}$-term applied to $\mathcal{S}$-terms. Letting $h$ denote the $\mathcal{P}$-term defined by

$$h(x_0, \ldots, x_6) := p(p(x_0, x_1, x_2), x_3, p(x_4, x_5, x_6))$$

then $m$ can be depicted by the following tree.

The main result to be presented in this talk is that it is impossible to disprove the modularity conjecture using terms of depth 2 alone:

**Theorem ([5]).** If two varieties $\mathcal{V}$ and $\mathcal{W}$ are such that terms of depth 2 in $\mathcal{V} \lor \mathcal{W}$ satisfy the well known Day identities for congruence-modularity, then either $\mathcal{V}$ or $\mathcal{W}$ is congruence-modular.

**References**


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Representing finite lattices as congruence lattices of finite algebras

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Call a finite lattice which is isomorphic to the congruence lattice of a finite algebra representable. In A constructive approach to the finite congruence lattice representation problem (Algebra Universalis 43, 2000), we began exploring constructions by which one can make new representable lattices from known representable lattices. In this talk, we will demonstrate how the techniques from that paper can be used to prove the following two theorems.

Theorem 1: Any subdirect product of a finite distributive lattice and a representable lattice is representable.

Theorem 2: Every finite lattice in the variety generated by $M_3$ is representable.

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On the congruence problem for polynomial semirings over natural numbers

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We apply rewriting techniques to solve the congruence problem for polynomial semirings over natural numbers. Given a relation $T$ on $\mathbb{N}[x_1, \ldots, x_n]$ a reduction relation is defined. A completion procedure is given. If the procedure terminates, then for a finite relation $T$ on $\mathbb{N}[x_1, \ldots, x_n]$ it gives a relation $T'$ such that $T'$ defines the same congruence relation as $T$ does, and the reduction relation defined by $T'$ is convergent. In contrast to the completion procedure for polynomial rings, this algorithm may not terminate. This is not unexpected since the congruence problem for this case is in general not solvable by using rewriting.

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Left symmetric left distributive groupoids

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Given a group $G$, the derived operation $x \ast y = x y^{-1} x$ is left symmetric (i.e. $x \ast (x \ast y) = y$), left distributive (i.e. $x \ast (y \ast z) = (x \ast y) \ast (x \ast z)$) and idempotent. Groupoids satisfying these identities (shortly LSLDI-groupoids) were widely studied and applied in geometry (see Nobusawa, Pierce, Joyce, etc.). We contribute with a description of the equational theory of several LSLDI-operations on a group.

On the other hand, almost nothing was known about generally non-idempotent LSLD-groupoids. We follow the investigations started by T. Kepka. Several non-idempotent LSLD-operations on a group are found and their equational theories are discussed. A structure of subdirectly irreducible non-idempotent LSLD-groupoids is described.

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Monotone retracts and NUF in finite posets
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Gábor Kun

We introduce a new version of the concept of order varieties, namely, in addition to retracts and product we require that the class of posets should be closed under taking idempotent subalgebras. As an application we prove that the variety generated by an order-primal algebra is congruence modular if and only if every idempotent subalgebra is connected.

We do this by developing a technique of dismantling retracts. We give a polynomial time algorithm to decide whether or not a variety generated by an order-primal algebra is congruence modular and we construct a NUF.

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Clones and Groups
Ágnes Szendrei
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In this talk we will discuss some results and open problems in clone theory that concern groups. Topics will include clones on finite sets whose unary part is a transitive permutation group, and term equivalence of groups.

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Some aspects of the interpretability lattice
Walter Taylor
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The talk will review some aspects of the varietal interpretability lattice, especially complete meet- and join-irreducibility questions, with some attention to contributions of Professor McKenzie in the area. Some open problems will be stated.

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Weak Congruences in Universal Algebra
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Branimir Seselja

A book "Weak Congruences in Universal Algebra" by B. Seselja and A. Tepavcevic, Institute of Mathematics, Symbol, Novi Sad, published in December 2001, will be presented. The book contains most of the results concerning weak congruences (symmetric, transitive and compatible relations on an algebra) from the appearance of the first papers on the topic fifteen years ago. New results about some identities valid
on a weak congruence lattice will be given. Open problems will be discussed, like abstract representation problem of lattices by weak congruences and a problem on the Congruence Intersection Property (CIP).

In structural investigations of algebras and varieties, the best known ordered structures are lattices of subalgebras and those of congruences. These two lattices are usually studied separately, since they consist of different kinds of object connected with an algebra: subsets of the universe and subsets of its square. Weak congruences appear to be a tool for common investigations of both, congruences and subalgebras of the same algebra. Being symmetric and transitive subalgebras of the square, these relations can be understood as congruences on all subalgebras, among which diagonal relations represent subalgebras themselves. Thus, the collection $\mathcal{C}_wA$ of weak congruences on an algebra $A$ is a lattice under inclusion, and its sublattices are $\text{Con}_A, \text{Sub}_A$, and $\text{Con}_B$, for every subalgebra $B$.

It turns out that $\mathcal{C}_wA$ provides information about properties of an algebra which could not be deduced from the lattices of its subalgebras and congruences (e.g., regularity, Hamiltonian property, extension of congruences etc.). In addition, all algebras representing a given lattice by weak congruences must fulfill these algebraic conditions.

Weak congruences were introduced in universal algebra together with other compatible relations generalizing the concept of a congruence (implicitly by F. Šik and his Ph.D. student T.D. Mai in 1974, and under the present name by Vojvodić and Šešelja in 1988).

The aim of this text is to present basic properties of weak congruences, particularly of lattices of these, and to highlight their applications in universal algebra. The book is appearing after systematic investigations conducted by the authors over several years, and contains the complete bibliography on these and related topics until 2001.

Chapter 1 is an overview of distributive and other special elements of lattices, which serve as a tool for describing properties of weak congruence lattices. Proofs are given only for results which could not be found in the literature.

Main definitions and properties of notions concerning weak congruences and the corresponding lattices are presented in Chapter 2. This part contains also some historical remarks about weak equivalences and compatible relations.

Weak congruences in known classes of algebras and varieties like groups, rings, Abelian and Hamiltonian algebras are discussed in Chapter 3.

Chapter 4 is devoted to different aspects of representations of lattices by weak congruences. As a concrete application of the results in this part, Appendix contains representations by weak congruences of all lattices with at most six elements.

Finally, a generalization of weak congruences to other compatible relations is presented in Chapter 5. Complete proofs are given mostly for theorems appearing here for the first time. In addition, formulations of some theorems and proofs differ from those in original papers. However, most of the theorems are commented on a sketch of the proof, or by a suitable example.

Open problems connected with weak congruences:
1. Which Abelian or Hamiltonian varieties possess the CIP (Congruence Intersection Property)?

   Congruence Intersection Property is equivalent with the distributivity of the diagonal element in the lattice of weak congruences.

2. Weak congruence lattice representation problem. Let $L$ be an algebraic lattice and $a \in L$. Is there an algebra such that its weak congruence lattice is isomorphic with $L$, the diagonal relation being the image of $a$ under the isomorphism.

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Residually Small Locally Finite Varieties
Matt Valeriote
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A variety is residually small if there is a cardinal bound to the size of its subdirectly irreducible members. I will discuss the progress that has been made in classifying the residually small locally finite varieties.

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Varieties with a property that is not uniform
Benjamin Wells
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Cupona et al. study* idempotent-type left and right power laws in groupoids. They prove characterizations, correspondences, and decision procedures for many varieties given by these equations. Most of their results have a uniform character across the different classes of varieties they consider. I set out to fix that.

* "Varieties of groupoids with axioms of the form \(x^{n+1}y = xy\) and/or \(xy^{n+1} = xy\)" by G. Cupona, N. Celakoshi, and B. Janeva; to appear in Glasnik matematiki.

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An answer to the question "Why develop general algebra as we do today?" can be seen in the desire to support humans in their thought, judgments and actions. A general claim is that algebra serves humans as a language for "good" descriptions of facts, systems, and relationships of our scientifically and technologically formed world. The specific nature of algebraic descriptions lies in the use of operations and their structural properties. Thus, the general understanding of algebra as the structure theory of sets with operations combines well with the understanding of algebra as language for operational descriptions and also with Peirce's understanding of mathematics as a cosmos of forms for potential realities. That algebra serves indeed as a good description language can be easily confirmed by many prominent examples. The lecture gives a systematic approach to those algebraic descriptions.

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Groups, Semigroups and Rings

General quantum polynomials
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Let $k$ be a field with a fixed matrix $Q = (q_{ij}) \in \text{Mat}(n, k), n \geq 2$, whose entries satisfy the relations $q_{ii} = 1$ for all $i, j$. Let also $r$ be an integer such that $0 \leq r \leq n$. Denote by $\Lambda$ the associative $k$-algebra with a unit generated by elements $X_1^{-1}, X_2^{-1}, \ldots, X_{r+1}^{-1}, x_{r+1}, \ldots, x_n$ subject to defining relations $X_i X_j^{-1} = X_j^{-1} X_i = 1$ for all $1 \leq i \leq r$ and $X_i X_j = q_{ij} X_j X_i$ for all $1 \leq i, j \leq n$. The algebra $\Lambda$ is a left and right Noetherian domain. Therefore it satisfies Ore condition and it has a division ring of fractions $F$.

The algebra $\Lambda$ is an algebra of quantum polynomials and the elements $q_{ij}$ are multiparameters. The algebra $\Lambda$ is of a considerable interest in noncommutative algebraic geometry [D]. If $r = n$ then $\Lambda$ is a coordinate algebra of affine quantum space $A^0_n$. If $r = n$ then $\Lambda$ is a coordinate algebra of a quantum torus [B]. An approach to an identification of points of $A^0_n$ is related to the study of prime ideals of $\Lambda$ [GL]. In this paper it is assumed that $r = 0$ and the multiparameters generate a torsion-free subgroup in the multiplicative group $k^*$ of the field $k$. An algorithm for a calculation of Krull dimension of $\Lambda$ in the case $r = n$ is presented in [MP]. Another characterization of Krull dimension of $\Lambda$ with $r = n$ in terms of commuting monomials is found [B].

In what follows we shall assume that the algebra $\Lambda$ is a general algebra of quantum polynomials that is all multiparameters $q_{ij}, 1 \leq i < j \leq n$, are independent in the multiplicative group $k^*$ of the field $k$ [MP].
We show that all valuations on the division ring of fractions $F$ are Abelian and all of them are classified by linear orders on $\mathbb{Z}^n$ [A3]. In some sense this is another classification of points of $A^0_n$.

With respect to a lexicographic order (and corresponding valuation $\nu$) we can correctly define the completion $F_\nu$ which is a division ring containing $F$ and which is known as Malcev-Neumann power series ring.

In our talk we shall show that all finitely generated projective modules over $\Lambda$ of a rank at least 2 are free [A1]. It will also be shown that if $\Lambda, \Lambda'$ are two Morita-equivalent quantum polynomial algebras and one of them is a general one, then $\Lambda \simeq \Lambda'$ [A2].

Theorem 1. If $n \geq 3$ then any injective endomorphism of $\Lambda$ is an automorphism. If $r < n$, then the automorphism group of $\Lambda$ is Abelian. If $r = n \geq 3$ then the automorphism group of $\Lambda$ is Abelian-by-cyclic of order 2. If $r = n = 2$ then the automorphism group of $\Lambda$ is Abelian-by-$\text{SL}(2, \mathbb{Z})$. If $r = n = 2$ then a finite automorphism group of $\Lambda$ is Abelian-by-cyclic of order 1, 2, 3, 4, 6.

The case $n \geq 3$ in Theorem 1 was considered in [AW].

Theorem 2. Any (injective) endomorphism of $F_\nu$ is an automorphism. It is a composition of an automorphism of multiplication of variables by scalars and a conjugation by an element $(1 - z)$, where $\nu(z) > 0$.

Any finite subgroup of automorphisms of $F_\nu$ is conjugate to the subgroup consisting of automorphisms of multiplications of variables by scalars.

The case $n \geq 3$ in Theorem 2 was considered in [A3]. The subalgebra $\text{Der} \Lambda$ (respectively, $\text{Der} F_\nu$) of inner derivations is always an ideal in the the Lie algebra $\text{Der} \Lambda$ (respectively, $\text{Der} F_\nu$) of derivations of $\Lambda$ (resp. $F_\nu$). We have derivation $\partial_1, \ldots, \partial_n$ such that $\partial_i(X_j) = q_{ij} X_j$. The span $\langle \partial_1, \ldots, \partial_n \rangle$ is an Abelian Lie algebra of dimension $n$.

Theorem 3. Let $\Lambda$ be a general quantum polynomial algebra. There is a direct decomposition of vector spaces $\text{Der} \Lambda = \text{Der} \Lambda \oplus L$. Similarly $\text{Der} F_\nu = \text{Der} F_\nu \oplus L$. Any finite dimensional Lie subalgebra in $\text{Der} \Lambda$ and in $\text{Der} F_\nu$ is Abelian. Let $k = 0$ and $\partial$ a derivation of $\Lambda$ or of $F_\nu$. Suppose that there exists a nonzero polynomial $f(T) \in k[T]$ such that $f(\partial) = 0$. Then $\partial = 0$.

Generalizing these results we prove
Similar theorem holds for an action of $H$ on $\mathcal{F}_\nu$. A classification of cocommutative Hopf algebras acting on $A_n$, $n \geq 3$ is given in [A4]. All these results can be viewed as a determination of finite and commutative quantum groups acting on quantum space $A^0$.

References


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Nondeterministic Methods in Group Theory
Alexandre Borovik
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I will give a survey of recent results on black box recognition of finite groups and emerging links with the combinatorial group theory.

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The tree lattice existence theorems
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Let $X$ be a locally finite tree. Then $G = \text{Aut}(X)$ is a locally compact group. Bass and Lubotzky asked the following question: When does $G$ contain lattices, that is, discrete subgroups of finite covolume? The answer to the question is difficult and complicated but in this talk we discuss a complete answer to the question, answering several conjectures that were formulated by Bass and Lubotzky. Our general strategy is to show that lattices exist by providing explicit constructions of them. The techniques involve an interesting mix of group theory, both finite and infinite, and graph theory.
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Correspondences and Partial Automorphisms of Regular Semigroups
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We will discuss the problem of characterizing regular semigroups by their partial automorphism monoids and bundles of correspondences.
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The lattice of convex subsemilattices of a semilattice
Peter R. Jones
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Kyeong Hee Cheong (Korea)

Given a (meet-) semilattice $E$, the subsemilattices that are convex with respect to the order form a complete lattice, denoted $\mathcal{CCV}(E)$. While the question of how the lattice properties of $\mathcal{CCV}(E)$ affect $E$ itself is of interest, we shall focus in this talk on the extent to which the lattice determines the semilattice: that is, given a semilattice $E$ and semilattice $F$, for which $\mathcal{CCV}(E)$ and $\mathcal{CCV}(F)$ are isomorphic, how is $F$ related to $E$? Since any three-element semilattices have isomorphic such lattices, the answer is clearly not "up to isomorphism". It is perhaps surprising, then, that we are able to answer this question completely and to construct all such semilattices $F$ from a given $E$.

There is a modest literature on the analogous question for the lattice of convex sublattices of a lattice.
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A constructing of finitely presented semigroups with some exotic properties.
Alexei Kanel-Belov
Moscow Center for Continuous Mathematical Education
Ivanov Ilya (Moscow State University)

This talk is devoted a construction of finitely presented semigroups with some exotic properties. The main result is as follows.
Theorem There exists a constant $M$ such that if $\alpha > M$ is a recursive real number (i.e. there is an algorithm for calculating each digit of its decimal presentation) then there exists a semigroup $G$ such that $GK(G) = \alpha$.

The proof is based on a technique of Minsky machines (which was learned by one author from conversations with Mark Sapir).

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Generic-case complexity of algorithmic problem in group theory
Ilya Kapovich
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Alexei Myasnikov, Paul Schupp, Vladimir Shpilrain

We introduce the notion of “generic-case” complexity for algorithmic problems, which reflects the performance of an algorithm on “most” inputs of a problem. We show that the known results about random walks on regular graphs imply that such traditional group theoretic problems as the word problem, the membership problem and the conjugacy problem, typically have linear “generic-case” complexity (even for the groups where the worst-case complexity is very high).

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The Automorphism Graph of the Free Group of Rank 2
Bilal Khan
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In the study of the automorphism group of a free group, J. H. C. Whitehead introduced a graph whose vertices are the elements of $F$, where two vertices are connected if and only if the corresponding elements of are related by one of a specially chosen set of generators of $Aut(F)$, which have come to be known as the so-called “Whitehead automorphisms”. In this paper we consider Whitehead’s graph, modulo inner automorphisms and the “obvious” automorphisms that are induced by permuting the basis of $F$—we term this combinatorial object the automorphism graph of the free group. We will use the automorphism graph to answer several questions about the action of $Aut(F)$ on $F$, and its relationship to the natural length function on $F$. In particular, we shall characterize the subgraphs of the automorphism graph that are induced by subsets of the form

$A(u) = \{v \in F \text{ s.t. } |v| = |u| \text{ and for some } \phi \in Aut(F) v = u\phi\}$

where $u$ is in $F$.

Here we will consider the case when $F$ has rank 2. We will show that there exist uniform constants $C$, $N$, such that for all in $u$ of length at least $N$, if the subgraph induced by $A(u)$ has more than $C$ vertices then it must be a chain containing at most $|u| - 5$ vertices. This implies that $|A(u)|$ is tightly bounded by $8|u|^2 - 40|u|$, resolving a question posed by A. Myasnikov and V. Shpilrain (see the preprint “Automorphic orbits in free groups” in relation to Open Problem F25 at http://www.grouptheory.org). This structural characterization of the automorphism graph yields an algorithm for testing automorphic conjugacy of elements in $F_2$ that surpasses the classical Whitehead algorithm.

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Algorithmic problems in fully residually free groups
Olga Kharlampovich
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I will talk about different algorithmic problems that are decidable in fully residually free groups, in particular about the conjugacy problem.

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Factorizable Submonoids of the Symmetric Inverse Monoid
Janusz Konieczny*
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Stephen Lipscott (Mary Washington College)

A monoid $M$ is called factorizable if for every element $a$ of $M$, there are an idempotent $e$ in $M$ and a unit $u$ in $M$ such that $a = eu$.

Let $I_n$ be the symmetric inverse monoid of degree $n$, that is, the monoid of all partial one-to-one transformations on the set $X_n = \{1, 2, \ldots, n\}$. The group of units of $I_n$ is the symmetric group $S_n$ of all permutations on $X_n$. For any permutation group $G$ (subgroup of $S_n$), we define the monoid $M(G)$ induced by $G$ by:

$$M(G) = \{a \text{ in } I_n : a \text{ is a restriction of some } g \text{ in } G\}.$$ 

For every permutation group $G$, $M(G)$ is a factorizable inverse monoid. It is the largest (with respect to inclusion) factorizable submonoid of $I_n$ that has $G$ as its group of units.

We study factorizable inverse submonoids of $I_n$ induced by subgroups of $S_n$. Let $G$ be a subgroup of $S_n$ and let $M = M(G)$. We give formulas for the order of $M$ for some classes of groups $G$ and investigate conjugacy classes of $M$ using a generalization of the class equation for finite groups to finite monoids. In particular, we characterize the groups $G$ for which the set of singleton conjugacy classes of $M$ is the union of $Z(G)$ and $\{0\}$, where $Z(G)$ is the center of $G$ and $0$ is the zero transformation.

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Some Varieties of Groups Closed to Being Just Non-Finitely Based
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A variety of groups $V$ is just non-finitely based (or limit) if all proper subvarieties of $V$ are finitely based but $V$ itself is not. It follows easily from Zorn's lemma that if a variety is not finitely based then it contains a just non-finitely based subvariety. In this sense varieties which are just non-finitely based form a "border" between those which are finitely based and those which are not. It is known that infinitely many such varieties exist (Newman [1]) although no explicit examples are known. The problem of the construction of such examples is one of the most important open problems in the theory of varieties of groups.

Three years ago C.K. Gupta and the present speaker [2] constructed a non-finitely based variety of groups which is closed, in a certain sense, to being just non-finitely based. In my talk I am going to give new examples of non-finitely based varieties which are smaller (and so closer to being just non-finitely based) than one constructed in [2]. I will also discuss some conjecture related to the problem above.


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Amalgams and dominions for nilpotent groups of class two

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A preprint may be found at

<http://www.matem.unam.mx/~magidin/preprints/amalgams.ps>

or as e-print math.GR/0105233 in the arXiv.

Let \( \mathcal{C} \) be a class of groups. An amalgam of \( \mathcal{C} \)-groups consists of two groups in \( \mathcal{C} \), \( G \) and \( K \), and a common subgroup \( H \). The amalgam is weakly embeddable in \( \mathcal{C} \) if there exists a group \( M \in \mathcal{C} \), with \( G \) and \( K \) subgroups of \( M \), and such that \( H \subset G \cap K \) in \( M \). The amalgam is strongly embeddable if we may choose \( M \) with \( H = G \cap K \).

Given a class \( \mathcal{C} \), and a group \( G \in \mathcal{C} \), Isbell defines the dominion of a subgroup \( H \) of \( G \) in \( \mathcal{C} \) as the collection of all elements \( a \) of \( G \) such that any two maps from \( G \) to a \( \mathcal{C} \)-group which agree on \( H \) also agree on \( a \).

When the class \( \mathcal{C} \) is a variety, there is a general algebraic connection between amalgams and dominions, provided by the special amalgams, which are the amalgams where \( G \) and \( K \) are isomorphic over \( H \). Namely, one may also characterize the dominion of \( H \) in \( G \) as the smallest subgroup \( D \) of \( G \) containing \( H \) such that the special amalgam of \( G \) and \( G \) over \( D \) is strongly embeddable.

Another related concept is that of amalgamation bases. A group \( H \) is a strong (resp. weak) amalgamation base for \( \mathcal{C} \) if every amalgam of two \( \mathcal{C} \)-groups \( G \) and \( K \) over \( H \) is strongly (resp. weakly) embeddable in \( \mathcal{C} \). A group \( H \) is a special amalgamation base if every special amalgam over \( H \) is strongly embeddable. It follows from general algebraic facts that a group is a strong amalgamation base for a variety \( \mathcal{V} \) if and only if it is both a weak and a special amalgamation base for \( \mathcal{V} \). The connection of amalgams and dominions also identifies the notion of special amalgamation base with that of being absolutely closed: a group \( H \) is absolutely closed in \( \mathcal{C} \) if and only if for every group \( G \) which contains \( H \), the dominion of \( H \) in \( G \) (in \( \mathcal{C} \)) equals \( H \).

In 1982, D. Saracino characterized the weak and strong amalgamation bases for the variety of all nilpotent groups of class two. In 1985, Berthold Meier gave necessary and sufficient conditions for weak embeddability of an amalgam in that variety, and in 1986 gave necessary and sufficient conditions for strong embeddability. In 2000, the author gave an explicit description of dominions, and also characterized the special amalgamation bases.

We present a generalization of these results to any subvariety of the variety of nilpotent groups of class two. These classes are known to correspond to pairs of nonnegative integers, \( (m,n) \), where \( n \) divides \( n/pcd(m,2) \); the pair \( (m,n) \) corresponds to the class of all groups satisfying

\[
x^m = [x,y]^n = [x,y,z] = e.
\]

For example, the characterization of strong embeddability is:

**Theorem.** Let \( G, K, H \) be an amalgam in \( (m,n) \). The amalgam is strongly embeddable in \( (m,n) \) if and only if

1. \( G^m[G,G] \cap H \subset Z(K) \) and \( K^n[K,K] \cap H \subset Z(G) \).

2. For every \( q > 0 \), \( q \mid n \), \( g \in G \), \( g' \in G^n[G,G] \), \( k \in K \), \( k' \in K^n[K,K] \), if \( g^q g', k^q k' \in H \), then

\[
[g^q g', k] = [g, k^q k'] \in H.
\]
By then looking at dominions, we may use the information they give to compare between embeddability of a single amalgam in different classes.

We also give a characterization of the weak, strong, and special bases in each subvariety, and compare the resulting classes. For example, we obtain the following result:

**Theorem.** Let \((m, n) \subset (m', n')\).

1. If \(G \in (m, n)\) is a strong (resp. weak, special) base in \((m', n')\), then it is also a strong (resp. weak, special) base in \((m, n)\).

2. Every strong (resp. weak) base in \((m, n)\) is also a strong (resp. weak) base in \((m', n')\) if and only if for each prime number \(p\), at least one of the following holds:
   
   (a) \(\text{ord}_p(n) = \text{ord}_p(n') = 0\).
   
   (b) \(\text{ord}_p(n) = \text{ord}_p(n)\).
   
   (c) \(\text{ord}_p(n) = \text{ord}_p(n')\) and \(\text{ord}_p(n) = \text{ord}_p(n')\).

3. Every special base in \((m, n)\) is also a special base in \((m', n')\) if and only if for each prime number \(p\), at least one of the following conditions hold:
   
   (a) \(\text{ord}_p(n), \text{ord}_p(n') \leq 1\).
   
   (b) \(\text{ord}_p(n) = \text{ord}_p(n) = \text{ord}_p(n')\).
   
   (c) \(\text{ord}_p(n) = \text{ord}_p(n')\) and \(\text{ord}_p(n) = \text{ord}_p(n')\).

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**Equations in Free Inverse Monoids**

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It is known that the problem of deciding whether a finite system of equations in a free group or a free monoid is decidable, but the corresponding problem for systems of equations in free inverse monoids is undecidable. Any solution to a system of equations in a free inverse monoid induces a solution to the same system of equations in the associated free group in an obvious way, but solutions to equations in free inverse monoids do not necessarily lift to solutions to the same equations in a free inverse monoid. We show, by use of Rabin's tree theorem, that it is decidable whether a solution to a system of equations in a free group lifts to a solution in a free inverse monoid. When combined with some results of Deis, this shows that the consistency problem for single variable equations in free inverse monoids is decidable in most cases.

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**Halfsubgroups**

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In this paper we have investigated some halfsubgroups of a group which generate a group. A halfsubgroup is a half groupoid \(S\) (a partially closed subset) of \(G\) in which associativity holds whenever the corresponding products are in \(S\); identity element is in \(S\) and each element of \(S\) has an inverse in \(S\). Different types of halfsubgroups generating the group (hsgrp) have been defined and certain groups have been studied in terms
of their hsggs. Order class of a group with unity forms a typical halfsubgroup which plays an important role in the further development of some important results. Finally it is proved that an abelian group whose order is product of distinct primes is a direct product of order classes with unity.

On elementary theories of free groups
Alexei Myasnikov*
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Olga Kharlampovich

In this talk I am going to discuss our solution of Tarski problems on elementary theories of free non-abelian groups. The methods involved here show a remarkable interplay between combinatorial group theory, algebraic geometry, and topology. A special attention will be paid to implicit function theorems and the role they play in the model theory of free groups.

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Existence varieties of locally inverse semigroups vs varieties of pseudosemilattices
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An existence variety (or, briefly, e-variety) of regular semigroups is a class of regular semigroups closed for homomorphic images, regular subsemigroups, and direct products. The class LI of all locally inverse semigroups is an example of an e-variety. Locally inverse semigroups can be characterized as regular semigroups where the sandwich set \( S(e, f) = fV(ef)e \) of any two idempotents \( e \) and \( f \) has exactly 1 element. A pseudosemilattice is an idempotent binary algebra \((E(S), \land)\) where \( S \) is a locally inverse semigroup, \( E(S) \) is the set of idempotents of \( S \), and \( f \land e \) is defined as the unique element of \( S(e, f) \). Usually, pseudosemilattices are not semigroups.

To each e-subvariety \( V \) of LI, we can associate the class \( E_V \) of all pseudosemilattices \((E(S), \land)\) such that \( S \) belongs to \( V \). This association defines a complete surjective homomorphism from the lattice of e-subvarieties of LI to the lattice of subvarieties of PS, where PS designates the variety of all pseudosemilattices. We will address some questions about the structure of the lattice of varieties of pseudosemilattices (cardinality, covers,...) which will lead to some results concerning on the lattice of e-varieties of locally inverse semigroups.

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Groups of finite Abelian width
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We say that a group \( G \) has finite Abelian width, if \( G = A_1A_2 \ldots A_k \) with suitable Abelian subgroups \( A_1, A_2, \ldots, A_k \). A remarkable result of M. Abért gives that the full symmetric group on any infinite set has finite Abelian width. We investigate this property for classical groups over local rings. In addition, for finite groups we give bounds on the number of factors needed to represent the group as a product of Abelian subgroups.

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Polyhedral convex cones and the equational theory of the bicyclic semigroup

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With any balanced semigroup identity a number of polyhedral convex cones is associated. This provides us with a tool to formulate an algorithm to verify whether or not the given identity is satisfied in the bicyclic semigroup. This algorithm may even be presented in terms of the computation of a number of determinants. As a consequence, given any semigroup identity, it is decidable whether or not the variety determined by this identity contains simple semigroups which are not completely simple. A variant of this latter result is obtained: given any semigroup identity, it is decidable whether or not the variety determined by this identity contains an idempotent free simple semigroup.

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Some problems in non-classical (universal) algebraic geometry

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1. Some basic notions of the classical algebraic geometry can be considered in arbitrary varieties of algebras A. This leads to universal algebraic geometry. On the other hand, algebraic geometry is considered in special varieties A, for example in varieties of groups, rings, etc. Correspondingly, there exist general problems of geometric nature and particular problems when we specify the variety A.

In the talk we pay attention on the case of the variety Comm = P of associative commutative algebras with unit over the field P, and on the case of the variety of associative algebras Ass = P. Algebraic geometry in Comm = P we define as the classical algebraic geometry.

For every algebra H ∈ A there are its algebraic structure, its logic and its geometry. Interactions of these three components is the key point of the theory in question. This leads to various new problems. For example, which algebras H1 and H2 has the same geometry? Clearly, the corresponding notions from the point of view of algebra and logic are isomorphism of algebras and elementary equivalence of theories.

2. Denote by K_A(H), C_A(H) the categories of algebraic sets and algebraic varieties over H, respectively. C_A(H) is the skeleton of K_A(H). Coincidence of geometries over H1 and H2 means existence of an isomorphism of categories K_A(H1) and K_A(H2) or an isomorphism of categories C_A(H1) and C_A(H2). The second condition means that K_A(H1) and K_A(H2) are equivalent. One can prove that for two non-periodic abelian groups H1 and H2 and A the variety of all groups the following conditions are equivalent 1. K_A(H1) and K_A(H2) are isomorphic, 2. K_A(H1) and K_A(H2) are equivalent, 3. Groups H1 and H2 have the same quasi-identities.

In the same spirit we consider other groups and algebras. The important role plays investigation of automorphisms and autoequivalences of categories of free algebras of varieties of algebras. This question and some others will be illuminated in the talk.

3. One can consider the category K_A of algebraic sets over different H ∈ A. Its skeleton C_A is the category of algebraic varieties over different H ∈ A. We consider conditions providing isomorphism or equivalence of K_A, and K_A. This question is related to the problem of categorical equivalence of varieties A1 and A2 considered by McKenzie.

Note that from geometrical and logical points of view it is more natural to consider varieties of the kind A^G, where G ∈ A is viewed as an algebra of constants.

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Algebraic geometry over groups: Logical Foundation.
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The object of this talk is to lay logical foundation of the algebraic geometry over groups. Exploiting links between the algebraic geometry over groups, model theory, and universal algebra, we solve two problems on geometrical equivalence of groups. These problems are due to B. Plotkin.

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Infinite words and length functions on groups
Denis Serbin
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Alexei Miasnikov (City University of New York), Vladimir Remeslevich (Umsk State University, Russia)

Let X be an alphabet and A be a discreetly ordered abelian group. We consider the set of sequences $w : [1, f_w] \to X^{\geq 1}$ defined on closed intervals $[1, f_w], f_w \geq 0 \in A^+$ and derive some interesting properties of it. In particular, if $Z[t]$ is the ring of integer polynomials $Z[t]$ then elements of Lyndon's free $Z[t]$-group $F(X)^{Z[t]}$ can be represented by such sequences over the the additive group of $Z[t]$. This gives a regular free Lyndon's length function $w \to f_w$ on $F(X)^{Z[t]}$ with values in $Z[t]^+$. This construction is very natural and provides a new method to construct length functions on various groups. Further generalizations also will be discussed.

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On the type of growth of semigroups
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In the paper [1] we gave a method for constructing different types of examples of finitely generated (f.g.) semigroups having intermediate growth. In particular, we found examples of nil-semigroups with the identity $x^2 = 0$ having arbitrarily small intermediate growth and relatively free nilpotent semigroup whose growth is intermediate, but smaller than the growth of Hardy-Ramanujan function. Here we discuss the following questions:

**QUESTION 1.** How large can the intermediate growth of semigroups be?

**QUESTION 2.** How large can the intermediate growth of relatively free semigroups be?

**THEOREM 1.** Let $f(m)$ be a monotone non-decreasing mapping from $N$ into $R^+$ such that $f(m) = o(e^c)$ for any $c > 1$. Then there exists a 2-generated semigroup whose growth is intermediate, but larger than the growth of the function $f$.

Define the sequence $\{ q_n(m) \}$ of the increasing mappings from the tails of $N$ into $R^+$ by the rule:

$q_1(m) = \ln m$, $q_{k+1}(m) = \ln q_k(m)$.

**THEOREM 2.** For any natural number $k$ there exists a relatively free semigroup whose growth is intermediate, larger than the growth of the function $exp(m/q_k(m))$.

**REFERENCES**


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Undecidability in extending partial permutations

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We show that there exists a pseudovariety $U$ of finite (in fact solvable) groups which has a decidable membership problem, but for which the following problem is undecidable: Given a finite set $S$ of partial permutations of a finite set $X$ as input, does there exist a finite set $Y$ containing $X$ and a collection $T$ of permutations of $Y$ extending $S$ such that $T$ generates a group in the pseudovariety $U$.

This implies that the pseudovariety of inverse semigroups $SI * U$ has an undecidable membership problem. It also implies that any pseudovariety of semigroups in the interval $[SI * U, DAmalcevLU]$ has undecidable membership problem. This includes the pseudovariety generated by Schützenberger products of groups in $U$.

We can also show that the pseudovariety of semigroups generated by power semigroups of groups in $U$ has an undecidable membership problem. Also the join of the pseudovariety of aperiodic semigroups with $U$ is undecidable. We can, in fact, find a semigroup variety generated by a single cyclic monoid whose join with $U$ is undecidable.

Joins, semidirect products and Malcev products were first shown to not preserve decidability (for semigroups) by resp. Albert, Baldwin and Rhodes (for joins) and Rhodes for the others. But none of these examples involve groups and the proofs are more involved. Also the join results used non-locally finite factors. The results for the power operator and the Schütz, product operator are completely new.

We note that $U$ satisfies no group identities and in fact $U$ contains solvable groups of arbitrary length (we mention that the undecidability of the identity problem for finite semigroups is used in all the above mentioned results). However, one can find similar decidable pseudovarieties of groups $U_n$ (provided $n > 1$) such that $U_n$ contains groups of solvable length $n$, but not $n + 1$, which lead to analogous undecidability results. The author has shown in the past that the extension problem for partial permutations is decidable for any decidable pseudovariety of Abelian groups.

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Products of commutators in free groups and alternating knots of given genus

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A Wicks form is a canonical form of a product of commutators in a free group $F$. The genus of a Wicks form $w$ is the least positive integer $g$ such that $w$ is a product of $g$ commutators in $F$. Wicks forms of genus $g$ are closely related to the structure of alternating knots of genus $g$: every knot can be interpreted as a special word. This gives, for example, an asymptotic behaviour of the number of alternating knots up to a constant.

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Semigroup varieties and quasivarieties from complexity-theoretical viewpoint
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Several 'obvious' questions in algebra become very hard if one looks at them from the standpoint of feasible computations. Here is an example of such a question: given a finite semigroup S and a (quasi)identity, does S satisfy the (quasi)identity? Kharlampovich and Sapir (in their well known survey 'Algorithmic problems in varieties', Int. J. Algebra Computation 5 (1995) 379-662) asked what is the complexity of this problem if one fixes S and considers the (quasi)identity as the input. Clearly, the problem belongs to the complexity class co-NP. I shall exhibit a 10-element semigroup for which the problem of checking quasidentities is co-NP-complete and a 19-element semigroup for which the problem of checking identities is co-NP-complete. Several results and problems of similar flavor will be presented in order to show how a complexity-theoretical analysis of certain familiar algebraic situations reveals unexpected and fascinating problems.

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Profinite Topologies on Free Products of Groups
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Let $\mathcal{C}$ be an extension closed pseudovariety of finite groups. Let $G$ be a group and $\mathcal{N}_G$ the collection of all normal subgroups $N$ of $G$ such that the quotient $G/N$ belongs to $\mathcal{C}$. Then there is a unique topology on $G$ making it into a topological group such that $\mathcal{N}_G$ is a fundamental system of neighborhoods of the identity element 1 of $G$. This topology is called the pro-$\mathcal{C}$ topology of $G$.

A group $G$ is said to be 2-product separable (with respect to its pro-$\mathcal{C}$ topology) if the product $HK$ of any two finitely generated closed subgroups $H$ and $K$ of $G$ is closed in the pro-$\mathcal{C}$ topology of $G$.

Theorem. Let $\mathcal{C}$ be an extension closed pseudovariety of finite groups. Let $G$ be a free product of 2-product separable groups (with respect to the pro-$\mathcal{C}$ topology). Then $G$ is 2-product separable.

When $\mathcal{C}$ is the pseudovariety of all finite groups the result was proved by T. Coulbois, Free products, profinite topology and finitely generated subgroups, Internat. J. Algebra Comput., 11 (2001) 171-184 for the product of an arbitrary number of subgroups.

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Infinite Dimensional Algebras
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Gelfand-Kirillov dimension (or growth) is an important invariant of an infinite dimensional algebra. We will discuss algebras of small dimensions and their classification theories.

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Profinite Groups
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The talk will focus on free pro-p groups and the problem if these groups are linear.

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Lattice Theory and Ordered Structures

Quasivarieties of idempotent semigroups
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A quasivariety K of algebras of finite type is said to be Q-universal if, for any quasivariety M of finite type, L(M) is a homomorphic image of a sublattice of L(K), where L(M) and L(K) are the lattices of quasivarieties contained in M and K, respectively.

It is shown that the varieties LSN of all left semi-normal idempotent semigroups and RSN all right semi-normal idempotent semigroups are Q-universal. It follows that a free lattice on ω generators may be embedded in each of L(LSN) and L(RSN). In particular, L(LSN) and L(RSN) each fail to satisfy any non-trivial lattice identity.

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Lattices of algebraic subsets
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Lattices of algebraic subsets of complete lattices arise naturally in the most general setting of continuous closure operators.

In the theory of quasivarieties they provide the model for the lattices of subquasivarieties, or Q-lattices for short. The problem of characterization of Q-lattices was raised independently by G. Birkhoff and A.I. Mal’cev. We give an overview of major results describing Q-lattices and further perspectives in that direction.

The lattices of algebraic subsets also provide a key example of a convex geometry, the notion that extends the combinatorial notion of (finite) convex geometry. In general lattice theory it places these lattices into the juxtaposition to the lattices of equivalence relations. If the latter play significant role in classification of lattices according to modular law, lattices of algebraic subsets do that with respect to join-semidistributive law. Both laws are important and somewhat “opposite” generalizations of distributive law: they reflect the nature of closure operators with exchange and anti-exchange property, correspondingly.

In this talk we will mention recent development in the lattice theory of convex geometries and also point to some open problems.

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Representable Integral Residuated Lattices
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A residuated lattice is a lattice-ordered monoid that is 'residuated' in the sense that it has binary operations / and \ which satisfy

\[ x \cdot z \leq y \iff z \leq x\backslash y \iff x \leq y/z. \]
An integral residuated lattice also satisfies \( x \leq 1 \). The class of integral residuated lattices is a variety which we denote by \( \mathcal{IR} \). These algebras originate in the study of ideal lattices of rings and as algebraic models of linear logic and Lambek's calculus with weakening.

Our objective is the axiomatization of the class of all members of \( \mathcal{IR} \) that may be represented as subalgebras of products of linearly ordered members of \( \mathcal{IR} \). Such integral residuated lattices are called representable; we shall prove that the class of all such algebras is axiomatized, relative to \( \mathcal{IR} \), by the identity

\[
(x/y) \lor (w/(w/((y/x)\lor x))) = 1.
\]

The identity uses only the operations /, \( \lor \), \( \lor \) and 1 so we shall also obtain axiomatizations of the class of representable algebras in subreducts classes of \( \mathcal{IR} \) whose languages contain /, \( \lor \), \( \lor \) and 1.

The congruence lattice of \( A \in \mathcal{IR} \) is isomorphic to the lattice of 'filters' of \( A \) (ie convex submonoids \( F \) of \( A \) satisfying \( [aF] = [Fa] \)). The method of proof utilizes the larger lattice of 'prefilters' of \( A \) (ie convex submonoids of \( A \)). In commutative examples, prefilters and filters coincide so the proof presented is an extension of the known result for commutative integral residuated lattices, where the identity \( (x/y) \lor (y/x) = 1 \) axiomatizes the representable subclass. This result is also known in the context of BCK-algebras, which are the \( (\land, \lor) \)-subreducts of commutative integral residuated lattices.

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Relation algebras and multigroupoids

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By a (semi-associative) multigroupoid we mean a relational structure the complex algebra of which is a (semi-associative) relation algebra. The complex algebra of a (semi-associative) multigroupoid \( \mathcal{U} \) will be denoted by \( \text{Cm}(\mathcal{U}) \).

Given a finite (semi-associative) multigroupoid \( \mathcal{U} = (U,C,r,I) \) and a (semi-associative) relation algebra \( \mathcal{R} \) with universe \( R \), the set \( U[R] \) of functions from \( U \) to \( R \) is the universe of a (semi-associative) relation algebra \( M_U(\mathcal{R}) \), called the \( \mathcal{U} \)-(semi-associative) relation algebra over \( \mathcal{R} \). The composition of two functions \( f \) and \( g \) is defined at \( w \in U \) as follows:

\[
(f;g)(w) := \sum \{ f(u); g(v) : u, v \in U \text{ and } (u, v, w) \in C \}.
\]

The composition is well-defined because \( U \) is finite. See [2]. Note that this definition is a generalization of matrix relation algebras, where \( \mathcal{U} \) is the standard multigroupoid on \( n^2 \) pairs \( (i,j) \) \( (i,j = 1, \ldots, n) \), refer to[1]. We use \( \text{deg}(\mathcal{R}) \) to denote the degree of \( \mathcal{R} \) in the sense of Maddux, see [3].

**Theorem 1.** If \( \mathcal{U} \) is a finite semi-associative multigroupoid and \( \mathcal{R} \) is a semi-associative relation algebra then

\[
\text{deg}(M_U(\mathcal{R})) = \min\{\text{deg}(\text{Cm}(\mathcal{U})), \text{deg}(\mathcal{R})\}.
\]

Our definition of algebras of type \( M_U(\mathcal{R}) \) works if \( \mathcal{U} \) is finite and \( \mathcal{R} \) is arbitrary. It can also be adopted with minor changes if \( \mathcal{U} \) is an arbitrary (semi-associative) multigroupoid and if \( \mathcal{R} \) is a complete (semi-associative) relation algebra. We denote the obtained (semi-associative) relation algebra by \( U(\mathcal{R}) \). The latter construction is clearly not a generalization of the former one. For this reason, different notations are used. One important result is:

**Theorem 2.** If \( \mathcal{U} \) and \( \mathcal{V} \) are semi-associative multigroupoids then

\[
\text{U}(\text{Cm}(\mathcal{V})) \cong \text{Cm}(\mathcal{U} \times \mathcal{V}) \cong \mathcal{V}(\text{Cm}(\mathcal{U})).
\]

We will present more properties of these constructions.

Assume now that both \( \mathcal{U} \) and \( \mathcal{R} \) are arbitrary. Then both the completion \( \mathcal{R}^* \) and the perfect extension \( \mathcal{R}^+ \) of \( \mathcal{R} \) are complete. We now can define two distinct (semi-associative) relation algebras:
• The subalgebra $U^+(\mathfrak{A})$ of $U(\mathfrak{A}^*)$ generated by the set $UR$.

• The subalgebra $U^+(\mathfrak{R})$ of $U(\mathfrak{R}^*)$ generated by the set $UR$.

In each situation, we could define an appropriate subalgebra generated by $UR$.

Problems 1. (P1) Given a finite cardinal $n$, we showed in [1] that the class of $n$-matrix relation algebras is an elementary class. Is the class of $U$-relation algebras for some fixed multigroupoid $U$ an elementary class?

(P2) Given a semi-associative multigroupoid $U$ and a semi-associative relation algebra, what are the degrees of $U^+(\mathfrak{A})$ and $U^+(\mathfrak{R})$?

(P3) Given two (semi-associative) relation algebras $\mathfrak{A}$ and $\mathfrak{R}$, is there a realistic kind of product that generalizes all these constructions?

References


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Forbidden Forests in Priestley Spaces
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We present a first order formula characterizing the distributive lattices $L$ the Priestley spaces $\mathcal{P}(L)$ of which contain no copy of a finite forest $T$. For Heyting algebras $L$, prohibiting a finite poset $T$ in $\mathcal{P}(L)$ is characterized by equations iff $T$ is a tree. We also give a condition characterizing the distributive lattices whose Priestley spaces contain no copy of a finite forest with a single additional point at the bottom.

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Residuated Ordered Structures in Logic and Algebra
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One of the central concepts of logic, that of implication, has been formalized in numerous ways with the aim of capturing certain intended interpretations; to mention a few, the implication of classical, intuitionistic and relevant logic, and, more recently, linear logic. An algebraic study of these systems leads to the consideration of ordered groupoids $(G, \cdot, \leq)$ in which the multiplication operation is left and right residuated. The multiplication corresponds to a notion of conjunction, the residuation operations represent the implications.

Many special classes of ordered residuated groupoids have been studied, and not only in algebraic work inspired by logic. The papers by Ward and Dilworth from the late thirties on residuated lattices were not motivated by logical considerations, but by interest in the structure of the ordered set of ideals of a ring, as a lattice with the operations of ideal multiplication and ideal quotients added.

In the talk, I will discuss and compare some important classes of ordered residuated groupoids, stemming both from logic and algebra, and highlight recent progress made by various authors. I will then focus on the
question whether—and how—some of the classes are generated by their finite members. A positive answer to that question will establish decidability of the equational or even universal theory of the class, and be of particular interest if it concerned a class associated with a logical system one is curious about.

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The role of bounded homomorphisms in lattice theory
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McKenzie's concept of a bounded homomorphism has played a crucial role in lattice theory. We survey the results in lattice theory that use this idea, from McKenzie's landmark paper up to recent work. Topics will include free lattices, varieties and finitely presented lattices.

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Continuum many idempotent minimal residuated-lattice varieties
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A residuated lattice is an algebra $L = \langle L, \land, \lor, e, /, \backslash \rangle$, such that $\langle L, \land, \lor \rangle$ is lattice, $\langle L, \lor, e \rangle$ is monoid and for all $a, b, c$ in $L$, $a \land b \leq c$ if $a \leq c/b$ if $b \leq a \backslash c$. Residuated lattices form a variety, which is denoted by $\mathcal{RL}$. We investigate the bottom of the subvariety lattice of $\mathcal{RL}$ and prove that there are continuum many atoms that satisfy the identity $x^2 = x$.

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Inequivalent representations of geometric relation algebras
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In a 1961 article, Roger Lyndon — building on earlier work of Bjarni Jónsson — showed how to construct a relation algebra $B(P)$ from any projective geometry $P$. When $P$ is Desarguesian (for instance, when $P$ has dimension at least 3), Lyndon showed that $B(P)$ can be represented as an algebra of binary relations; in fact, he showed that there is a unique complete representation, up to equivalence — i.e., up to renaming the elements of the base set. (A representation is “complete” if it preserves all infinite joins as unions. For finite geometries $P$, every representation of $B(P)$ is necessarily complete.) When $P$ is a non-Desarguesian plane, an argument due to Jónsson shows that $P$ cannot be represented as — i.e., is not isomorphic to — a (set-theoretically defined) algebra of binary relations. For the case of a projective line $P$ of order $n$, Lyndon showed that $B(P)$ has a complete representation if and only if there is a projective plane of order $n$. However, he did not investigate the number of inequivalent complete representations that $B(P)$ can have in this case.

In this talk, we present a general formula for computing the exact number of inequivalent complete representations of the algebras $B(P)$. We use it to compute the number of inequivalent representations of the algebras for projective lines of orders $n < 11$ (the only orders for which the requisite information about
projective planes of order n is currently known). For example, there are 120 inequivalent representations of \( B(P) \) when \( P \) has order 7, there are 240 when \( P \) has order 8, and there are 56, 760 when \( P \) has order 9. The formula can also be used to give lower bounds on the number of inequivalent representations when the requisite information about projective planes of order n is not known. For instance, there are at least 1.08888869 times \( 10^{28} \) inequivalent representations of \( B(P) \) when \( P \) has order 29.

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Embedding finitely generated Abelian lattice-ordered groups: Higman’s Theorem and a realisation of \( \pi \)
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Graham Higman proved that a finitely generated group can be embedded in a finitely presented group iff it has a recursively enumerable set of defining relations. We consider the analogue for lattice-ordered groups. Clearly, the finitely generated lattice-ordered groups that can be \( \ell \)-embedded in finitely presented lattice-ordered groups must have recursively enumerable sets of defining relations. We prove the converse direction for a special class of lattice-ordered groups:

Theorem. Every finitely generated Abelian lattice-ordered group that has finite rank and a recursively enumerable set of defining relations can be \( \ell \)-embedded in a finitely presented lattice-ordered group.

As a consequence we obtain that \( D(\pi) \) the Abelian rank 2 group \( \mathbb{Z}^2 \) with order \((m, n) > 0 \) iff \( m + n \pi > 0 \) can be \( \ell \)-embedded in a finitely presented lattice-ordered group, whence \( \pi \) is \"\( \ell \)-algebraic\" in that it can be captured by finitely many relations in this language. Indeed,

Corollary. The recursive reals are precisely those real numbers \( \xi \) for which \( D(\xi) \) can be \( \ell \)-embedded in a finitely presented lattice-ordered group.

The technique is an amalgamation of three disparate areas: (1) continued fractions, (2) recent advances in direct limits of Abelian lattice-ordered groups, and (3) using permutation groups to encode the necessary information (a technique whose origins can be found in work of Ralph McKenzie and Richard Thompson).

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Finite lattices and congruences: The good, the bad, and the ugly
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Many of our earliest papers dealt with congruences and ideals of lattices. There were two major themes.

1. Lattice congruences are not as nice as congruences of groups and rings: There is no single class, in general, determining a congruence. We were interested when is there such a class and which types of ideals occur naturally in these discussions. This lead, in particular, to the introduction of the concept of standard ideals.

2. In the early forties, R. P. Dilworth proved his famous result: Every finite distributive lattice \( D \) can be represented as the congruence lattice of a finite lattice \( L \). In one of our early papers, we presented the first published proof of this result in the stronger form: We proved that the finite lattice \( L \) constructed was sectionally complemented.

We have been publishing papers on these two topics for 45 years. In this lecture, we are going to review some of our results. Many of them deal with the second topic, making \( L \) “nice”.

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If being “nice” is an algebraic property such as being semimodular or sectionally complemented, then we
have tried in many instances to prove a much stronger form of these results by verifying that every finite
lattice has a congruence-preserving extension that is “nice”. We shall discuss some of the techniques we use
to construct congruence-preserving extensions.

We shall pay special attention to finite, sectionally complemented lattices. It is really odd that many of
these constructions go through “freely generated” join-semilattices.

We shall conclude with some recent results on the spectrum of a congruence of a finite, sectionally
complemented lattice, measuring the sizes of the congruence classes. With very few restrictions, this can be
as bad as we wish.

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Equational theory of regular rings with involution and complemented modular polarity
lattices
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Florence Micol, Michael Roddy

Universal algebraic points of view have contributed to the understanding of the relationship between
*-regular rings, modular ortholattices, and spaces with an anisotropic form. Yet, the main question re-
mained unsolved in these equivalent versions: Is the variety of modular ortholattices generated by its finite
dimensional members? Is every subdirectly irredicible *-regular ring a *-ring of operators? Is the free *-
regular ring residually arithmetical? In the more general framework of regular rings with involution, orthogonal
geometries, and polarity lattices these questions become more easily accessible — and contribute to the
understanding of forms in general.

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Boolean products and canonical extensions of bounded distributive lattice expansions
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The talk will be based on a joint paper with Mai Gehrke, *Bounded distributive lattice expansions*, that is
to appear in Mathematica Scandinavica. The paper, still under revision, is available on the authors’ home
pages. In this abstract, the construction of canonical extensions will be described in some detail, but little
space will be given to their study. In the talk itself, this will be reversed; the construction will be dealt with
very sketchily, but certain techniques used in investigating canonical extensions will be the principal topic.

A bounded distributive lattice expansion \( A = (A_0, \omega^A, \omega \in O) \) is a bounded distributive lattice \( A_0 =
(A, \vee, \wedge, 0, 1) \) with auxiliary operations \( \omega^A \). Canonical extensions of bounded distributive lattice expansions
have been defined and investigated under some conditions on the auxiliary operations, but here these oper-
ations will be completely arbitrary.

It is convenient to have names for the several categories involved in the definition:

- DL: The category of all bounded distributive lattices and their homomorphisms.
- PS: The category of all Priestley spaces, with continuous isotone maps as the morphisms
- PO: The category of all partially ordered sets (posets), with isotone maps as the morphisms.
• DL⁺: The category of all doubly algebraic, bounded distributive lattices, with complete homomorphisms as the morphisms.

• DLEμ: The category of all bounded distributive lattice expansions of similarity type μ, and their homomorphisms. (When the similarity type is clear from the context, or is irrelevant, the subscript may be omitted.)

The name of a category will also be used as a collective name for its objects. E.g., a bounded distributive lattice will be referred to as a DL.

As in the earlier work, canonical extension will be defined in three stages. For a DL A, the canonical extension A⁺ is defined, as before, using Priestley duality between DL and PS and the duality between DL⁺ and PO. The Priestley dual of a DL A is a PS A↓ = (A↓, ≤↓), whose dual (A↓) deser, also referred to as the second dual of A↓, is a DL that is isomorphic to A↓. On the other hand, if F is the functor that forgets the topology, then F(A↓) = (A↓, ≤↓) is a poset whose dual F(A↓)* is a DL⁺ containing (A↓) deser as a sublattice. Combined with the natural isomorphism from A↓ to its second dual, this yields an injective homomorphism from A↓ into the DL⁺ (A↓)*. Since we prefer to work with extensions rather than injective homomorphisms, we define the canonical extension A⁺ of a DL A up to equivalence to be a DL⁺ containing A as a sublattice, such that the natural isomorphism A ≤ (A↓) deser extends to an isomorphism A⁺ ≅ (A↓)*. Denoting by F∞(A⁺) and by M∞(A⁺) the sets consisting, respectively, of all the strictly join irreducible elementary of A⁺ and of all the strictly meet irreducible elements of A⁺, we can characterize A⁺ abstractly by the condition that it is a DL⁺ containing A as a sublattice, with the following two properties:

(Sep) For all p ∈ F∞(A⁺) and u ∈ M∞(A⁺), if p ≤ u, then there exists a ∈ A with p ≤ a ≤ u.

(Comp) For all X, Y ⊆ A, if X ◦ Y, then there exist finite sets F ⊆ X and C ⊆ Y with X ◦ Y.

An element x ∈ A⁺ is said to be open (x ∈ O(A⁺)) if x = X for some X ⊆ A, and x is said to be closed (x ∈ K(A⁺)) if x = X for some X ⊆ A. This terminology is suggested by the isomorphism between A⁺ and (A↓)*. The elements of (A↓)* can be thought of as order filters in the Priestley space A↓, and the open and the closed elements of A⁺ are precisely the elements that correspond to open and to closed order filters, respective.

We next define the canonical extension f⁺ : A⁺ → B⁺ of a map f : A → B between DL’s. This is where the present approach differs fundamentally from earlier work, allowing us to consider completely arbitrary maps. We want to define f⁺(x) and its dual f⁺(x) for x ∈ A⁺ in terms of the values of f at “nearby” points of A. To make this idea precise, we need a topology on A⁺. We define σ or σ(A⁺) to be the topology having as a basis the intervals [p, u] with p closed and u open. Using this topology, we define

f⁺(x) = \bigvee (f(A ∩ U) : U ∈ σ),
f⁺(x) = \bigwedge (f(A ∩ U) : U ∈ σ).

Observe that every member a of A is an isolated point of A⁺ because {a} = [a, a] is open. Therefore f⁺(a) = f(a) = f⁺(a), i.e., f⁺ and f⁺ are in fact extensions of f. Also, it follows from (comp) that every non-empty basic interval [p, u] contains a member of A, so A is dense in A⁺. Hence, f⁺(x) and f⁺(x) can be thought of as the limit inferior and the limit superior of the values of f at nearby points in A.

Finally we define the canonical extension of a DLE A = (A₀, ω⁺, ω ∈ O) to be the DLE A⁺ = (A₀⁺, (ω⁺)⁺, ω ∈ O). I.e., we set ω⁺⁺⁺ = (ω⁺)⁺. The dual canonical extension A⁺⁺⁺ = (A₀⁺⁺⁺, ω⁺⁺⁺, ω ∈ O) is defined similarly. (For DL’s, the notion of a canonical extension is self-dual.) There is an apparent problem with these definitions. An n-ary operation ωⁿ is a map from A⁺ into A, and the domain of (ωⁿ)⁺⁺⁺ is therefore A⁺⁺⁺, not A⁺⁺⁺. However, using the obvious isomorphism between (A₀⁺⁺⁺)⁺⁺⁺⁺⁺ and (A⁺⁺⁺)⁺⁺⁺, we can harmlessly identify the two sets.

This completes the description of the canonical extension of a DLE. What has been accomplished is that an arbitrary DLE A has been embedded in a DLE A⁺⁺⁺ of a special kind. Most notably, the DL reduct A₀⁺⁺⁺ of A⁺⁺⁺ has the very strong property of being doubly algebraic. Also, the way in which A sits within A⁺⁺⁺
gives rise to three topologies \( \sigma, \sigma^* \) and \( \sigma^+ \) on the set \( A^\sigma \). These are defined by taking as bases intervals of the forms \([p, u)\), \([p, 1]\) and \([0, u]\), respectively, with \( p \in K(A^\sigma) \) and \( u \in O(A^\sigma) \). Three other topologies, \( \iota, \iota^* \) and \( \iota^+ \) are defined similarly, except that now \( p \) and \( u \) are required to be, respectively, compact elements and dually compact elements of the doubly algebraic lattice \( A^\sigma \). The last three topologies are therefore intrinsic to the lattice \( A^\sigma \). The topology \( \sigma \) has already been used in the definition of canonical extensions of maps and of operations. All six topologies are used in showing that many properties are preserved by canonical extensions.

This is the good news. The bad news is that some important properties are not always preserved. A spectacular example is the fact that, given a DLE homomorphism \( h : A \to B \), the canonical extension \( h^\sigma : A^\sigma \to B^\sigma \), is not always a homomorphism. Therefore, for the categories DLE\(\Sigma\), the maps \( A \to A^\sigma \) and \( h \mapsto h^\sigma \) for objects \( A \) and morphisms \( h \) do not constitute a functor. It is therefore important to show that for large classes of DLE’s homomorphisms are preserved. Every homomorphism can of course be factored into a surjective homomorphism and an injective homomorphism. By showing that surjective homomorphisms are always preserved, the problem is reduced to the special case of injective homomorphisms, and this makes the preservation a property of the target algebra. An algebra \( B \) is said to have the property (PH) if every injective homomorphism into \( B \) is preserved, and a class of algebras is said to have (PH) if all its members have (PH). The problem then becomes: Which DLE’s have (PH)? This is where Boolean products play a major role, and in the talk it will be explained how this comes about. Here we merely state one major result and two of its consequences.

**Theorem.** For every class \( K \) of DLE\(\Sigma\)'s, \( \text{Var}(K) \) has (PH) iff \( \text{Pu}(K) \) has (PH).

**Corollary.** Every finitely generated variety of DLE's has (PH).

**Corollary.** If two varieties of DLE\(\Sigma\)'s have (PH), then so does their lattice join.

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**Computable Isomorphisms of Boolean Algebras with Operators**

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In computable algebra and model theory computable isomorphism types of structures have been studied intensively over almost three decades. These include a number of natural classes of structures, such as Boolean algebras, Abelian groups, and lattices. The Handbook of Recursive Mathematics is a good source of results in the area. Here we present two results about computable isomorphisms of Boolean algebras with operators (BAOs).

A computable BAO \( A \) is one whose domain is a computable subset of \( \mathbb{N} \), and whose Boolean operations and the operators are computable functions. If a BAO \( B \) is isomorphic to a computable BAO \( A \) then \( B \) is called computably presentable and \( A \) is a computable copy of \( B \). A computable function that sets up an isomorphism between computable BAOs \( A \) and \( B \) is a computable isomorphism, in which case \( A \) and \( B \) have the same computable isomorphism types.

A central definition in the study of computable isomorphism is one of computable dimension. The computable dimension of a BAO \( A \), denoted by \( \text{dim}(A) \), is the number of its computable isomorphism types. If \( \text{dim}(A) = 1 \) then \( A \) is called computably categorical.

In mid 70s Goncharov and independently Remmel showed that any computable Boolean algebra \( B \) is either computably categorical or has infinite computable dimension. Moreover, \( \text{dim}(B) = 1 \) iff \( B \) has finitely many atoms. The following two theorems describe computable dimensions of BAOs and show that the situation for computable BAO is very different in comparison to computable Boolean algebras without operators.
Theorem 1. For every natural number $n$ there exists a BAO whose computable dimension is $n$.

The next theorem shows that extending BAO with a finite number of constants affects the computable isomorphism types of the original BAO.

Theorem 2. For any natural number $n$ there exists a computably categorical BAO $B$ such that the expansion of $B$ by any constant atom $c$ has computable dimension $n$.

Theorem 1 codes the family of computably enumerable sets constructed in [1]. Theorem 2 uses the family of pairs of computably enumerable sets constructed in [2].

References.


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Commutative bounded integral residuated lattices
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The interest in residuated lattices, i.e., algebras that combine a lattice and a residuated monoid has been recently revived. To this revival we would like to contribute a few remarks about commutative bounded integral residuated lattices, which by we understand residuated lattices that are bounded as lattices, and: (1) the bottom and top elements are constants in the type; (2) the unit of the monoid is the top of the lattice; (3) the monoid is commutative.

Varieties of these correspond in a natural way to a class of substructural logics, known by (some) logicians as "extensions of $FL_{cu}$". Thus, for want of a better name, we will call our structures $FL_{cu}$-algebras. We will present some facts about the lattice of subvarieties of $FL_{cu}$-algebras.

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Finite-dimensional algebras that do not admit a lattice order
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In 1956, G. Birkhoff and R.S. Pierce showed that the field of complex numbers cannot be lattice-ordered as a lattice-ordered algebra over the totally ordered field of real numbers; and the field of rational complex numbers cannot be made into a lattice-ordered ring. Later in 1968, R.A. McHaffey showed that the division algebra of real quaternions cannot be made into a real lattice-ordered algebra. Recently, P. Wojciechowski and the author have obtained some results concerning lattice orders on matrix algebras. Motivated by their work, in this talk we provide some conditions to ensure that a subalgebra of a matrix algebra over a subfield of the totally ordered field of real numbers cannot be lattice-ordered as a lattice-ordered algebra. In particular, we show that the matrix algebra over the field of complex numbers and the matrix algebra over the division algebra of real quaternions cannot be lattice-ordered as a real lattice-ordered algebra. Therefore, a finite-dimensional simple real lattice-ordered algebra is isomorphic to a lattice-ordered matrix algebra over the totally ordered field of real numbers, and hence is completely determined. For instance, if the identity element of a finite-dimensional simple real lattice-ordered algebra is positive, then it is isomorphic to the lattice-ordered matrix algebra over the totally ordered field of real numbers with the positive cone consisting of all matrices in which each entry is nonnegative.
Nonrepresentable relation algebras generated by functional elements
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There are nonrepresentable relation algebras generated by their functional elements. This solves a problem posed many years ago. The number of generating functional elements can be as low as 2. An open problem is to determine whether there is a nonrepresentable relation algebra generated by a single functional element.

Cardinal functions on Boolean algebras
Don Monk
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After a survey of results and problems concerning infinite cardinal-valued functions on Boolean algebras, I will concentrate on two particular cardinals, the boundedness and dominating numbers, which are mainly of set-theoretic interest.

MV-algebras: a variety for magnitudes with archimedean elements
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Ralph McKenzie dealt with the question when two classes of algebras are equivalent as categories. He considered not only varieties and quasivarieties, but also more general classes. MV-algebras and lattice ordered abelian groups with strong unit yield an interesting example of a variety being categorically equivalent to a non-elementary class of algebras. A strong unit u in a lattice ordered abelian group G is an archimedean element, in the sense that elements of the form u, 2u, 3u, ... are cofinal in the positive cone of G. An MV-algebra is an abelian semigroup with neutral element 1 and an operation * satisfying the equations \( x^{**} = x, x1^* = 1^* \) and \( (yx)^*y = (xy)^*x \). MV-algebras were invented by C.C. Chang to prove the completeness theorem for the infinite-valued propositional calculus of Łukasiewicz. We shall discuss this categorical equivalence and its neighbours, linking (via Grothendieck's \( K_0 \) functor) MV-algebras to the AF C*-algebras of infinite spin systems.

REFERENCES
A Sahlqvist Theorem for Distributive Modal Logics
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We present a generalization of Sahlqvist theorem within the framework of a generalized modal logic, which we call distributive modal logic. The algebraic counterpart of this logic is bounded distributive lattices with unary operators. The operators that preserve joins or meets of the lattice are generalized modal operators. The operators that flip joins to meets or vice versa are generalized negations that are weaker than the negation of Boolean algebras. We prove that canonicity holds in a large class of distributive modal logics using the theory of canonical extension of lattices that was recently developed. We also give a proper Kripke-style semantics for these logics. In this setting Sahlqvist correspondence theorem readily follows from the classical correspondence theorem. Putting these results together, we get a generalized Sahlqvist completeness theorem. Along with the classical modal logic, many distributive lattice based logics such as positive modal logic can be captured in this framework.

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Sublattices of lattices
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This is a continuation of Freese’s talk. We will emphasize sublattice problems of various types.

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On characterized varieties and quasi-varieties of lattices
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The classical results in Lattice Theory by Dedekind [1] and Birkhoff [2] - a lattice is modular (distributive) if and only if it does not contain the pentagon $N_5$ (resp. $N_3$ and the 3-diamond $M_3$) as a sublattice - have been generalized by McKenzie [3] to notion of splitting algebras. We will consider varieties and quasi-varieties defined by finite algebras not embeddable into algebras from those classes. It’s easy to see that modular and distributive varieties are such varieties.

Let $R, K$ are classes of algebras and $R \subseteq K$. A class of algebras $R$ is $K$-characterized if there exist nonempty set $M$ of finite algebras belonging to $K$ such that $R$ coincides with all algebras from $K$ that do not contain $R$-algebras from $M$ as subalgebras. If set $M$ does not exist then we say that $R$ is non $K$-characterized. If the set $M$ is empty then algebra from the set $M$ are called $R$-forbidden $K$-algebra. We set $M$ is finite then $R$ is called finitely $K$-characterized. If $K$ is well-known class of algebras then we say that $R$ is characterized (non-characterized, finitely characterized).

Further we will consider lattices classes. But many definitions and results are true for any classes of algebras.

By R. Dedekind [1] (G. Birkhoff[2]) characterization we have that variety of modular lattices (variety of distributive lattices) is finitely characterized. Examples of non-characterized varieties are variety generated
by all finite modular lattices and variety generated by all finite Desargues lattices. Nation's[4] counterexample to Jönsson's finite height conjecture give us possibility to construct non-characterized variety generated by finite lattice.

In [5] was constructed continuum non characterized varieties of lattices. We prove that there is continuum non characterized varieties of modular lattices and there is continuum non characterized locally finite varieties of lattices.

Other results are concerned to finite characterized quasivarieties, covers in lattice (quasi)varieties and $R$-forbidden lattices.


Duality theory for lattice-ordered algebras: the present, the past, and the future

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From the perspective of an introverted algebraist, dualities for varieties and quasivarieties provide a valuable tool for solving algebraic problems. But this perspective does not give due weight to the interdisciplinary nature of duality and the ways in which duality theory's connections with logic and computer science have enriched it and contributed to its development. This talk will give a survey of points of contact between various approaches to the representation of lattice-ordered algebras and to algebraic and relational semantics for non-classical logics. In particular, bounded lattice extensions will be looked at from a duality angle. [Recent major advances made by Mai Gehrke and Bjarni Jónsson in the theory of canonical extensions will be described in Bjarni Jónsson's talk; details can be found in his abstract.]

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Lattice-ordered fields of quotients

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The group ring of a totally ordered integral domain over a torsion-free commutative group is a lattice-ordered integral domain with respect to the coefficientwise order. If the field of quotients of this integral domain has a compatible lattice-order extending the coefficientwise order, then the group ring is convex in the field of quotients and has trivial polar. This restricts the possible orders on the field of quotients.

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On a representation of posets by subsets
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For an arbitrary poset $P$ and its nonempty subset $Q$, consider the following collection of subsets of $P$: $P(Q) := \{ \uparrow p \cap Q \mid p \in P \}$.

Such collections often appear in the framework of various set representations of poset. Usually, $Q$ is the collection of particular (e.g., irreducible) elements of $P$, and instead of the principal filter, the corresponding ideal may appear. If this collection is ordered by inclusion, then $f : p \mapsto \uparrow p \cap Q$ is an anti-isotone map from $P$ onto $P(Q)$. This map may be a dual order isomorphism depending on properties of the subset $Q$ and of general properties of $P$.

We give conditions under which $f$ is a dual isomorphism, in some particular, and also in general case.

We also present some applications of the foregoing concept in representation of posets and in lattice valued (fuzzy) structures.

Let $S$ be a nonempty set and $(P, \leq)$ a poset. Consider a map $f : S \to P$, and for every $p \in P$ the set $f_p \subseteq S$ defined by

$x \in f_p$ if and only if $f(x) \geq p$.

The collection $F_P := \{ f_p \mid p \in P \}$ can be ordered by inclusion and the following hold.

(T1) For every $x \in S$ $f(x) = \bigvee \{ p \in P \mid x \in f_p(x) \}$,

i.e., the least upper bound of the set $\{ p \in P \mid x \in f_p(x) \}$ exists and is equal to $f(x)$.

(T2) If $p, q \in P$ and $p \leq q$, then $f_q \subseteq f_p$.

(T3) (i) $\bigcup \{ f_p \mid p \in P \} = S$;

(ii) for every $x \in S$, $\bigcap \{ f_p \mid x \in f_p \} \in F_P$, i.e., the collection $F_P$ is a point closure system under inclusion.

Let $\approx$ be an equivalence relation on $P$, defined by: for $p, q \in P$ $p \approx q$ if and only if $f_p = f_q$.

For $p \in P$, let $[p]_{\approx} := \{ q \in P \mid p \approx q \}$.

(T4) $p \approx q$ if and only if $\uparrow p \cap f(S) = \uparrow q \cap f(S)$.

On the collection of $P/\approx = \{ [p]_{\approx} \mid p \in P \}$ of $\approx$-classes, define the relation $\leq$ as follows:

$[p]_{\approx} \leq [q]_{\approx}$ if and only if $\uparrow q \cap f(S) \subseteq \uparrow p \cap f(S)$.

(T5) (i) If for $x \in S$, $f(x) = p$, then $p$ is the greatest element of the $\approx$-class to which it belongs.

(ii) The relation $\leq$, defined by (1) is an ordering on $P/\approx$ and also $[p]_{\approx} \leq [q]_{\approx}$ if and only if $f_q \subseteq f_p$.

(iii) The poset $(P/\approx, \leq)$ is dually isomorphic with the poset $(F_P, \subseteq)$.

The converse of the above constructions also holds, i.e., point closure system on a set can be represented by a single function on its subset.

(T6) Let $f : S \to L$ be a map from a nonempty set $S$ to a lattice $L$. Then:

(a) The collection $F_L := \{ f_p \mid p \in L \}$ is closed under intersections and a lattice under set inclusion;

(b) for every $x \in S$, $f(x) = \bigvee \{ f(x) \}$;

(c) for any $p, q \in L$, $[p]_{\approx} \leq [q]_{\approx}$ if and only if $\bigvee [p]_{\approx} \leq \bigvee [q]_{\approx}$

(the order on the right is the one from $L$).

The above representation of posets by functions can be generalized in the following way.

Theorem Let $(P, \leq)$ be a poset and $Q \subseteq P$. Then $(P, \leq)$ is dually isomorphic to the collection $\{ \uparrow p \cap Q \mid p \in P \}$ ordered by inclusion if and only if $Q$ is inf-dense in $P$.

By using characteristic functions of subsets arising in the foregoing constructions, particular classes of binary block-codes can be presented by a single function or a set.
Sublattices of lattices of convex sets
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For a poset \( P \), the set \( \text{Co}(P) \) of all order-convex subsets of \( P \) forms a lattice under inclusion. The lattices of the form \( \text{Co}(P) \) have been characterized by Birkhoff and Bennett. Furthermore, it is easy to see that the class \( \text{SUB} \) of all lattices that are embeddable into some \( \text{Co}(P) \) forms a quasivariety. The description of the members of \( \text{SUB} \) was an open problem from the survey paper from Adaricheva, Gorbunov, and Tumanov on convex geometries.

We outline a proof of the fact that \( \text{SUB} \) is a (finitely based) variety, defined by three identities, \( (S) \), \( (U) \), and \( (B) \), with special geometrical meaning. In particular, we obtain the nontrivial consequence that \( \text{SUB} \) is closed under homomorphic image. We discuss some further properties of \( \text{SUB} \), for example, if a finite lattice with a join-irreducibles embeds into some \( \text{Co}(P) \), then it embeds into some finite \( \text{Co}(P) \) with \( P \) of size at most \( 2n^2 - 5n + 4 \).

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