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PARTIAL DIFFERENTIAL EQUATIONS

On Blow-Up of Solutions to Cauchy Problems for Nonlinear Ferrite Equations in $(N+1)$-Dimensional Case

M. O. Korpusov[a,b,*] **and V. M. Ozornin**[a]

[a] *Faculty of Physics, Lomonosov Moscow State University, Moscow, 119991 Russia*

[b] *RUDN University, Moscow, 117198 Russia*

**e-mail: korpusov@gmail.com*

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Abstract—Three Cauchy problems for $(N+1)$-dimensional nonlinear Sobolev-type equations arising in the theory of magnetic oscillations in ferrites are considered. Results concerning the existence and uniqueness of local-in-time weak solutions to these problems and the blow-up of these solutions are obtained.

Keywords: nonlinear Sobolev-type equations, blow-up, local solvability, nonlinear capacity, blow-up time estimates

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1. INTRODUCTION

In this paper, we consider the following three nonlinear equations with model nonlinearities:

$$\frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i=1}^{N}\omega_i^2\frac{\partial^2 u(x,t)}{\partial x_i^2} + |\nabla u(x,t)|^q = 0, \tag{1.1}$$

$$\begin{gathered}\frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i=1}^{N}\omega_i^2\frac{\partial^2 u(x,t)}{\partial x_i^2} - \frac{\partial|\nabla u|^q}{\partial t} = 0,\\ \frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i=1}^{N}\omega_i^2\frac{\partial^2 u(x,t)}{\partial x_i^2} + \frac{\partial^2|\nabla u|^q}{\partial t^2} = 0,\end{gathered} \tag{1.2}$$

where $q > 1$, $(x,t)\in\mathbb{R}^N\times[0,T]$, and $u = u(x,t)$. These equations arise in the study of magnetostatic waves in ferrites (see [1]). This paper continues the research begun in [2–6], where critical exponents for Sobolev-type equations were found. Note that the case of $N = 3$ was studied in [2]. Below, the results of [2] are extended to the case $N \geqslant 3$. It turns out that the solutions of Cauchy problems exhibit a significantly different behavior for

$$1 < q \leqslant \frac{N}{N-1} \quad \text{and} \quad q > \frac{N}{N-1}, \quad N \geqslant 3.$$

Most of the statements are trivial generalizations of the results from [2], which we present without detailed proof.

Equations (1.1), (1.2) are nonlinear Sobolev-type ones. Linear and nonlinear Sobolev-type equations have been addressed in numerous works. For example, initial-boundary value problems for a wide variety of classes of linear and nonlinear Sobolev-type equations were studied in the general form and as examples by Sviridyuk, Zagrebina, and Zamyshlyaeva (see [7–9]).

Potential theory for nonclassical Sobolev-type equations was first considered by Kapitonov [10]. Later, potential theory was studied by Gabov and Sveshnikov [11, 12] and their students (see Pletner's work [13]).

In their classical work [14], Pohozaev and Mitidieri used the fairly simple method of nonlinear capacity to obtain profound results concerning the role of critical exponents. Note also the papers of Galakhov and Salieva [15, 16].

2. DESIGNATIONS

Define

$$I_N = \{1, 2, \ldots, N\}.$$

In what follows, we will systematically use abstract Banach spaces of the form $C^{(n)}([0,T];\mathbb{B})$, where $\mathbb{B}$ is a Banach space with respect to the norm $|\cdot|_{\mathbb{B}}$, $n \in \mathbb{Z}_+$. This space is defined inductively. First, the Banach space $C([0,T];\mathbb{B})$ is defined as the linear space of functions $f : [0,T] \to \mathbb{B}$ such that

$$|f(t_2) - f(t_1)|_{\mathbb{B}} \to +0 \quad \text{for any} \quad t_1, t_2 \in [0,T], \quad |t_2 - t_1| \to +0.$$

The linear space $C([0,T];\mathbb{B})$ is a Banach one with respect to the norm

$$\|f\|_T := \sup_{t\in[0,T]} |f(t)|_{\mathbb{B}}.$$

A function f belongs to the class $C^{(1)}([0,T];\mathbb{B})$ if $f \in C([0,T];\mathbb{B})$ and there exists a strong derivative $df/dt \in C([0,T];\mathbb{B})$ defined as

$$\lim_{|\Delta t|\to+0} \left|\frac{1}{\Delta t}[f(t+\Delta t) - f(t)] - \frac{df}{dt}(t)\right|_{\mathbb{B}} = 0.$$

Then the Banach space $C^{(n)}([0,T];\mathbb{B})$ is inductively defined for an arbitrary $n \in \mathbb{N}$.

The symbol $C_b(\mathbb{R}^N)$ stands for the linear space of continuous and bounded functions, which is a Banach space with respect to the norm

$$|f(x)|_0 := \sup_{x\in\mathbb{R}^N} |f(x)|. \tag{2.1}$$

For all $x \in \mathbb{R}^N$, the symbol $C_b\left(\rho(x);\mathbb{R}^N\right)$ with $\rho(x) \geqslant \rho_0 > 0$ denotes the linear subspace of functions from $C_b(\mathbb{R}^N)$ for which the following norm is finite:

$$|f(x)|_{0,\rho} := \sup_{x\in\mathbb{R}^N} \rho(x)|f(x)|,$$

where $C_b\left(\rho(x);\mathbb{R}^N\right)$ is a Banach space with respect to this norm.

Consider also the linear space $C_b^{\gamma_1,\gamma_2}\left(\rho(x);\mathbb{R}^N\right)$, where $\rho(x) \geqslant \rho_0 > 0$ for all $x \in \mathbb{R}^N$ and $\gamma_1 > 0, \gamma_2 > 0$, which is defined as the linear subspace of functions f from the Banach space $C_b^{(1)}(\mathbb{R}^N)$ with a finite norm

$$|f(x)|_{\gamma_1,\gamma_2} := \sup_{x\in\mathbb{R}^N} \rho^{\gamma_1}(x)|f(x)| + \sum_{j\in I_N} \sup_{x\in\mathbb{R}^N} \rho^{\gamma_2}(x)\left|\frac{\partial f(x)}{\partial x_j}\right|. \tag{2.2}$$

Lemma 1. *The linear space* $C_b^{\gamma_1,\gamma_2}\left(\rho(x);\mathbb{R}^N\right)$ *is a Banach space with respect to norm* (2.2).

We introduce the Banach spaces

$$C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right) = C_b^{\gamma_1,\gamma_2}\left(\rho(x);\mathbb{R}^N\right)$$

for $\rho(x) = \left(1+|x|^2\right)^{1/2}$, $\gamma_1 = 0$, and $\gamma_2 = 1$; its norm is denoted by the symbol $|\cdot|_1$:

$$|f(x)|_1 := |f(x)|_0 + \sum_{j\in I_N} \left|\left(1+|x|^2\right)^{1/2} \frac{\partial f(x)}{\partial x_j}\right|_0, \quad C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right) = C_b^{\gamma_1,\gamma_2}\left(\rho(x);\mathbb{R}^N\right) \tag{2.3}$$

for $\rho(x) = (1 + |x|^2)^{1/2}$, $\gamma_1 = N - 2$, and $\gamma_2 = N - 1$. In what follows, we will use the Banach space $C_b^{(2)}\left(\left(1 + |x|^2\right)^\beta; \mathbb{R}^N\right)$ with $\beta > 0$, which is a linear subspace of the Banach space $C_b^{(2)}(\mathbb{R}^N)$ with a finite norm

$$|f|_2 := \sup_{x\in\mathbb{R}^N}\left(1 + |x|^2\right)^{(N-2)/2}|f(x)| + \sum_{j\in I_N}\sup_{x\in\mathbb{R}^N}(1 + |x|)^{(N-1)/2}\left|\frac{\partial f(x)}{\partial x_j}\right|$$
$$+ \sum_{j,k\in I_N}\sup_{x\in\mathbb{R}^N}\left(1 + |x|^2\right)^\beta\left|\frac{\partial^2 f(x)}{\partial x_j \partial x_k}\right|, \quad \beta > N/2.$$

The symbol $C_0^{(2)}[0,T]$ denotes the linear space of functions f from $C^{(2)}[0,T]$, $f(T) = f'(T) = f''(T) = 0$, which is a Banach space with respect to the norm

$$\|f(t)\| := |f(x)|_0 + |f'(x)|_0 + |f''(x)|_0,$$

where the norm $|\cdot|_0$ is given by formula (2.1). The following notation is used for the norms of special Banach spaces:

$$C\left([0,T]; C_b^{0,1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right) : \|f\|_{0,T} := \sup_{t\in[0,T]}|f(x,t)|_1,$$

$$C^{(1)}\left([0,T]; C_b^{0,1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right) : \|f\|_{1,T} := \sup_{t\in[0,T]}\left[|f(x,t)|_1 + \left|\frac{\partial f(t)}{\partial t}\right|_1\right],$$

$$C^{(2)}\left([0,T]; C_b^{0,1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right) : \|f\|_{2,T} := \sup_{t\in[0,T]}\left[|f(x,t)|_1 + \left|\frac{\partial f(t)}{\partial t}\right|_1 + \left|\frac{\partial^2 f(t)}{\partial t^2}\right|_1\right],$$

where the norm $|\cdot|_1$ is defined by equality (2.3). The symbol $C_{x,t}^{(0,n)}\left(\mathbb{R}^N \times [0,T]\right)$, where $n \in \mathbb{N} \cup \{0\}$, stands for the space of functions f such that

$$f, \frac{\partial^j f}{\partial t^j} \in C\left(\mathbb{R}^N \times [0,T]\right) \quad \text{for} \quad j = \overline{0,n}.$$

The symbol $\mathcal{D}(\Omega)$ denotes the topological vector space of test functions with a compact support, and $\mathcal{D}'(\Omega)$ designates the corresponding spaces of generalized functions.

The symbol $W_{q,\text{loc}}^{1,0}\left(\mathbb{R}^N \times [0,T]\right)$ denotes the space of measurable functions f with weak partial derivatives; moreover, for any compact set $K \in \mathbb{R}^N$, we have

$$f, \frac{\partial f}{\partial x_j} \in L^q(K \times [0,T]).$$

The symbol $W_{q,\text{loc}}^{1,0}\left(\mathbb{R}^N \times [0,+\infty)\right)$ denotes the space of measurable functions f with weak partial derivatives; moreover, for any compact set $K \in \mathbb{R}^N$ and any $T > 0$, we have

$$f, \frac{\partial f}{\partial x_j} \in L^q(K \times [0,T]).$$

The symbol $W_{q,\text{loc}}^{1,0}(\mathbb{R}^N)$ denotes the space of measurable functions f having all weak derivatives such that, for any compact set $K \subset \mathbb{R}^N$, it is true that

$$f, \frac{\partial f}{\partial x_j} \in L^q(K).$$

The symbols $W_q^1(D)$ and $H^2\left(O(x_0, R_0)\right)$ denote classical Sobolev spaces, where $O(x_0, R_0)$ is the open ball of radius $R_0 > 0$ in $\mathbb{R}^N$ centered at the point $x_0 \in \mathbb{R}^N$.

Finally, for a function $f \in C^{(n)}(-\infty,0] \cup C^{(n)}[0,+\infty)$, where $n \in \mathbb{N}$ and the derivatives at the point $t = 0$ are understood as one-sided limits, we will use the following notation:

$$\left\{\frac{\partial^j f(t)}{\partial t^j}\right\} = \begin{cases} f^{(j)}(t) & \text{if} \quad \{t > 0\} \cup \{t < 0\}, \quad j = \overline{1,n}; \\ \text{arbitrary} & \text{if} \quad t = 0. \end{cases}$$

Let f be a function defined for $t \in [0,T]$. Then $\tilde{f}$ denotes its extension by zero to $t < 0$:

$$\tilde{f}(t) = \begin{cases} f(t) & \text{if} \quad t \in [0,T]; \\ 0 & \text{if} \quad t < 0. \end{cases}$$

3. CONSTRUCTION OF A FUNDAMENTAL SOLUTION AND ITS PROPERTIES

Let us construct a fundamental solution of the operator

$$\mathfrak{M}_{x,t}[u](x,t) = \frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i \in I_N} \omega_i^2 \frac{\partial^2 u(x,t)}{\partial x_i^2}, \quad N \geqslant 3. \tag{3.1}$$

Applying the Laplace transform with respect to t, we obtain the equation

$$\sum_{i=1}^{N} (p^2 + \omega_i^2) \frac{\partial^2 \hat{\mathcal{E}}(x,t)}{\partial x_i^2} = \prod_{i=1}^{N} \delta(x_i)$$

One of the solutions of this equation is the function

$$\hat{\mathcal{E}}(x,p) = -\frac{1}{S_N(N-2)}\left[\sum_{i}^{\overline{1,N}} \prod_{j \neq i}^{\overline{1,N}} (p^2 + \omega_j^2)^{1/(N-2)} (p^2 + \omega_i^2)^{\frac{3-N}{N-2}} x_i^2\right]^{1-N/2}$$

$$= -\frac{1}{S_N |x|^{N-2}(N-2)}\left[\frac{1}{|x|^2}\sum_{i}^{\overline{1,N}} \prod_{j \neq i}^{\overline{1,N}} (p^2 + \omega_j^2)^{1/(N-2)} (p^2 + \omega_i^2)^{\frac{3-N}{N-2}} x_i^2\right]^{1-N/2}.$$

Assume that $\operatorname{Re} p = \sigma > R_{\mathcal{L}}$ for sufficiently large $R_{\mathcal{L}} > 0$. Then $|p| > R_{\mathcal{L}}$. The notation $\sum_{k_1,\cdots,k_N}^{\overline{0,+\infty}}$ implies repeated summation over all the indicated indices in the range $\overline{0,+\infty}$:

$$\prod_{i \neq j}^{\overline{1,N}} (p^2 + \omega_i^2)^{1/(N-2)} (p^2 + \omega_j^2)^{\frac{3-N}{N-2}} x_j^2 = p^{\frac{4}{N-2}} \sum_{k_1,\cdots,k_N}^{\overline{0,+\infty}} p^{-2\sum_i^{\overline{1,N}} k_i} \left[\frac{\left(\frac{3-N}{N-2}\right)_{k_q} \prod_{i \neq q}^{\overline{1,N}} \left(\frac{1}{N-2}\right)_{k_i}}{\prod_i^{\overline{1,N}} k_i! \omega_i^{-2k_i}} x_q^2\right],$$

$$(m)_k := \begin{cases} \prod_r^{\overline{0,k-1}} (m - r), & k > 0, \\ 1, & k = 0. \end{cases}$$

We use the notation

$$\alpha_{k_1,\cdots,k_N}(x) := \sum_q^{\overline{1,N}} \left[\frac{\left(\frac{3-N}{N-2}\right)_{k_q} \prod_{i \neq q}^{\overline{1,N}} \left(\frac{1}{N-2}\right)_{k_i}}{\prod_i^{\overline{1,N}} k_i! \omega_i^{-2k_i}}\right] \frac{x_q^2}{|x|^2}.$$

Then, according to these expressions, the original function can be rewritten as

$$\hat{\mathcal{E}}(x,p) = -\frac{1}{S_N |x|^{N-2}(N-2)}\left[p^{\frac{4}{N-2}}\sum_{k_1,\cdots,k_N}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^{1-N/2}$$

$$= -\frac{1}{S_N |x|^{N-2}(N-2)}\left[p^{\frac{4}{N-2}} + p^{\frac{4}{N-2}}\sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^{1-N/2}$$

$$= -\frac{p^{-2}}{S_N |x|^{N-2}(N-2)}\left[1 + \sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^{1-N/2}.$$

Consider the series expansion

$$\frac{1}{p^2}\left[1 + \sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^{1-N/2}$$

$$= \sum_{a=0}^{+\infty}\binom{1-N/2}{a}\frac{1}{p^{2a+2}}\left[\sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^a = \frac{1}{p^2} + \hat{\Phi}(x,p),$$

where

$$\hat{\Phi}(x,p) := \sum_{a=1}^{+\infty}\binom{1-N/2}{a}\frac{1}{p^{2a+2}}\left[\sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \alpha_{k_1,\cdots,k_N}(x)p^{-2\sum_i^{\overline{1,N}} k_i}\right]^a.$$

Since $|p| > R_{\mathcal{L}} > 0$, applying the inverse Laplace transform yields the expression

$$\Phi(x,t) = \theta(t)\sum_{a=0}^{+\infty}\binom{1-N/2}{a}\left[\sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \frac{\alpha_{k_1,\cdots,k_N}(x)t^{2\sum_i^{\overline{1,N}} k_i - 1}}{\left(2\sum_i^{\overline{1,N}} k_i - 1\right)!} *\right]^a \frac{t^{2a+1}}{(2a+1)!}, \tag{3.2}$$

where

$$\left[\sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \frac{\alpha_{k_1,\cdots,k_N}(x)t^{2\sum_i^{\overline{1,N}} k_i - 1}}{\left(2\sum_i^{\overline{1,N}} k_i - 1\right)!} *\right]\phi(t) = \sum_{k_1+\cdots+k_N>0}^{\overline{0,+\infty}} \frac{\alpha_{k_1,\cdots,k_N}(x)}{\left(2\sum_i^{\overline{1,N}} k_i - 1\right)!}\int_0^t (t-\tau)^{2\sum_i^{\overline{1,N}} k_i - 1}\phi(\tau)d\tau.$$

For $a = 0$, this operator degenerates into the operator id. Thus, the following relation holds for the fundamental solution:

$$\mathcal{E}(x,t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} \hat{\mathcal{E}}(x,p)e^{pt}dp, \quad \sigma > R_{\mathcal{L}} > 0.$$

Taking into account equality (3.2), we have

$$\mathcal{E}(x,t) = -\frac{t\theta(t) + \Phi(x,t)}{S_N |x|^{N-2}(N-2)}, \tag{3.3}$$

where S_N is the area of an N-dimensional unit sphere.

Theorem 1. *The fundamental solution $\mathcal{E}$ has the following properties*:

(1) $\mathcal{E} \in C^{(\infty)}(\mathbb{R}^N \backslash \{0\} \times [0,+\infty))$.

(2) *For $x \neq 0$ and $t \in [0,T]$, the fundamental solution satisfies the estimates*

$$\left|\frac{\partial^k \mathcal{E}}{\partial t^k}(x,t)\right| \leqslant \frac{K_1(T)}{|x|^{N-2}},$$

$$\left|\frac{\partial^{k+1} \mathcal{E}}{\partial t^k \partial x_j}(x,t)\right| \leqslant \frac{K_2(T)}{|x|^{N-1}}, \quad \left|\frac{\partial^{k+2} \mathcal{E}}{\partial t^k \partial x_j \partial x_i}(x,t)\right| \leqslant \frac{K_3(T)}{|x|^{N}},$$

where all mixed derivatives are permutable, $k \in \mathbb{N} \cup \{0\}$, *and* $i, j = \overline{1,N}$.

(3) *For $x \neq 0$* and *$t \geqslant 0$, it is true that*

$$\frac{\partial \mathcal{E}}{\partial x_j}(x,0) = \mathcal{E}(x,0) = 0,$$

$$\frac{\partial^n}{\partial x_j^n}\frac{\partial \mathcal{E}}{\partial t}(x,0) = -\frac{1}{S_N(N-2)}\frac{\partial^n}{\partial x_j^n}\frac{1}{|x|^{N-2}}, \quad n = 0,1,$$

(4) *For $x \neq 0$ and $t \geqslant 0$*,

$$\frac{\partial^{k+2} \mathcal{E}}{\partial t^2 \partial x_j^k}(x,0) = 0.$$

4. ESTIMATE FOR AN INTEGRAL

Consider the function

$$J_N(x) := \int\limits_{\mathbb{R}^N} K(x-y) f(y) dy,$$

$$K \in C^{(l)}(\mathbb{R}^N \backslash \{0\}), \quad f \in C_b^{(l)}(\mathbb{R}^N), \quad l \in \mathbb{N} \cup \{0\},$$

$$\left|D_y^k K(x-y)\right| \leqslant \frac{C}{|x-y|^{N-\delta+k}}, \quad x \neq y, \quad k = \overline{0,l},$$

$$\left|D_y^k f(y)\right| \leqslant \frac{C}{(1+|y|^2)^{(\gamma+k)/2}}, \quad k = \overline{0,l}.$$

Lemma 2. *For $N \geqslant 3$, $\gamma > N$, and $0 < \delta < N$, we have the estimate*

$$\left|D_x^l J_N(x)\right| \leqslant \frac{C}{\left(1+|x|^2\right)^{(N-\delta+l)/2}}. \tag{4.1}$$

Proof. Note that

$$D_x^l J_N(x) = \int\limits_{\mathbb{R}^N} \left(D_y^l f(y)\right) K(x-y) dy.$$

Now we obtain the first estimate for $D_x^l J_N(x)$. It is true that

$$D_x^l J_N(x) = \int\limits_{|x-y|\leqslant 1} \left(D_y^l f(y)\right) K(x-y) dy + \int\limits_{|x-y|\geqslant 1} \left(D_y^l f(y)\right) K(x-y) dy,$$

which implies the estimate

$$\left|D_x^l J_N(x)\right| \leqslant C\left(\int\limits_{|x-y|\leqslant 1} \frac{1}{|x-y|^{N-\delta}} dy + \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{(\gamma+l)/2}} dy\right) \leqslant C. \tag{4.2}$$

Now we derive the second estimate:

$$D_x^l J_N(x) = \int\limits_{|y|\leqslant|x|/2} \left(D_y^l f(y)\right) K(x-y)dy + \int\limits_{|y|\geqslant|x|/2} \left(D_y^l f(y)\right) K(x-y)dy = K_1 + K_2. \tag{4.3}$$

If $|y| \leqslant |x|/2$, then

$$|x-y| \geqslant |x| - |y| \geqslant |x|/2.$$

Therefore, integration by parts in K_1 yields the equality

$$K_1 = (-1)^l \int\limits_{|y|\leqslant|x|/2} f(y)\left(D_y^l K(x-y)\right)dy + \sum_{k=0}^{l-1} \int\limits_{|y|=|x|/2} \left(D_y^k f(y)\right)\left(D_y^{l-1-k} K(x-y)\right)\chi_k\left(\frac{y}{|y|}\right)dS_y,$$

where χ_k are bounded functions. Then we have the estimate

$$|K_1| \leqslant \frac{C}{|x|^{N-\delta+l}} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{\gamma/2}}dy + C\sum_{k=0}^{l-1} \frac{1}{(1+|x|^2)^{(\gamma+k)/2}} \frac{1}{|x|^{N-\delta+l-1-k}} \leqslant \frac{C}{|x|^{N-\delta+l}}. \tag{4.4}$$

Additionally, K_2 satisfies the estimate

$$|K_2| \leqslant C \int\limits_{|y|\geqslant|x|/2} \frac{1}{(1+|y|^2)^{(\gamma+l)/2}} \frac{1}{|x-y|^{N-\delta}}dy; \tag{4.5}$$

moreover, if $|y| \geqslant |x|/2$, then

$$|x-y| \leqslant |x| + |y| \leqslant 3|y|.$$

Therefore, for $|y| \geqslant |x|/2$, we have the estimate

$$\frac{1}{(1+|y|^2)^{(\gamma+l)/2}} = \frac{1}{(1+|y|^2)^{(N-\delta+l)/2}} \frac{1}{(1+|y|^2)^{(\gamma+\delta-N)/2}} \leqslant C\frac{1}{(1+|x|^2)^{(N-\delta+l)/2}} \frac{1}{(1+|x-y|^2)^{(\gamma+\delta-N)/2}}. \tag{4.6}$$

Thus, combining (4.5) with (4.6) yields the inequality

$$\begin{aligned} |K_2| &\leqslant \frac{C}{(1+|x|^2)^{(N-\delta+l)/2}} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|x-y|^2)^{(\gamma+\delta-N)/2}} \frac{1}{|x-y|^{N-\delta}}dy \\ &= \frac{C}{(1+|x|\,|^2)^{(N-\delta+l)/2}} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^{(\gamma+\delta-N)/2}} \frac{1}{|y|^{N-\delta}}dy \leqslant \frac{C}{(1+|x|^2)^{(N-\delta+l)/2}} \end{aligned} \tag{4.7}$$

Therefore, from (4.3), taking into account (4.4) and (4.7), we derive the estimate

$$\left|D_x^l J_N(x)\right| \leqslant \frac{C}{|x|^{N-\delta+l}}. \tag{4.8}$$

In turn, estimate (4.1) follows from (4.2) and (4.8).

The lemma is proved.

5. WEAK SOLUTIONS OF THE CAUCHY PROBLEMS

In this section, we consider weak formulations of the following Cauchy problems:

$$\begin{gathered} \frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i=1}^N \omega_i^2 \frac{\partial^2 u(x,t)}{\partial x_i^2} + |\nabla u(x,t)|^q = 0, \quad (x,t) \in \mathbb{R}^N \times (0,T], \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \mathbb{R}^N; \end{gathered} \tag{5.1}$$

$$\begin{gathered} \frac{\partial^2}{\partial t^2}\Delta u(x,t) + \sum_{i=1}^N \omega_i^2 \frac{\partial^2 u(x,t)}{\partial x_i^2} - \frac{\partial |\nabla u|^q}{\partial t} = 0, \quad (x,t) \in \mathbb{R}^N \times (0,T], \\ u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x), \quad x \in \mathbb{R}^N; \end{gathered} \tag{5.2}$$

$$\frac{\partial^2}{\partial t^2}\Delta u(x,t)+\sum_{i=1}^{N}\omega_i^2\frac{\partial^2 u(x,t)}{\partial x_i^2}+\frac{\partial^2|\nabla u|^q}{\partial t^2}=0,\quad (x,t)\in\mathbb{R}^N\times(0,T],$$
$$u(x,0)=u_0(x),\quad \frac{\partial u}{\partial t}(x,0)=u_1(x),\quad x\in\mathbb{R}^N. \tag{5.3}$$

Definition 1. A function $u\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N\times[0,T])$ is called a *local-in-time weak solution of the Cauchy problem* (5.1) if for any function $\phi\in\mathcal{D}(\mathbb{R}^N\times(-\infty,T))$

$$\int_0^T\int_{\mathbb{R}^N}\left[\left(\nabla u(x,t),\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right)+\sum_{i=1}^{N}\omega_i^2\frac{\partial u(x,t)}{\partial x_i}\frac{\partial\phi(x,t)}{\partial x_i}\right]dxdt$$
$$+\int_{\mathbb{R}^N}\left[\left(\nabla\frac{\partial\phi(x,0)}{\partial t},\nabla u_0(x)\right)-(\nabla\phi(x,0),\nabla u_1(x))\right]dx=\int_0^T\int_{\mathbb{R}^N}\phi(x,t)|\nabla u(x,t)|^q\,dxdt,$$

where $u_0,u_1\in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

Definition 2. A function $u\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N\times[0,+\infty))$ is called a global-in-time weak solution of the Cauchy problem (5.1) if for any function $\phi\in\mathcal{D}(\mathbb{R}^N\times(-\infty,+\infty))$

$$\int_0^{+\infty}\int_{\mathbb{R}^N}\left[\left(\nabla u(x,t),\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right)+\sum_{i=1}^{N}\omega_i^2\frac{\partial u(x,t)}{\partial x_i}\frac{\partial\phi(x,t)}{\partial x_i}\right]dxdt$$
$$+\int_{\mathbb{R}^N}\left[\left(\nabla\frac{\partial\phi(x,0)}{\partial t},\nabla u_0(x)\right)-(\nabla\phi(x,0),\nabla u_1(x))\right]dx=\int_0^{+\infty}\int_{\mathbb{R}^N}\phi(x,t)|\nabla u(x,t)|^q\,dxdt,$$

where $u_0,u_1\in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

Definition 3. A function $u\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N\times[0,T])$ is called a local-in-time weak solution of the Cauchy problem (5.2) if for any function $\phi\in\mathcal{D}(\mathbb{R}^N\times(-\infty,T))$

$$+\int_0^T\int_{\mathbb{R}^N}\left[\left(\nabla u(x,t),\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right)+\sum_{i=1}^{N}\omega_i^2\frac{\partial u(x,t)}{\partial x_i}\frac{\partial\phi(x,t)}{\partial x_i}\right]dxdt$$
$$+\int_{\mathbb{R}^N}\left[\left(\nabla\frac{\partial\phi(x,0)}{\partial t},\nabla u_0(x)\right)-(\nabla\phi(x,0),\nabla u_1(x))+\phi(x,0)|\nabla u_0(x)|^q\right]dx=\int_0^T\int_{\mathbb{R}^N}\frac{\partial\phi(x,t)}{\partial t}|\nabla u(x,t)|^q\,dxdt,$$

where $u_0\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N)$ and $u_1\in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

Definition 4. A function $u\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N\times[0,+\infty))$ is called a global-in-time weak solution of the Cauchy problem (5.2) if for any function $\phi\in\mathcal{D}(\mathbb{R}^N\times(-\infty,+\infty))$

$$\int_0^{+\infty}\int_{\mathbb{R}^N}\left[\left(\nabla u(x,t),\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right)+\sum_{i=1}^{N}\omega_i^2\frac{\partial u(x,t)}{\partial x_i}\frac{\partial\phi(x,t)}{\partial x_i}\right]dxdt$$
$$+\int_{\mathbb{R}^N}\left[\left(\nabla\frac{\partial\phi(x,0)}{\partial t},\nabla u_0(x)\right)-(\nabla\phi(x,0),\nabla u_1(x))+\phi(x,0)|\nabla u_0(x)|^q\right]dx=\int_0^{+\infty}\int_{\mathbb{R}^N}\frac{\partial\phi(x,t)}{\partial t}|\nabla u(x,t)|^q\,dxdt,$$

where $u_0\in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N)$ and $u_1\in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

Definition 5. A function $u \in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N \times [0,T])$ is called a local-in-time weak solution of the Cauchy problem (5.3) if for any function $\phi \in \mathcal{D}(\mathbb{R}^N \times (-\infty, T))$

$$\int_0^T \int_{\mathbb{R}^N} \left[\left(\nabla u(x,t), \nabla \frac{\partial^2 \phi(x,t)}{\partial t^2} \right) + \sum_{i=1}^N \omega_i^2 \frac{\partial u(x,t)}{\partial x_i} \frac{\partial \phi(x,t)}{\partial x_i} \right] dxdt$$

$$+ \int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi(x,0)}{\partial t}, \nabla u_0(x) \right) + (\nabla \phi(x,0), \nabla u_1(x)) + |\nabla u_0(x)|^q \frac{\partial \phi}{\partial t}(x,0) \right.$$

$$\left. - \phi(x,0) q \left(\nabla u_1(x), |\nabla u_o(x)|^{q-2} \nabla u_o(x) \right) \right] dx = \int_0^T \int_{\mathbb{R}^N} \frac{\partial^2 \phi(x,t)}{\partial t^2} |\nabla u(x,t)|^q \, dxdt,$$

where $u_0(x), u_1(x) \in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

Definition 6. A function $u \in W^{1,0}_{q,\mathrm{loc}}(\mathbb{R}^N \times [0,+\infty))$ is called a global-in-time weak solution of the Cauchy problem (5.3) if for any function $\phi \in \mathcal{D}(\mathbb{R}^N \times (-\infty, +\infty))$

$$\int_0^{+\infty} \int_{\mathbb{R}^N} \left[\left(\nabla u(x,t), \nabla \frac{\partial^2 \phi(x,t)}{\partial t^2} \right) + \sum_{i=1}^N \omega_i^2 \frac{\partial u(x,t)}{\partial x_i} \frac{\partial \phi(x,t)}{\partial x_i} \right] dxdt$$

$$+ \int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi(x,0)}{\partial t}, \nabla u_0(x) \right) - (\nabla \phi(x,0), \nabla u_1(x)) + |\nabla u_0(x)|^q \frac{\partial \phi}{\partial t}(x,0) \right.$$

$$\left. - \phi(x,0) q \left(\nabla u_1(x), |\nabla u_0(x)|^{q-2} \nabla u_0(x) \right) \right] dx = \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{\partial^2 \phi(x,t)}{\partial t^2} |\nabla u(x,t)|^q \, dxdt,$$

where $u_0(x), u_1(x) \in W^{1,0}_{1,\mathrm{loc}}(\mathbb{R}^N)$.

The following result was proved as Lemma 7.1 in [2].

Lemma 3. *Any global-in-time weak solutions in the sense of Definitions* 2, 4, *and* 6 *are local-in-time weak solutions for any* $T > 0$ *in the sense of Definitions* 1, 3, *and* 5, *respectively.*

The following theorems were proved as Theorems 7.1–7.3 in [2].

Theorem 2. *Let u be a weak solution of the Cauchy problem* (5.1) *in the sense of Definition* 1. *Then, in the class of functions* u, u_0, *and* u_1 *such that there exist convolutions*

$$U_1(x,t) := -\int_{-\infty}^{t} \int_{\mathbb{R}^N} \mathcal{E}(x-y, t-\tau) |\nabla \tilde{u}(y,\tau)|^q \, dy d\tau \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$V(x,t) := \int_{\mathbb{R}^N} \mathcal{E}(x-y, t) \Delta u_1(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$W(x,t) := \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t} \Delta u_0(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

we have the following representation in the form of a sum of three potentials:

$$\tilde{u}(x,t) = U_1(x,t) + V(x,t) + W(x,t) \quad \text{for a.e.} \quad (x,t) \in \mathbb{R}^N \times (-\infty, T),$$

where $\mathcal{E}(x,t)$ *is the fundamental solution of the operator* $\mathfrak{M}_{x,t}$ *defined by equality* (3.3).

Theorem 3. *Let u be a weak solution of the Cauchy problem* (5.2) *in the sense of Definition* 3. *Then, in the class of* $|\nabla_x u|^q \in C^{(0,1)}_{x,t}(\mathbb{R}^N \times [0,T])$ *and functions* u, u_0, *and* u_1 *such that there exist convolutions*

$$U_2(x,t) := \int_{-\infty}^{t}\int_{\mathbb{R}^N} \mathcal{E}(x-y,t-\tau)\left\{\frac{\partial |\nabla \tilde{u}(y,\tau)|^q}{\partial \tau}\right\} dy d\tau \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$V(x,t) := \int_{\mathbb{R}^N} \mathcal{E}(x-y,t)\Delta u_1(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$W(x,t) := \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t}\Delta u_0(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

we have the following representation in the form of a sum of three potentials:

$$\tilde{u}(x,t) = U_2(x,t) + V(x,t) + W(x,t) \quad \text{for almost all} \quad (x,t) \in \mathbb{R}^N \times (-\infty, T),$$

where $\mathcal{E}(x,t)$ *is the fundamental solution of the operator* $\mathfrak{M}_{x,t}$ *defined by equality* (3.3).

Theorem 4. *Let u be a weak solution of the Cauchy problem* (5.3) *in the sense of Definition* 5. *Then, in the class of* $|\nabla u|^q \in C^{(0,2)}_{x,t}\left(\mathbb{R}^N \times [0,T]\right)$ *and functions* u, u_0, *and* u_1 *such that there exist convolutions*

$$U_3(x,t) := -\int_{-\infty}^{t}\int_{\mathbb{R}^N} \mathcal{E}(x-y,t-\tau)\left\{\frac{\partial^2 |\nabla \tilde{u}(y,\tau)|^q}{\partial \tau^2}\right\} dy d\tau \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$V(x,t) := \int_{\mathbb{R}^N} \mathcal{E}(x-y,t)\Delta u_1(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

$$W(x,t) := \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t}\Delta u_0(y) dy \in L^1_{\mathrm{loc}}\left(\mathbb{R}^N \times (-\infty, T)\right),$$

we have the following representation in the form of a sum of three potentials:

$$\tilde{u}(x,t) = U_3(x,t) + V(x,t) + W(x,t) \quad \text{for a.e.} \quad (x,t) \in \mathbb{R}^N \times (-\infty, T),$$

where $\mathcal{E}(x,t)$ *is the fundamental solution of the operator* $\mathfrak{M}_{x,t}$ *defined by equality* (3.1).

6. PROPERTIES OF VOLUME AND SURFACE POTENTIALS

We introduce the notation

$$G_\beta(x,y,t) := \frac{\left(1+|x|^2\right)^{1/2}}{\left(1+|y|^2\right)^{\beta/2}}\mathcal{E}(x-y,t),$$

where $\mathcal{E}(x,t)$ is the fundamental solution defined by formula (3.3).

Consider the following volume and surface potentials, which play an important role in the study of local-in-time solvability of the Cauchy problems formulated in Section 5:

$$U(x,t) := U[\rho](x,t) = \int_0^t \int_{\mathbb{R}^N} G_\beta(x,y,t-\tau)\rho(y,\tau) dy d\tau,$$

$$V(x,t) := V[\mu](x,t) = \int_{\mathbb{R}^N} G_\beta(x,y,t-\tau)\mu(y) dy,$$

$$W(x,t) := W[\sigma](x,t) = \int_{\mathbb{R}^N} \frac{\partial G_\beta(x,y,t)}{\partial t}\sigma(y) dy.$$

The following main theorem holds (see Theorem 8.1 in [2]).

Theorem 5. *If* $\rho \in C\left([0,T];C_b(\mathbb{R}^N)\right)$ *and* $\beta > N$, *then the volume potential* U *belongs to the class* $C^{(2)}\left([0,T];C_b^{0,1}\left((1+|x|^2)^{1/2};\mathbb{R}^N\right)\right)$.

Theorem 6. *The following estimates hold:*

$$\sup_{(x,t)\in\mathbb{R}^N\times[0,T]} \int_{\mathbb{R}^N} \left|\frac{\partial^k G_\beta(x,y,t)}{\partial t^k}\right| dy = B_2(\beta,T) < +\infty,$$

$$\sup_{(x,t)\in\mathbb{R}^N\times[0,T]} \left(1+|x|^2\right)^{1/2} \int_{\mathbb{R}^N} \left|\frac{\partial^{k+1} G_\beta(x,y,t)}{\partial t^k \partial x_j}\right| dy = B_3(\beta,T) < +\infty, \quad j=\overline{1,N},$$

where $\beta > N$ *and* $k = 0,1,2,3$.

Now consider the volume potentials

$$U_0^{(1)}(x,t) = \int_0^t \int_{\mathbb{R}^N} \frac{\partial G_\beta(x,y,t-\tau)}{\partial t} \rho(y,\tau)\,dy\,d\tau,$$

$$U_0^{(2)}(x,t) = \int_0^t \int_{\mathbb{R}^N} \frac{\partial^2 G_\beta(x,y,t-\tau)}{\partial t^2} \rho(y,\tau)\,dy\,d\tau.$$

The following result holds (see Theorem 8.2 in [2]).

Theorem 7. *If* $\beta > N$ *and* $\rho \in C\left([0,T];C_b(\mathbb{R}^N)\right)$, *then*

$$U_0^{(1)} \in C^{(1)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right),$$

$$U_0^{(2)} \in C\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right).$$

The following result holds (see Theorem 8.3 in [2]).

Theorem 8. *Assume that* $\mu,\sigma \in C_b(\mathbb{R}^N)$ *and* $\beta > N$. *Then, for an arbitrary* $T > 0$, *the surface potentials satisfy*

$$V,W \in C^{(2)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right).$$

Consider volume and surface potentials (with no weight) of the form

$$U^1(x,t) := \int_0^t \int_{\mathbb{R}^N} \mathcal{E}(x-y,t-\tau)\rho_1(y,\tau)\,dy\,d\tau,$$

$$V^1(x,t) := \int_{\mathbb{R}^N} \mathcal{E}(x-y,t)\mu_1(y)\,dy,$$

$$W^1(x,t) := \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t} \sigma_1(y)\,dy,$$

where $\mathcal{E}(x,t)$ is the fundamental solution defined by equality (3.3). The following theorem on the smoothness of these potentials holds (see Theorem 8.4 in [2]).

Theorem 9. *If* $\rho_1 \in C\left([0,T];C_b\left(\left(1+|x|^2\right)^{\beta/2};\mathbb{R}^N\right)\right)$ *and*

$$\mu_1,\sigma_1 \in C_b\left(\left(1+|x|^2\right)^{\beta/2};\mathbb{R}^N\right),$$

then, for $\beta > N$, *we have*

$$U^1 \in C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right),$$

$$V^1, W^1 \in C^{(2)}\left([0, T_1]; C_b^{N-2,N-1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$$

for any $T_1 > 0$.

The following result holds (see Theorem 8.4 in [2]).

Lemma 4. *Assume that* $\rho_1 \in C\left([0, T]; C_b\left(\left(1 + |x|^2\right)^{\beta/2}; \mathbb{R}^N\right)\right)$ *for* $\beta > N$. *Then*

$$\left\langle \mathfrak{M}_{x,t}[U^1](x,t), \phi(x)\right\rangle = \langle \rho_1(x,t), \phi(x)\rangle \quad \text{for all} \quad t \in [0, T]$$

and all $\phi \in \mathcal{D}(\mathbb{R}^N)$, *where* $\langle \cdot, \cdot\rangle$ *denotes the duality brackets between* $\mathcal{D}(\mathbb{R}^N)$ *and* $\mathcal{D}'(\mathbb{R}^N)$ *and the operator* $\mathfrak{M}_{x,t}$ *is defined by the equality*

$$\mathfrak{M}_{x,t}[w](x,t) := \Delta_x \frac{\partial^2 w(x,t)}{\partial t^2} + \sum_{j \in I_N} \omega_j^2 w_{x_j x_j}(x,t), \tag{6.1}$$

in which the spatial derivatives are understood as generalized functions from $\mathcal{D}'(\mathbb{R}^N)$. *Moreover,*

$$U^1(x, 0) = \frac{\partial U^1}{\partial t}(x, 0) = 0 \quad \text{for all} \quad x \in \mathbb{R}^N.$$

The following result is proved in a similar manner (see Theorem 8.4 in [2]).

Lemma 5. *Assume that* $\mu_1, \sigma_1 \in C_b\left(\left(1 + |x|^2\right)^{\beta/2}; \mathbb{R}^N\right)$ *for* $\beta > N$. *Then*

$$\left\langle \mathfrak{M}_{x,t}[V^1](x,t), \phi(x)\right\rangle = 0, \quad \left\langle \mathfrak{M}_{x,t}[W^1](x,t), \phi(x)\right\rangle = 0 \quad \text{for} \quad t \in [0, +\infty)$$

and for all $\phi \in \mathcal{D}(\mathbb{R}^N)$, *where the operator* $\mathfrak{M}_{x,t}$ *is defined by equality* (6.1).

The following result holds.

Lemma 6. *Assume that* $u_0, u_1 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta/2}; \mathbb{R}^N\right)$ *for* $\beta > N$. *Then*

$$W^1[\Delta u_0](x, 0) = -\int_{\mathbb{R}^N} \frac{\Delta u_0(y)}{S_N (N-2)|x-y|^{N-2}} dy = u_0(x),$$

$$\frac{\partial V^1[\Delta u_1]}{\partial t}(x, 0) = -\int_{\mathbb{R}^N} \frac{\Delta u_1(y)}{S_N (N-2)|x-y|^{N-2}} dy = u_1(x)$$

for all $x \in \mathbb{R}^N$. *Additionally, for all* $\mu_1, \sigma_1 \in C_b\left(\left(1 + |x|^2\right)^{\beta/2}; \mathbb{R}^N\right)$ *with* $\beta > N$, *the following equalities hold*:

$$V^1[\mu_1](x, 0) = 0, \quad \frac{\partial W^1[\sigma_1]}{\partial t}(x, 0) = 0 \quad \text{for all} \quad x \in \mathbb{R}^N.$$

Define

$$L(x,t) := U^1[\rho_1](x,t) + V^1[\Delta u_1](x,t) + W^1[\Delta u_0](x,t).$$

Theorem 9 and Lemmas 4–6 imply the following result.

Theorem 10. *Assume that* $\rho_1 \in C\left([0, T]; C_b\left(\left(1 + |x|^2\right)^{\beta/2}; \mathbb{R}^N\right)\right)$,

$$u_0 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta_1/2}; \mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta_2/2}; \mathbb{R}^N\right)$$

for $\beta > N$, $\beta_1 > N$, *and* $\beta_2 > N$. *Then*

$$L \in C^{(2)}\left([0, T]; C_b^{N-2,N-1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$$

and

$$\langle \mathfrak{M}_{x,t}[L](x,t), \phi(x)\rangle = \langle \rho_1(x,t), \phi(x)\rangle \quad \textit{for all} \quad t \in [0,T]$$

and for any $\phi \in \mathcal{D}(\mathbb{R}^N)$, *where* $\langle \cdot,\cdot \rangle$ *is the duality brackets between* $\mathcal{D}(\mathbb{R}^N)$ *and* $\mathcal{D}'(\mathbb{R}^N)$ *and the operator* $\mathfrak{M}_{x,t}$ *is defined by the equality*

$$\mathfrak{M}_{x,t}[w](x,t) := \Delta_x \frac{\partial^2 w(x,t)}{\partial t^2} + \sum_{j \in I_N} \omega_j^2 w_{x_j x_j}(x,t),$$

in which the spatial derivatives are understood as generalized functions from $\mathcal{D}'(\mathbb{R}^N)$. *Moreover, the following initial conditions are satisfied*:

$$L(x,0) = u_0(x), \quad \frac{\partial L}{\partial t}(x,0) = u_1(x) \quad \textit{for all} \quad x \in \mathbb{R}^N.$$

7. CAUCHY PROBLEM (5.1): EXISTENCE OF LOCAL-IN-TIME WEAK SOLUTIONS AND BLOW-UP OF GLOBAL-IN-TIME WEAK SOLUTIONS

Consider the auxiliary integral equation

$$u(x,t) = -\int_0^t \int_{\mathbb{R}^N} \mathcal{E}(x-y,t-\tau)|\nabla u(y,\tau)|^q \, dy d\tau + \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t} \Delta u_0(y) dy + \int_{\mathbb{R}^N} \mathcal{E}(x-y,t)\Delta u_1(y) dy. \quad (7.1)$$

In this equation, passing to the new function

$$v(x,t) = \left(1 + |x|^2\right)^{(N-2)/2} u(x,t) \quad (7.2)$$

and taking into account the equality

$$|\nabla u(x,t)|^q = \left|\nabla \frac{v(x,t)}{\left(1+|x|^2\right)^{(N-2)/2}}\right|^q = \left|\frac{1}{\left(1+|x|^2\right)^{(N-2)/2}} \nabla v(x,t) - (N-2)\frac{x}{\left(1+|x|^2\right)^{(N-1)/2}} v(x,t)\right|^q$$
$$= \frac{1}{\left(1+|x|^2\right)^{q(N-1)/2}} \left|\left(1+|x|^2\right)^{1/2} \nabla v(x,t) - (N-2)\frac{x}{\left(1+|x|^2\right)^{1/2}} v(x,t)\right|^q. \quad (7.3)$$

in the class of differentiable functions, we obtain the following integral equation for the new function v:

$$v(x,t) = \int_0^t \int_{\mathbb{R}^N} G_q(x,y,t-\tau)\rho(y,\tau) dy d\tau + \int_{\mathbb{R}^N} \frac{\partial G_\alpha(x,y,t)}{\partial t} \mu_\alpha(y) dy + \int_{\mathbb{R}^N} G_\beta(x,y,t)\sigma_\beta(y) dy, \quad (7.4)$$

$$G_\gamma(x,y,t) := \frac{\left(1 + |x|^2\right)^{1/2}}{\left(1 + |y|^2\right)^{\gamma}} \mathcal{E}(x-y,t), \quad \gamma := (N-1)q/2,$$

$$\rho(x,t) := -\left|\left(1+|x|^2\right)^{1/2} \nabla v(x,t) - \frac{(N-2)x}{\left(1+|x|^2\right)^{1/2}} v(x,t)\right|^q,$$

$$\mu_\alpha(x) := \left(1+|x|^2\right)^{\alpha} \Delta u_0(x), \quad \sigma_\beta(x) := \left(1+|x|^2\right)^{\beta} \Delta u_1(x).$$

Lemma 7. *If* $q > 1$ *and* $v_1, v_2 \in C\left([0,T]; C_b^{0,1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$, *then the function* ρ *belongs to the class* $C\left([0,T]; C_b\left(\mathbb{R}^N\right)\right)$ *and the following estimate holds*:

$$\sup_{t\in[0,T]} \left|\rho^1(x,t) - \rho^2(x,t)\right|_0 \leqslant q \max\left\{\|v_1\|_{0,T}^{q-1}, \|v_2\|_{0,T}^{q-1}\right\} \|v_1 - v_2\|_{0,T},$$

$$\rho^k(x,t) := \left|\left(1 + |x|^2\right)^{1/2} \nabla v_k(x,t) - \frac{(N-2)x}{\left(1 + |x|^2\right)^{1/2}} v_k(x,t)\right|^q, \quad k = 1,2,$$

$$\|v_k\|_{0,T} := \sup_{t\in[0,T]} |v_k(x,t)|_1 .$$

The following result holds (see Lemma 9.2 in [2]).

Lemma 8. *If* $q > N/(N-1)$, $N \geqslant 3$, *and*

$$u_0 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\alpha}; \mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta}; \mathbb{R}^N\right)$$

for $\alpha > N/2$ *and* $\beta > N/2$, *then there exists a maximum* $T_0 = T_0(u_0, u_1) > 0$ *such that for each* $T \in (0, T_0)$ the integral equation (7.4) *has a unique solution* v in the class $C\left([0,T]; C_b^{0,1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$; *moreover, either* $T_0 = +\infty$ *or* $T_0 < +\infty$, *and, in the latter case,*

$$\lim_{T\uparrow T_0} \|v\|_{0,T} = +\infty.$$

This lemma implies the following solvability result for the integral equation (7.1).

Lemma 9. *If* $q > N/(N-1)$, $N \geqslant 3$, *and*

$$u_0 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\alpha}; \mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta}; \mathbb{R}^N\right)$$

for $\alpha > N/2$ *and* $\beta > N/2$, *then there exists a maximum* $T_0 = T_0(u_0, u_1) > 0$ *such that for each* $T \in (0, T_0)$ the integral equation (7.4) *has a unique solution u in the class* $C\left([0,T]; C_b^{N-2,N-1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$; *moreover, either* $T_0 = +\infty$ *or* $T_0 < +\infty$, *and, in the latter case,*

$$\lim_{T\uparrow T_0} \left\|\left(1 + |x|^2\right)^{(N-2)/2} u(x,t)\right\|_{0,T} = +\infty.$$

Additionally, for $q > N/(N-1)$,

$$\rho_1(x,t) = -|\nabla u(x,t)|^q \in C\left([0,T]; C_b\left(\left(1 + |x|^2\right)^{q(N-1)/2}; \mathbb{R}^N\right)\right) \tag{7.5}$$

for each $T \in (0, T_0)$.

Theorem 10 implies the following result.

Theorem 11. *If* $q > N/(N-1)$, $N \geqslant 3$, *and*

$$u_0 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\alpha}; \mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1 + |x|^2\right)^{\beta}; \mathbb{R}^N\right)$$

for $\alpha > N/2$ *and* $\beta > N/2$, *then the Cauchy problem* (5.1) *has a unique local-in-time weak solution u in the sense of Definition 1 in the class* $C\left([0,T]; C_b^{N-2,N-1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$, *which for every* $T \in (0, T_0)$ *belongs to the class*

$$u \in C^{(2)}\left([0,T]; C_b^{N-2,N-1}\left(\left(1 + |x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$$

and satisfies the initial conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \textit{for all} \quad x \in \mathbb{R}^N$$

where the time $T_0 = T_0(u_0, u_1) > 0$ *is given in Lemma* 9.

Now the task is to prove the nonexistence of local-in-time weak solutions of the Cauchy problem (5.1) in the sense of Definition 1 for $q \in (1, N/(N-1)]$. First, we note that the following result holds (see Lemma 9.4 in [2]).

Lemma 10. *If u is a local-in-time weak solution of the Cauchy problem* (5.1) *in the sense of Definition* 1, *then the equality from Definition* 1 *holds for any function* $\phi = \phi_1\phi_2$, *where* $\phi_1 \in \mathcal{D}(\mathbb{R}^N)$ *and* $\phi_2 \in C_0^{(2)}[0,T]$.

Definition 7. We say that a pair of functions $u_0, u_1 \in W_q^1(\mathbb{R}^N)$ belongs to the class of initial functions M^1 (designated as $(u_0, u_1) \in M^1$) if there is a ball $O(x_0, R_0) \subset \mathbb{R}^N$ of positive radius such that

$$\left(\Delta u_0(x)\right)^2 + \left(\Delta u_1(x)\right)^2 > 0$$

on a subset of $O(x_0, R_0)$ of positive Lebesgue measure.

Theorem 12. *If* $1 < q \leqslant N/(N-1)$, *then the Cauchy problem* (5.1) *has no local-in-time weak solution in the sense of Definition* 1 *in the class of initial functions* $(u_0, u_1) \in M^1$ *for any* $T > 0$.

Proof. Let u be a local-in-time weak solution of the Cauchy problem (5.1) in the sense of Definition 1 for some $T > 0$. In view of Lemma 10, as a test function ϕ in the equality in Definition 1, we use the product

$$\phi(x,t) = \phi_1(x)\phi_2(t), \tag{7.6}$$

where

$$\phi_1(x) = \phi_0\left(\frac{|x|^2}{R^2}\right), \quad \phi_0(s) = \begin{cases} 1 & \text{if} \quad 0 \leqslant s \leqslant 1, \\ 0 & \text{if} \quad s \geqslant 2, \end{cases}$$

here, $\phi_0 \in C_0^\infty[0, +\infty)$ is a monotonically nonincreasing function and the function ϕ_2 is given explicitly by

$$\phi_2(t) = \left(1 - \frac{t}{T}\right)^\lambda \in C_0^{(2)}[0,T] \quad \lambda > 2q', \quad q' = \frac{q}{q-1}. \tag{7.7}$$

The following estimates hold:

$$\begin{gathered} \left| \int_0^T \int_{\mathbb{R}^N} \left(\nabla u(x,t), \nabla \frac{\partial^2 \phi(x,t)}{\partial t^2} \right) dx dt \right| \leqslant I^{1/q} c_1(R,T), \\ \left| \sum_{j \in I_N} \omega_j^2 \int_0^T \int_{\mathbb{R}^N} u_{x_j}(x,t) \phi_{x_j}(x,t) dx dt \right| \leqslant \sum_{j \in I_N} \omega_j^2 c_{2j}(R,T) I^{1/q}, \\ \left| \int_{\mathbb{R}^N} (\nabla \phi(x,0), \nabla u_1(x)) dx \right| \leqslant c_3(R) \left\| |\nabla u_1| \right\|_q, \\ \left| \int_{\mathbb{R}^N} \left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x) \right) dx \right| \leqslant \frac{\lambda}{T} c_3(R) \left\| |\nabla u_0| \right\|_q, \end{gathered} \tag{7.8}$$

where

$$c_1(R,T) := \left(\int_0^T \int_{\mathbb{R}^N} \frac{\left| \nabla \dfrac{\partial^2 \phi(x,t)}{\partial t^2} \right|^{q'}}{\phi^{q'/q}(x,t)} dx dt \right)^{1/q'} = c_{11}(T) \left(\int_{\mathbb{R}^N} \frac{|\nabla \phi_1(x)|^{q'}}{\phi_1^{q'/q}(x)} dx \right)^{1/q'} = c_{11}(T) c_{10} R^{(N-q')/q'},$$

$$c_{2j}(R,T) := \left(\int_0^T \int_{\mathbb{R}^N} \frac{\left|\phi_{x_j}(x,t)\right|^{q'}}{\phi(x,t)^{q'/q}} dxdt \right)^{1/q'} \leqslant \left(\int_0^T \int_{\mathbb{R}^N} \frac{|\nabla \phi(x,t)|^{q'}}{\phi(x,t)^{q'/q}} dxdt \right)^{1/q'} = c_{20}(T) R^{(N-q')/q'}, \tag{7.9}$$

$$c_3(R) := \left(\int_{\mathbb{R}^N} |\nabla \phi_1(x)|^{q'} dx \right)^{1/q'} = c_{30} R^{(N-q')/q'},$$

$$I := \int_0^T \int_{\mathbb{R}^N} \phi(x,t) |\nabla u(x,t)|^q dxdt. \tag{7.10}$$

Taking into account estimates (7.8), (7.9), and notation (7.10), from the equality in Definition 1, we derive the inequality

$$R^{(N-q')/q'} \left(c_{11}(T)c_{10} + N\omega^2 c_{20}(T)\right) I^{1/q} + R^{(N-q')/q'} \left(c_{30} \left\|\nabla u_1\right\|_q + c_{30} \frac{\lambda}{T} \left\|\nabla u_0\right\|_q\right) \geqslant I, \quad \omega^2 := \max_j \left\{\omega_j^2\right\}. \tag{7.11}$$

Using the arithmetic Hölder inequality

$$a^{1/q'} b^{1/q} \leqslant \frac{a}{q'} + \frac{b}{q}, \quad a,b \geqslant 0, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

we find from (7.11) that

$$R^{N-q'} \left(c_{11}(T)c_{10} + 3\omega^2 c_{20}(T)\right)^{q'} + q' R^{(N-q')/q'} \left(c_{30} \left\|\nabla u_1\right\|_q + c_{30} \frac{\lambda}{T} \left\|\nabla u_0\right\|_q\right) \geqslant I. \tag{7.12}$$

Consider the case where $1 < q$ and $q' \leqslant N$. Then

$$1 < q \leqslant \frac{N}{N-1}.$$

Now we separately consider the case $1 < q < N/(N-1)$ and the critical case $q = N/(N-1)$. In general, the line of reasoning is similar to the one used in [2]. In particular, by applying the classical Beppo Levi theorem, in the case $1 < q < N/(N-1)$, the following equality is obtained from (7.12) in the limit as $R \to +\infty$:

$$\int_0^T \int_{\mathbb{R}^N} \left(1 - \frac{t}{T}\right)^{\lambda} |\nabla u(x,t)|^q dxdt = 0. \tag{7.13}$$

The same equality is also derived in the critical case $q = N/(N-1)$. It follows from (7.13) that

$$u(x,t) = F(t) \quad \text{for a.e.} \quad (x,t) \in \mathbb{R}^N \times [0,T]. \tag{7.14}$$

Combining the equality from Definition 1 with (7.14) yields

$$\int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - \left(\nabla \phi(x,0), \nabla u_1(x)\right) \right] dx = 0 \tag{7.15}$$

for an arbitrary function $\phi(x,t) = \phi_1(x)\phi_2(t)$, where $\phi_2(t)$ is defined by equality (7.7) and $\phi_1 \in \mathcal{D}(\mathbb{R}^N)$. Since $(u_0, u_1) \in M^1$, it follows from (7.15) that

$$\int_{\mathbb{R}^N} \left[\frac{\lambda}{T} \Delta u_0(x) + \Delta u_1(x) \right] \phi_1(x) dx = 0$$

for any function $\phi_1 \in \mathcal{D}(\mathbb{R}^N)$ with a support $\operatorname{supp} \phi_1 \subset O(x_0, R_0)$. In view of the fundamental lemma of the calculus of variations, we obtain the relation

$$\frac{\lambda}{T} \Delta u_0(x) + \Delta u_1(x) = 0 \quad \text{for a.e.} \quad x \in O(x_0, R_0). \tag{7.16}$$

Note that if $u(x,t)$ is a weak solution of the Cauchy problem for some $T = T_1 > 0$, then $u(x,t)$ is also a weak solution for any $T \in (0,T_1]$; then equality (7.16) must hold for all $T \in (0,T_1]$. Therefore,

$$\Delta u_0(x) = 0, \quad \Delta u_1(x) = 0 \quad \text{for a.e.} \quad x \in O(x_0, R_0),$$

which contradicts the definition of the class of functions M^1. The theorem is proved.

Now our task is to prove that in some class N^1 of initial functions $u_0(x)$, $u_1(x)$ for $q > N/(N-1)$ the local-in-time weak solution of the Cauchy problem (5.1) in the sense of Definition 1 blows up in a finite time $T_0 = T_0(u_0,u_1) > 0$ defined in Lemma 9.

Definition 8. We say that initial functions u_0, u_1 belong to the class N^1 (designated as $(u_0,u_1) \in N^1$) if there is a nonnegative test function $\phi \in \mathcal{D}\left(\mathbb{R}^N \times (-\infty,+\infty)\right)$ such that the following nonlinear capacity is finite:

$$E_1(\phi) := \int_0^{+\infty}\int_{\mathbb{R}^N} \frac{\left|\nabla \dfrac{\partial^2 \phi(x,t)}{\partial t^2}\right|^{q'}}{\phi^{q'/q}(x,t)}\,dxdt + \sum_{j\in I_N} \omega_j^2 \int_0^{+\infty}\int_{\mathbb{R}^N} \frac{\left|\phi_{x_j}(x,t)\right|^{q'}}{\phi^{q'/q}(x,t)}\,dxdt < +\infty \tag{7.17}$$

and it satisfies the inequality

$$\frac{1}{q'}\left(\frac{2}{q}\right)^{q'/q} E_1(\phi) + \int_{\mathbb{R}^N}\left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - (\nabla\phi(x,0), \nabla u_1(x))\right]dx < 0. \tag{7.18}$$

Note that the existence of a test function such that the nonlinear capacity (7.17) is finite was proved in [14]. We fix this test function $\phi(x,t)$. Assume that

$$\int_{\mathbb{R}^N}\left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - (\nabla\phi(x,0), \nabla u_1(x))\right]dx < 0.$$

If the initial functions u_0 and u_1 are replaced by ru_0 and ru_1 with $r > 0$, then, for sufficiently large $r > 0$, inequality (7.18) holds. Now assume that

$$\int_{\mathbb{R}^N}\left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - (\nabla\phi(x,0), \nabla u_1(x))\right]dx > 0.$$

If the initial functions $u_0(x)$ and $u_1(x)$ are replaced by $ru_0(x)$ and $ru_1(x)$ with $r < 0$, then, for sufficiently large $r < 0$, inequality (7.18) holds true. Thus, Definition 8 is well defined.

Theorem 13. *If $q > N/(N-1)$ and $(u_0,u_1) \in N^1$, then the Cauchy problem* (5.1) *has a unique local-in-time weak solution in the sense of Definition* 1 *in the class* $C\left([0,T]; C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$ *for all* $T \in (0,T_0)$ *of smoothness class* $C^{(2)}\left([0,T]; C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$*; moreover,* $0 < T_0 < +\infty$ *and the weak solution blows up at the time* T_0 *in the sense that*

$$\lim_{T\uparrow T_0}\left\|\left(1+|x|^2\right)^{(N-2)/2} u(x,t)\right\|_{0,T} = +\infty.$$

Proof. Assume that $T_0 = +\infty$, where T_0 is defined in Lemma 9. Then, in the same manner as in the proof of Theorem 11, we can show that, for any $T > 0$ in the class

$$C\left([0,T]; C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$$

the Cauchy problem (5.1) has a unique global-in-time weak solution u in the sense of Definition 4 of smoothness class $C^{(2)}\left([0,T]; C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2}; \mathbb{R}^N\right)\right)$ for any $T > 0$. Let us prove that, for initial functions $(u_0, u_1) \in N^1$, we obtain a contradictory inequality. Indeed, using the Hölder inequality and the three-parameter Young inequality

$$ab \leqslant \varepsilon a^q + \frac{1}{q'(\varepsilon q)^{q'/q}}, \quad a, b \geqslant 0, \quad \varepsilon > 0, \quad \frac{1}{q} + \frac{1}{q'} = 1,$$

the following estimate can be derived from the equality of Definition 4 (by applying the standard technique from [14]):

$$\begin{gathered}\frac{1}{q'}\frac{1}{(\varepsilon q)^{q'/q}} E_1(\phi) + \int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - (\nabla \phi(x,0), \nabla u_1(x))\right] dx \\ \geqslant (1 - 2\varepsilon) \int_0^{+\infty} \int_{\mathbb{R}^N} \phi(x,t) |\nabla u(x,t)|^q \, dx dt\end{gathered} \tag{7.19}$$

for any function $\phi \in \mathcal{D}\left(\mathbb{R}^N \times (-\infty, +\infty)\right)$. Setting $\varepsilon = 1/2$ in (7.19), we obtain

$$\frac{1}{q'}\left(\frac{2}{q}\right)^{q'/q} E_1(\phi) + \int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x)\right) - (\nabla \phi(x,0), \nabla u_1(x))\right] dx \geqslant 0 \tag{7.20}$$

for any nonnegative function $\phi \in \mathcal{D}\left(\mathbb{R}^N \times (-\infty, +\infty)\right)$. Since $(u_0, u_1) \in N^1$, there is a nonnegative test function ϕ such that inequality (7.18) holds, which contradicts (7.20). Thus, we have proved that the time T_0 in Lemma 9 is finite. Then inequality (7.5) holds. The blow-up of a local-in-time weak solution to the Cauchy problem (5.1) in the sense of Definition 1 is proved, which completes the proof of the theorem.

8. CAUCHY PROBLEM (5.2): EXISTENCE OF LOCAL-IN-TIME WEAK SOLUTIONS AND BLOW-UP OF GLOBAL-IN-TIME WEAK SOLUTIONS

Consider the auxiliary integral equation

$$\begin{gathered}u(x,t) = \int_0^t \int_{\mathbb{R}^N} \mathcal{E}(x-y, t-\tau) \frac{\partial |\nabla u(y,\tau)|^q}{\partial \tau} dy d\tau \\ + \int_{\mathbb{R}^N} \frac{\partial \mathcal{E}(x-y,t)}{\partial t} \Delta u_0(y) dy + \int_{\mathbb{R}^N} \mathcal{E}(x-y,t) \Delta u_1(y) dy.\end{gathered} \tag{8.1}$$

Passing to the new function (7.2) in Eq. (8.1), we obtain the integral equation

$$\begin{gathered}v(x,t) = \int_0^t \int_{\mathbb{R}^N} G_q(x,y,t-\tau) \rho(y,\tau) dy d\tau + \int_{\mathbb{R}^N} \frac{\partial G_\alpha(x,y,t)}{\partial t} \mu_\alpha(y) dy \\ + \int_{\mathbb{R}^N} G_\beta(x,y,t) \sigma_\beta(y) dy := U(x,t) + W(x,t) + V(x,t),\end{gathered}$$

$$G_\gamma(x,y,t) := \frac{\left(1 + |x|^2\right)^{1/2}}{\left(1 + |y|^2\right)^\gamma} \mathcal{E}(x-y,t), \tag{8.2}$$

$$\rho(x,t) := \frac{\partial}{\partial t} \left| \left(1 + |x|^2\right)^{1/2} \nabla v(x,t) - \frac{(N-2)x}{\left(1 + |x|^2\right)^{1/2}} v(x,t) \right|^q,$$

$$\mu_\alpha(x) := \left(1 + |x|^2\right)^\alpha \Delta u_0(x), \quad \sigma_\beta(x) := \left(1 + |x|^2\right)^\beta \Delta u_1(x).$$

Lemma 11. *If* $q \geqslant 2$ *and* $v \in C^{(1)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$, *then the function* ρ *belongs to the class* $C\left([0,T];C_b(\mathbb{R}^N)\right)$; *moreover, for any* $v_k(x,t) \in C^{(1)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$ *with* $k=1,2$, *the following inequality holds*:

$$\sup_{t\in[0,T]}\left|\rho_1(x,t)-\rho_2(x,t)\right|_0 \leqslant d(q)\max\left\{\|v_1\|_{1,T}^{q-1},\|v_2\|_{1,T}^{q-1}\right\}\|v_1-v_2\|_{1,T},$$

where

$$\rho_k(x,t) := \frac{\partial}{\partial t}\left|\left(1+|x|^2\right)^{1/2}\nabla v_k(x,t)-\frac{(N-2)x}{\left(1+|x|^2\right)^{1/2}}v_k(x,t)\right|^q, \quad k=1,2,$$

$$\|v\|_{1,T} := \|v\|_{0,T}+\left\|\frac{\partial v(x,t)}{\partial t}\right\|_{0,T}, \quad \|w\|_{0,T} := \sup_{t\in[0,T]}|w(x,t)|_1.$$

The following solvability result holds for the integral equation (8.2) (see Lemma 10.1 in [2]).

Lemma 12. *If* $q \geqslant 2$ *and*

$$u_0 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right)$$

for $\alpha > N/2$ *and* $\beta > N/2$, *then there exists a maximum* $T_0 = T_0(u_0,u_1) > 0$ such that for each $T \in (0,T_0)$ *the integral equation* (8.2) *has a unique solution* v *in the class* $C^{(1)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$; *moreover, either* $T_0 = +\infty$ *or* $T_0 < +\infty$, *and the following limit property holds in the latter case*:

$$\lim_{T\uparrow T_0}\|v\|_{1,T} = +\infty.$$

Lemma 12 implies the following solvability result for the integral equation (8.1) (see Lemma 10.2 in [2]).

Lemma 13. *If* $q \geqslant 2$ *and*

$$u_0 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right)$$

for $\alpha > N/2$ *and* $\beta > N/2$, *then there exists a maximum* $T_0 = T_0(u_0,u_1) > 0$ *such that for each* $T \in (0,T_0)$ *the integral equation* (8.1) *has a unique solution* u in the class $C^{(1)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$; *moreover, either* $T_0 = +\infty$ *or* $T_0 < +\infty$, *and the following limit property holds in the latter case*:

$$\lim_{T\uparrow T_0}\left\|\left(1+|x|^2\right)^{(N-2)/2}u(x,t)\right\|_{1,T} = +\infty.$$

Lemma 12 implies that

$$|\nabla u|^q \in C^{(1)}\left([0,T];C_b\left(\left(1+|x|^2\right)^{(N-1)q/2};\mathbb{R}^N\right)\right) \subset C_{x,t}^{(0,1)}\left(\mathbb{R}^N\times[0,T]\right)$$

$$\rho_1 = \frac{\partial}{\partial t}|\nabla u|^q \in C\left([0,T];C_b\left(\left(1+|x|^2\right)^{(N-1)q/2};\mathbb{R}^N\right)\right).$$

The following result holds (see Theorem 10.1 in [2]).

Theorem 14. *If $q \geqslant 2$ and*

$$u_0 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right)$$

for $\alpha > N/2$ and $\beta > N/2$, then the Cauchy problem (5.2) *has a unique local-in-time weak solution u in the sense of Definition* 4 *in the class $C^{(1)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$, which for every $T \in (0,T_0)$ belongs to the class $u \in C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$ and satisfies the initial conditions*

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for all} \quad x \in \mathbb{R}^N,$$

where the time $T_0 = T_0(u_0,u_1) > 0$ is defined in Lemma 13.

Lemma 14. *If u is a local-in-time weak solution of the Cauchy problem* (5.2) *in the sense of Definition* 4, *then the corresponding equality holds for any function $\phi(x,t) = \phi_1(x)\phi_2(t)$, where $\phi_1 \in \mathcal{D}(\mathbb{R}^N)$ and $\phi_2 \in C_0^{(2)}[0,T]$.*

The proof is entirely similar to the proof of Lemma 10.

Definition 9. We say that functions u_0 and u_1 belong to the class M^2 of initial functions (designated as $(u_0,u_1) \in M^2$) if $u_0,u_1 \in W_q^1(\mathbb{R}^N)$, $|\nabla u_0(x)| = 0$ for almost all $x \in \mathbb{R}^N$, and there is a ball $O(x_0,R_0) \subset \mathbb{R}^N$ of positive radius such that $(\Delta u_1(x))^2 > 0$ on a subset of $O(x_0,R_0)$ of positive Lebesgue measure.

Theorem 15. *If $1 < q \leqslant N/(N-1)$ and $(u_0,u_1) \in M^2$, then the Cauchy problem* (5.2) *has no local-in-time weak solution in the sense of Definition* 3 *for any $T > 0$.*

Proof. Step 1. Derivation of the a priori estimate. Let u be a weak solution of the Cauchy problem (5.2) in the sense of Definition 3 for some $T > 0$. As a test function ϕ in the equality of Definition 3, we use the function defined by (7.6) and (7.7), where $\lambda > q'$. The following estimates hold:

$$\left|\int_0^T\int_{\mathbb{R}^N}\left(\nabla u(x,t),\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right)dxdt\right| \leqslant b_1(R,T)I^{1/q},$$

$$\left|\sum_{j\in I_N}\int_0^T\int_{\mathbb{R}^N}\omega_j^2 u_{x_j}(x,t)\phi_{x_j}(x,t)dxdt\right| \leqslant \sum_{j\in I_N}\omega_j^2 b_{2j}(R,T)I^{1/q},$$

$$\left|\int_{\mathbb{R}^N}\left(\nabla\frac{\partial\phi}{\partial t}(x,0),\nabla u_0(x)\right)dx\right| \leqslant \frac{\lambda}{T}b_3(R)\|\nabla u_0\|_q = \frac{\lambda}{T}b_{30}R^{(N-q')/q'}\|\nabla u_0\|_q, \tag{8.3}$$

$$\left|\int_{\mathbb{R}^N}(\nabla\phi(x,0),\nabla u_1(x))\,dx\right| \leqslant b_3(R)\|\nabla u_1\|_q = b_{30}R^{(N-q')/q'}\|\nabla u_1\|_q,$$

$$b_1(R,T) := \left(\int_0^T\int_{\mathbb{R}^N}\frac{\left|\nabla\frac{\partial^2\phi(x,t)}{\partial t^2}\right|^{q'}}{\left(-\frac{\partial\phi(x,t)}{\partial t}\right)^{q'/q}}dxdt\right)^{1/q'} = b_{10}(T)R^{(N-q')/q'},$$

$$b_{2j}(R,T) := \left(\int_0^T\int_{\mathbb{R}^N}\frac{\left|\phi_{x_j}(x,t)\right|^{q'}}{\left(-\frac{\partial\phi(x,t)}{\partial t}\right)^{q'/q}}dxdt\right)^{1/q'} = b_{20j}(T)R^{(N-q')/q'},$$

$$I := \int_0^T \int_{\mathbb{R}^N} \left(-\frac{\partial \phi(x,t)}{\partial t}\right) |\nabla u(x,t)|^q \, dxdt. \tag{8.4}$$

Now we consider the case where $1 < q$ and $q' \geqslant N$. This case is equivalent to the inequalities $1 < q \leqslant N/(N-1)$. Combining the equality from Definition 3 with estimates (8.3) and (8.4), we obtain the estimate

$$\begin{gathered} b_{10}(T) R^{(N-q')/q'} I^{1/q} + \sum_{j \in I_N} \omega_j^2 b_{20j}(T) R^{(N-q')/q'} I^{1/q} \\ + \left(\frac{\lambda}{T} b_{30} \|\nabla u_0\|_q + \|\nabla u_1\|_q\right) R^{(N-q')/q'} + \int_{\mathbb{R}^N} \phi_1(x) |\nabla u_0(x)|^q \, dx \geqslant I. \end{gathered} \tag{8.5}$$

Now using the three-parameter Young inequality with $\varepsilon \in (0,1/2)$, we derive the following inequality from (8.5):

$$\begin{gathered} R^{N-q'} c_1(\varepsilon) \left(b_{10}^{q'}(T) + \left(\sum_{j \in I_N} \omega_j^2 b_{20j}(T) \right)^{q'} \right) + \left(\frac{\lambda}{T} b_{30} \|\nabla u_0\|_q + \|\nabla u_1\|_q\right) R^{(N-q')/q'} \\ + \int_{\mathbb{R}^N} \phi_1(x) |\nabla u_0(x)|^q \, dx \geqslant (1-2\varepsilon) I, \quad c_1(\varepsilon) := \frac{1}{q'} \frac{1}{(q\varepsilon)^{q'/q}}. \end{gathered} \tag{8.6}$$

Separately, we need (as before) to consider the case $1 < q < N/(N-1)$ and the critical case $q = N/(N-1)$. For example, if $1 < q < N/(N-1)$, then we can apply the Beppo Levi theorem. As a result, in the limit as $R \to +\infty$, the following a priori estimate can be derived from (8.6):

$$\int_{\mathbb{R}^N} |\nabla u_0(x)|^q \, dx \geqslant \frac{\lambda}{T}(1-2\varepsilon) \int_0^T \int_{\mathbb{R}^N} \left(1 - \frac{t}{T}\right)^{\lambda - 1} |\nabla u(x,t)|^q \, dxdt, \quad \varepsilon \in (0,1/2). \tag{8.7}$$

In the limit as $\varepsilon \to +0$, relation (8.7) implies the desired a priori estimate

$$\int_{\mathbb{R}^N} |\nabla u_0(x)|^q \, dx \geqslant \frac{\lambda}{T} \int_0^T \int_{\mathbb{R}^N} \left(1 - \frac{t}{T}\right)^{\lambda - 1} |\nabla u(x,t)|^q \, dxdt, \quad \lambda > q'. \tag{8.8}$$

Step 2. Blow up of local solutions. Since $(u_0, u_1) \in M^2$, it follows from (8.8) that

$$u(x,t) = F(t) \quad \text{for a.e.} \quad (x,t) \in \mathbb{R}^N \times [0,T].$$

Substituting this equality into Definition 3 yields

$$\int_{\mathbb{R}^N} \left[\left(\nabla \frac{\partial \phi}{\partial t}(x,0), \nabla u_0(x) \right) - (\nabla \phi(x,0), \nabla u_1(x)) \right] dx = 0,$$

which holds for all $T > 0$ and all functions $\phi(x,t) = \phi_1(x)\phi_2(t)$ defined by equalities (7.6), (7.7). The rest of the argument is similar to the proof of Theorem 12. We obtain a contradiction with the fact that $(u_0, u_1) \in M^2$. The theorem is proved.

Now we prove that, for $q \geqslant 2$ in some class of initial functions, the unique weak solution of the Cauchy problem (5.2) in the sense of Definition 3 blows up in a finite time.

Definition 10. We say that initial functions u_0, u_1 belong to the class N^2 (designated as $(u_0, u_1) \in N^2$) if there is a test function $\phi \in \mathcal{D}\left(\mathbb{R}^N \times (-\infty, +\infty)\right)$ such that $\phi_t \leqslant 0$ for all $(x,t) \in \mathbb{R}^N \times [0, +\infty)$, the following nonlinear capacity is finite:

$$E_2(\phi) := \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{\left|\nabla \dfrac{\partial^2 \phi(x,t)}{\partial t^2}\right|^{q'}}{\left(-\dfrac{\partial \phi(x,t)}{\partial t}\right)^{q'/q}} dxdt + \sum_{j \in I_N} \omega_j^2 \int_0^{+\infty} \int_{\mathbb{R}^N} \frac{|\phi_{x_j}(x,t)|^{q'}}{\left(-\dfrac{\partial \phi(x,t)}{\partial t}\right)^{q'/q}} dxdt < +\infty \tag{8.9}$$

and it is true that

$$\frac{1}{q'}\left(\frac{2}{q}\right)^{q'/q} E_2(\phi) - \int\limits_{\mathbb{R}^N}\left[\left(\nabla\frac{\partial\phi}{\partial t}(x,0),\nabla u_0(x)\right) - (\nabla\phi(x,0),\nabla u_1(x)) - \phi(x,0)|\nabla u_0(x)|^q\right]dx < 0. \tag{8.10}$$

It was proved in [14] that there exists a test function ϕ such that capacity (8.9) is finite. We fix this test function. If $u_1(x)$ is replaced by $ru_1(x)$, then inequality (8.10) can be satisfied for some fixed function $u_0(x)$ due to the large value of $|r| > 0$.

The following result is valid.

Theorem 16. *If $q \geqslant 2$ and $(u_0,u_1)\in N^2$, then there exists a unique local-in-time weak solution of the Cauchy problem* (5.2) *in the sense of Definition* 3 *in the class $C^{(1)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$ for all $T\in(0,T_0)$ of smoothness class $C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$, where $0<T_0<+\infty$ and the weak solution blows up at the time T_0 in the sense that*

$$\lim_{T\uparrow T_0}\left\|\left(1+|x|^2\right)^{(N-2)/2}u\right\|_{1,T} = +\infty.$$

The proof is similar to that of Lemma 9. Namely, the equality from Definition 4 of a global-in-time weak solution to the Cauchy problem (5.2) should be applied to the individual terms.

9. CAUCHY PROBLEM (5.3): EXISTENCE OF LOCAL-IN-TIME WEAK SOLUTIONS AND BLOW-UP OF GLOBAL-IN-TIME WEAK SOLUTIONS

Consider the auxiliary integral equation

$$\begin{gathered} u(x,t) = -\int\limits_0^t\int\limits_{\mathbb{R}^N}\mathcal{E}(x-y,t-\tau)\frac{\partial^2|\nabla u(y,\tau)|^q}{\partial\tau^2}dyd\tau \\ + \int\limits_{\mathbb{R}^N}\frac{\partial\mathcal{E}(x-y,t)}{\partial t}\Delta u_0(y)dy + \int\limits_{\mathbb{R}^N}\mathcal{E}(x-y,t)\Delta u_1(y)dy. \end{gathered} \tag{9.1}$$

Passing to the new function (7.2) in Eq. (9.1) yields the integral equation

$$v(x,t) = \int\limits_0^t\int\limits_{\mathbb{R}^N}G_q(x,y,t-\tau)\rho(y,\tau)dyd\tau + \int\limits_{\mathbb{R}^N}\frac{\partial G_\alpha(x,y,t)}{\partial t}\mu_\alpha(y)dy$$

$$+ \int\limits_{\mathbb{R}^N}G_\beta(x,y,t)\sigma_\beta(y)dy := U(x,t)+W(x,t)+V(x,t),$$

$$G_\gamma(x,y,t) := \frac{\left(1+|x|^2\right)^{1/2}}{\left(1+|y|^2\right)^\gamma}\mathcal{E}(x-y,t), \tag{9.2}$$

$$\rho(x,t) := -\frac{\partial^2}{\partial t^2}\left|\left(1+|x|^2\right)^{1/2}\nabla v(x,t) - \frac{(N-2)x}{\left(1+|x|^2\right)^{1/2}}v(x,t)\right|^q,$$

$$\mu_\alpha(x) := \left(1+|x|^2\right)^\alpha\Delta u_0(x), \quad \sigma_\beta(x) := \left(1+|x|^2\right)^\beta\Delta u_1(x).$$

The following result holds (see Lemma 11.1 in [2]).

Lemma 15. *If* $q=2$ *and* $v \in C^{(2)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right)$, *then the function* ρ *belongs to the class* $C\left([0,T];C_b(\mathbb{R}^N)\right)$; *moreover, for any*

$$v_k \in C^{(2)}\left([0,T];C_b^{0,1}(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right), \quad k=1,2,$$

we have the inequality

$$\sup_{t\in[0,T]}|\rho_1(x,t)-\rho_2(x,t)|_0 \leqslant 6\max\left\{\|v_1\|_{2,T},\|v_2\|_{2,T}\right\}\|v_1-v_2\|_{2,T},$$

$$\rho_k(x,t) := -\frac{\partial^2}{\partial t^2}\left|\left(1+|x|^2\right)^{1/2}\nabla v_k(x,t)-\frac{(N-2)x}{\left(1+|x|^2\right)^{1/2}}v_k(x,t)\right|^2,$$

$$\|v\|_{2,T} := \|v\|_{0,T}+\left\|\frac{\partial v}{\partial t}\right\|_{0,T}+\left\|\frac{\partial^2 v}{\partial t^2}\right\|_{0,T}, \quad \|w\|_{0,T} := \sup_{t\in[0,T]}|w(x,t)|_1.$$

The following solvability result holds for the integral equation (9.2) (see Lemma 11.2 in [2]).

Lemma 16. *If* $q=2$, *then, for any* $T>0$, *there are small* $R_1>0$, $R_2>0$, *and* $R>0$ *such that, for any functions*

$$u_0 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right),$$

where $\alpha > N/2$ *and* $\beta > N/2$, *such that*

$$|u_0|_2 \leqslant R_1, \quad |u_1|_2 \leqslant R_2,$$

there exists a unique solution v of the integral equation (9.2) *in the ball*

$$D_{R,T} := \left\{v \in C^{(2)}\left([0,T];C_b^{0,1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right) : \|v(x,t)\|_{2,T} \leqslant R\right\}.$$

Lemma 16 implies the following solvability result for the integral equation (9.1).

Lemma 17. *If* $q=2$, *then, for any* $T>0$, *there are small* $R_1>0$, $R_2>0$, *and* $R>0$ *such that, for any functions*

$$u_0(x) \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1(x) \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right),$$

where $\alpha > N/2$ *and* $\beta > N/2$, *for which*

$$|u_0|_2 \leqslant R_1, \quad |u_1|_2 \leqslant R_2,$$

there exists a unique solution u of the integral equation (9.1) *in the ball*

$$B_{R,T} := \left\{u(x,t) \in C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right) : \left\|\left(1+|x|^2\right)^{(N-2)/2}u\right\|_{2,T} \leqslant R\right\}.$$

From Lemma 17 and equality (7.3), it follows that

$$|\nabla u|^2 \in C^{(2)}\left([0,T];C_b\left(\left(1+|x|^2\right)^{N-1};\mathbb{R}^N\right)\right) \subset C_{x,t}^{(0,2)}\left(\mathbb{R}^N\times[0,T]\right), \tag{9.3}$$

$$\rho_1 = -\frac{\partial^2}{\partial t^2}|\nabla u(x,t)|^2 \in C\left([0,T];C_b\left(\left(1+|x|^2\right)^{N-1};\mathbb{R}^N\right)\right). \tag{9.4}$$

Therefore, Lemma 17, properties (9.3) and (9.4), Theorem 4, and Lemma 5 imply the following result (see Theorem 11.1 in [2]).

Theorem 17. *If* $q = 2$, *then, for any* $T > 0$, *there are small numbers* $R_1 > 0$, $R_2 > 0$, *and* $R > 0$ *such that for any functions*

$$u_0 \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\alpha};\mathbb{R}^N\right), \quad u_1(x) \in C_b^{(2)}\left(\left(1+|x|^2\right)^{\beta};\mathbb{R}^N\right),$$

where $\alpha > N/2$ *and* $\beta > N/2$, *satisfying the inequalities*

$$|u_0|_2 \leqslant R_1, \quad |u_1|_2 \leqslant R_2,$$

there exists a unique weak local-in-time solution u of the Cauchy problem (5.3) *in the sense of Definition* 5 *in the class*

$$C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right) \cap B_{R,T},$$

$$B_{R,T} := \left\{u \in C^{(2)}\left([0,T];C_b^{N-2,N-1}\left(\left(1+|x|^2\right)^{1/2};\mathbb{R}^N\right)\right) : \left\|\left(1\ +|x|^2\right)^{(N-2)/2} u\right\|_{2,T} \leqslant R\right\};$$

moreover, u satisfies the initial conditions

$$u(x,0) = u_0(x), \quad \frac{\partial u}{\partial t}(x,0) = u_1(x) \quad \text{for all} \quad x \in \mathbb{R}^N.$$

The following result holds (see Lemma 11.5 in [2]).

Lemma 18. *If* $1 < q \leqslant N/(N-1)$ *and u is a local-in-time weak solution of the Cauchy problem* (5.3) *in the sense of Definition* 5 *with initial functions* $u_0, u_1 \in W_q^1(\mathbb{R}^N)$, *then the following a priori estimate is valid*:

$$\int_{\mathbb{R}^N}\left[\frac{\lambda}{T}|\nabla u_0(x)|^q + q\left(\nabla u_1(x), |\nabla u_0(x)|^{q-2}\nabla u_0(x)\right)\right]dx \geqslant \frac{\lambda(\lambda-1)}{T^2}\int_0^T\int_{\mathbb{R}^N}\left(1-\frac{t}{T}\right)^{\lambda-2}|\nabla u(x,t)|^q\,dxdt, \quad \lambda > 2.$$

The following result holds (see Theorem 11.2 in [2]).

Theorem 18. *If* $1 < q \leqslant N/(N-1)$ *and* $u_0, u_1 \in W_q^1(\mathbb{R}^N)$, *then, under the condition*

$$\int_{\mathbb{R}^N}\left(\nabla u_1(x), |\nabla u_0(x)|^{q-2}\nabla u_0(x)\right)dx < 0,$$

the Cauchy problem (5.3) *has no global-in-time weak solution u* in the sense of Definition 6. *Moreover, the lifespan of the local-in-time weak solution to the Cauchy problem* (5.3) *in the sense of Definition* 5 *satisfies the estimate*

$$T \leqslant -\frac{2}{q}\frac{I_1}{I_2}, \quad I_1 := \int_{\mathbb{R}^N}|\nabla u_0(x)|^q\,dx, \quad I_2 := \int_{\mathbb{R}^N}\left(\nabla u_1(x), |\nabla u_0(x)|^{q-2}\nabla u_0(x)\right)dx.$$

If the initial functions (u_0, u_1) *belong to* M^2, *then the Cauchy problem* (5.3) *has no local-in-time weak solutions in the sense of Definition* 5 *for any* $T > 0$.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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