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Unsteady Wave Processes in a Cylinder Made of Viscoelastic Functionally Graded Material

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Abstract—The paper considers the problem of unsteady wave propagation in the cross section of an infinite hollow cylinder made of a viscoelastic functionally graded material with nonmonotonically varying properties along the radius. The cylinder is replaced by a piecewise-homogeneous one with a large number of coaxial homogeneous layers approximating the properties of the source material. Based on the previously constructed solution for a layered cylinder, wave processes in a cylinder made of a viscoelastic functionally graded material with different types of nonmonotonic inhomogeneities are studied.

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INTRODUCTION

Inhomogeneous materials are widely used in different fields of the modern production and are a subject of study in many fields of natural sciences. Of special interest is the research of the behavior of functionally graded materials (FGMs) with a continuous dependence of physicomechanical parameters on the spatial coordinates under various external excitation, including the unsteady dynamical ones. The publications devoted to studying the dynamics of continuous-inhomogeneous elastic bodies with application to analytical and numerical-analytical methods have arisen since several decades [1–5]. For the survey of modern achievements in the considered field, see works [6–9]. Among the methods for studying the wave processes in FGMs, matrix methods in various variants based on reduction of dynamic equations to a system of first-order differential equations have received wide application [10]. Researchers use the power series and orthogonal polynomial methods, as well as the special semianalytical finite element method (SAFE) (in addition to traditional one), see description in surveys [6, 7]. In studying harmonic oscillations and harmonic waves in FGM, the well-known approach is in approximating its continuous inhomogeneity of layered-homogeneous structure with the continuity conditions at the contact of layers. Such approach is expressed in different modifications of the transfer matrix method, using which researchers studied steady wave processes in elastic [11] and piezoelectroelastic FGMs [12]. In recent years, the results in the field of studying the diffraction problems of sound harmonic waves by solid bodies with continuous-inhomogeneous elastic and thermoelastic coatings have been obtained [13, 14]. Note that in a majority of publications on the FGM dynamics, steady wave processes are considered, mainly within the linear elasticity, thermoelasticity, and piezoelectroelasticity, although studying unsteady elastic waves in FGMs started already in the last century [1–3] and continues in the recent decades [15].

There are far fewer publications devoted to the dynamics of viscoelastic FGMs. Researchers mainly studied steady waves within the viscoelastic Kelvin–Voigt model [16, 17] and the models with fractional-order operators [18, 19], as well as harmonic oscillations and waves with the model of standard

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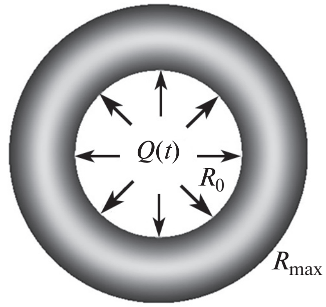

Fig. 1. Cross section and loading scheme.

viscoelastic body [20, 21]. In the cited works there are references to other few works on this topic. There are almost no publications on the unsteady dynamics of viscoelastic FGMs, except for several works [22–24], in which the results were obtained in the case of a monotonous dependence of the properties of viscoelastic FGMs on the coordinate.

The current work is aimed at studying the unsteady wave processes in a cylinder made of a viscoelastic FGM at nonmonotonic variation in its properties. We here continue studies [22–24] and propagate the above mentioned method applied by other authors only in stationary problems and consisting in replacement of FGM by a structure including approximating homogeneous layers to unsteady problems. Advantage of the method is the possibility of using the constructed solutions to non-stationary dynamic problems for layered-homogeneous bodies.

PROBLEM FORMULATION AND INVESTIGATION METHOD

Consider a problem about propagation of unsteady waves in the cross section of a viscoelastic infinite hollow cylinder whose material parameters continuously depend on the radial coordinate R (R and θ are the polar coordinates in the cross-section plane, $R_0 \leqslant R \leqslant R_{\max}$). The cylinder is initially at rest, its external surface $R = R_{\max}$ is free, and the internal surface ($R = R_0$) is subjected to uniformly distributed normal loading $Q(t)$ beginning from the time instance $t = 0$ (Fig. 1).

We introduce the dimensionless values:

$$\tau = t/t_0, \quad r = R/R_{\max}, \quad r_0 = R_0/R_{\max}, \quad \sigma_r(r,\tau) = P_R/[2G_0(r)], \quad \sigma_\theta(r,\tau) = P_\theta/[2G_0(r)],$$
$$u(r,\tau) = u_R/R_{\max}, \quad Q_0 f(\tau) = Q/[2G_0(r_0)], \quad \gamma_{\mathrm{v}}(r,\tau) = t_0 T_{\mathrm{v}}, \quad \gamma_s(r,\tau) = t_0 T_s,$$

where u_R is the displacement; P_R, P_θ are the radial and circumferential stresses; $t_0 = R_{\max}/c(1)$; $G_0(r)$, $\nu_0(r)$, $\rho(r)$, $T_s(r,\tau)$, $T_{\mathrm{v}}(r,\tau)$ are the instantaneous values of the shear modulus and Poisson's ratio, the density, and the kernels of shear and bulk relaxation satisfying the condition of constrained creep of material; $c(r) = \sqrt{2w(r)G_0(r)/\rho(r)}$ is the speed of longitudinal elastic waves; $w(r) = [1 - \nu_0(r)]/[1 - 2\nu_0(r)]$; Q_0 is the dimensionless constant (if the function $Q(t)$ is bounded, then we think that $\sup(|f(\tau)|) = 1$, $\tau > 0$).

When we seek $\sigma_r(r,\tau)$, $\sigma_\theta(r,\tau)$, $u(r,\tau)$, as the complete system of independent original data, we can take the values r_0, $Q_0 f(\tau)$, $\rho(r)/\rho(r_0)$, $G_0(r)/G_0(r_0)$, $\nu_0(r)$, $\gamma_s(r,\tau)$, $\gamma_{\mathrm{v}}(r,\tau)$.

Consider an analogous problem for the cylinder with the same dimensions R_0 and $R_{\mathbf{max}}$ consisting of N coaxial homogeneous viscoelastic layers: $R_{n-1} \leqslant R \leqslant R_n$ ($n = 1, 2, 3, \ldots, N$, $R_N = R_{\mathbf{max}}$, $N \gg 1$) with the continuity conditions of the displacement and stress vectors at contact of adjacent layers $R = R_m$ ($m = 1, 2, 3, ..., N-1$). By $G_0^{(n)}$, $\nu_0^{(n)}$, ρ_n, c_n, $T_s^{(n)}$, $T_{\mathrm{v}}^{(n)}$ we denote the instantaneous values of the shear modulus and Poisson's ratio, as well as the density, speed of longitudinal elastic waves, and hereditary kernels in the nth layer. By choosing the characteristic time $t_* = R_N/c_N$, we introduce the dimensionless values ($n = 1, 2, 3, \ldots, N$):

$$\tau = t/t_*, \quad r = R/R_N, \quad r_0 = R_0/R_N, \quad r_n = R_n/R_N \quad (r_N = 1), \quad \alpha_n = c_N/c_n,$$
$$\sigma_r^{(n)}(r,\tau) = P_R^{(n)}/(2G_0^{(n)}), \quad \sigma_\theta^{(n)}(r,\tau) = P_\theta^{(n)}/(2G_0^{(n)}), \quad u^{(n)}(r,\tau) = u_R^{(n)}/R_N,$$

$$Q_0 f(\tau) = Q/(2G_0^{(1)}), \quad \gamma_{\text{v}}^{(n)}(\tau) = t_* T_{\text{v}}^{(n)}, \quad \gamma_{\text{s}}^{(n)}(\tau) = t_* T_{\text{s}}^{(n)}, \quad w_n = (1-\nu_0^{(n)})/(1-2\nu_0^{(n)}),$$

where $u_R^{(n)}$ is the displacement; $P_R^{(n)}$ and $P_\theta^{(n)}$ are the corresponding stresses in the nth layer.

In contrast to works [22–24], we propose approximating the parameters of the FGM taking into account the specific features of the further accepted laws of variation in some of them using the relations $(n = 1, 2, 3, \dots, N)$

$$G_0^{(n)} = 0.5[G_0(r_{n-1}) + G_0(r_n)], \quad \rho_n = 0.5[\rho(r_{n-1}) + \rho(r_n)], \quad \nu_0^{(n)} = 0.5[\nu_0(r_{n-1}) + \nu_0(r_n)],$$
$$\gamma_s^{(n)} = 0.5[\gamma_s(r_{n-1}, \tau) + \gamma_s(r_n, \tau)], \quad \gamma_{\text{v}}^{(n)} = 0.5[\gamma_{\text{v}}(r_{n-1}, \tau) + \gamma_{\text{v}}(r_n, \tau)],$$

and, as N grows and the thickness of the Nth (outer) layer simultaneously increases, the value t_* tends to t_0. The thicknesses of the approximating layer can be chosen following from the specific properties of the FGMs, but in the current work these thicknesses these thicknesses were considered identical.

The formulation of the problem for a layered cylinder in the dimensionless form is formed by the dynamic equations $(r_{n-1} \leqslant r \leqslant r_n, n = 1, 2, \dots, N)$

$$\left(1 - \hat{d}_1^{(n)}\right) \frac{\partial}{\partial r}\left[\frac{\partial u^{(n)}(r,\tau)}{\partial r} + \frac{u^{(n)}(r,\tau)}{r}\right] - \alpha_n^2 \frac{\partial^2 u^{(n)}(r,\tau)}{\partial \tau^2} = 0, \tag{1}$$

the constitutive relations of linear viscoelasticity (the strains are expressed through displacements):

$$\sigma_r^{(n)}(r,\tau) = w_n\left(1 - \hat{d}_1^{(n)}\right)\frac{\partial u^{(n)}(r,\tau)}{\partial r} + (w_n - 1)\left(1 - \hat{d}_2^{(n)}\right)\frac{u^{(n)}(r,\tau)}{r},$$
$$\sigma_\theta^{(n)}(r,\tau) = w_n\left(1 - \hat{d}_1^{(n)}\right)\frac{u^{(n)}(r,\tau)}{r} + (w_n - 1)\left(1 - \hat{d}_2^{(n)}\right)\frac{\partial u^{(n)}(r,\tau)}{\partial r}, \tag{2}$$

$$\hat{d}_j^{(n)}\xi(r,\tau) = \int_0^\tau d_j^{(n)}(\tau - \chi)\xi(r,\chi)\,d\chi, \quad j = 1, 2,$$

$$d_1^{(n)}(\tau) = \frac{1}{3(1-\nu_0^{(n)})}\left[(1+\nu_0^{(n)})\gamma_{\text{v}}^{(n)}(\tau) + 2(1-2\nu_0^{(n)})\gamma_{\text{s}}^{(n)}(\tau)\right],$$
$$d_2^{(n)}(\tau) = \frac{1}{3\nu_0^{(n)}}\left[(1+\nu_0^{(n)})\gamma_{\text{v}}^{(n)}(\tau) - (1-2\nu_0^{(n)})\gamma_{\text{s}}^{(n)}(\tau)\right],$$

the initial conditions:

$$u^{(n)}(r,0) = 0, \quad \frac{\partial u^{(n)}}{\partial \tau}(r,0) = 0, \tag{3}$$

the boundary conditions:

$$\sigma_r^{(1)}(r_0,\tau) = -Q_0 f(\tau), \quad \sigma_r^{(N)}(1,\tau) = 0, \quad \tau > 0, \tag{4}$$

the continuity conditions at the contact of adjacent layers $(m = 1, 2, \dots, N-1)$:

$$u^{(m)}(r_m,\tau) = u^{(m+1)}(r_m,\tau), \quad G_0^{(m)}\sigma_r^{(m)}(r_m,\tau) = G_0^{(m+1)}\sigma_r^{(m+1)}(r_m,\tau). \tag{5}$$

The solution to problem (1)–(5) is constructed using the integral Laplace transform with respect to time with subsequent inversion. In the most convenient form, the solution in images is presented in work [25]. At regular hereditary kernels $\gamma_s^{(n)}$, $\gamma_{\text{v}}^{(n)}$ in form of a finite sum of exponents, the solution in originals is constructed in form of a residue series [25, 26], and for kernels of a more general form it is convenient to write it in another form (formula (5.2) of paper [27]).

In the previous works [22–24], the substantiation of applying the considered method to studying the unsteady FGM dynamics is confirmed for bodies with plane-parallel and cylindrical boundaries at monotonically varying FGM properties. By means of the calculations at specific initial data, we established the convergence of the results as the number of approximating layers N increases under the condition $f(\tau) \to 0$ as $\tau \to 0$, and, as N increases, we observe a uniform in τ tendency to zero of the jumps of circumferential stress $|\sigma_\theta^{(m)}(r_m,\tau)-\sigma_\theta^{(m+1)}(r_m,\tau)|$ at the contact between adjacent cylindrical layers. In the current work, a similar convergence is confirmed by the corresponding calculations for the case of nonmonotonic variation in the properties of viscoelastic FGM. As in work [24], we here could limit ourselves by 80 layers of the same thickness.

CALCULATION RESULTS

Using a numerical implementation of the solution to the unsteady dynamic problem for a layered structure (1)–(5), we studied the wave processes in the cross section of cylinder made of FGM with different initial parameters. In all presented figures, we draw just individual characteristic results for the case $r_0 = 0.5$ for the external load function in form of a smoothed step $f(\tau) = 1 - e^{-50\tau}$, $\tau > 0$. Note that at such function $f(\tau)$ the external load $Q(t)$ tends to the value $2G_0(r_0)Q_0$ as $t \to \infty$; therefore, for an FGM cylinder it is convenient to analyze the behavior of relative stresses:

$$\kappa_\theta(r,\tau) = G_0(r)\sigma_\theta(r,\tau)/[G_0(r_0)Q_0] = P_\theta/[2G_0(r_0)Q_0],$$
$$\kappa_r(r,\tau) = G_0(r)\sigma_r(r,\tau)/[G_0(r_0)Q_0] = P_R/[2G_0(r_0)Q_0].$$

According to the applied approach, as an approximation of such values we took the relative stresses for a layered-homogeneous cylinder ($n = 1, 2, 3, \ldots, N$):

$$\kappa_\theta^{(n)}(r,\tau) = G_0^{(n)}\sigma_\theta^{(n)}(r,\tau)/(G_0^{(1)}Q_0) = P_\theta^{(n)}/(2G_0^{(1)}Q_0),$$
$$\kappa_r^{(n)}(r,\tau) = G_0^{(n)}\sigma_r^{(n)}(r,\tau)/(G_0^{(1)}Q_0) = P_R^{(n)}/(2G_0^{(1)}Q_0)$$

and in all figures the results referred to any FGM are obtained at $N = 80$. The negative relative stresses always correspond to compression (we assume $Q_0 > 0$). To compute the relative stresses as an information about external loading, it suffices to prescribed only $f(\tau)$.

Consider two types of FGM. One type is characterized by the following parameters:

$$\nu_0 \equiv 0.3; \quad \gamma_{\rm v} \equiv 0; \quad \gamma_{\rm s} = 0.06e^{-0.05\tau}\tau^{-0.7};$$
$$\rho(r)/\rho(r_0) = G_0(r)/G_0(r_0) = b + (1-b)(r-r_b)^2/(r_0-r_b)^2; \quad r_b = 0.5(1+r_0); \quad b = 0.25, \quad (6)$$

that is, only the density and instantaneous shear modulus depend on the coordinate r. As r increases, they vary by the same quadratic law, firstly, decreasing and then increasing up to the initial value symmetrically with respect to the coordinate r_b equidistant from the boundaries of the cross section. The speed of longitudinal elastic waves is independent of r, which is convenient in analysis of wave processes.

Another FGM is characterized by the piecewise-linear variation in the density and instantaneous shear modulus

$$\rho(r)/\rho(r_0) = G_0(r)/G_0(r_0) = \begin{cases} b + (1-b)(r-r_b)/(r_0-r_b) & \text{for} \quad r \in [r_0; r_b), \\ b + (1-b)(r-r_b)/(1-r_b) & \text{for} \quad r \in [r_b; 1] \end{cases} \quad (7)$$

in the same range as the first material, under preservation of other initial data (6).

One of the advantages of FGMs compared to traditional layered materials being a package of homogeneous constituents is a possibility of avoiding jumps in stresses in structural elements which can lead to delamination. In view of this, at the same $f(\tau)$ and r_0, for the purpose of comparison, we performed the studies of wave process in a cylinder consisting of three homogeneous layers ($N = 3$) with the following data (the properties of the first and three layers are the same):

$$r_1 = 0.6; \quad r_2 = 0.9; \quad \nu_0^{(n)} = 0.3; \quad \gamma_{\rm v}^{(n)} \equiv 0; \quad \gamma_{\rm s}^{(n)} = 0.06e^{-0.05\tau}\tau^{-0.7}, \quad n = 1,2,3;$$
$$\rho_3/\rho_1 = G_0^{(3)}/G_0^{(1)} = 1; \quad \rho_2/\rho_1 = G_0^{(2)}/G_0^{(1)} = 0.25. \quad (8)$$

Note that the initial data (6) and (8), as well as (7) and (8), are coupled by the relations

$$\rho_2/\rho_1 = G_0^{(2)}/G_0^{(1)} = \rho(r_b)/\rho(r_0) = G_0(r_b)/G_0(r_0),$$

that is, nonconstant FGM characteristics of both types vary continuously in the range from the smallest value corresponding to the properties of the middle layer of the three-layer cylinder to the largest one

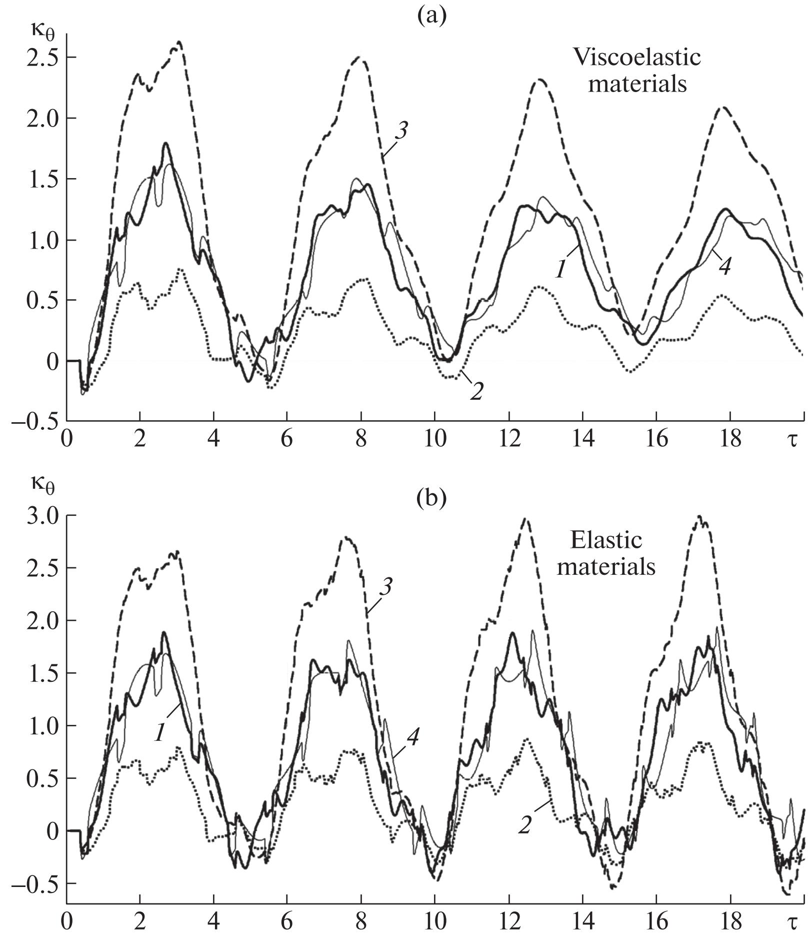

Fig. 2. Time variation of the relative circumferential stress at $r = 0.9$: (*1*) cylinder made of FGM; three-layer cylinder: (*2*) middle layer, (*3*) outer layer; and (*4*) homogeneous cylinder.

corresponding to the properties of its two outer layers. The results of studying the dynamics of three-layer cylinder were compared with the results for an FGM cylinder.

At the contact of layers with different properties, the value κ_r is continuous, in contrast to κ_θ, and, as the calculations show, under the external load function $f(\tau)$ in form of a smoothed step and initial data of the specified types (6)–(8) among the values characterizing the dynamic stress state, the behavior of the value κ_θ is more interesting compared to κ_r. The relative circumferential stress varies in a larger range than κ_r, and it is generally more sensitive to change in the initial data; therefore, we here demonstrate the characteristic results only for κ_θ.

In Fig. 2a the thick solid curve *1* corresponds to time variation in the value κ_θ at the point $r = 0.9$ for a cylinder made of viscoelastic FGM with quadratic variation in the properties (6). Figure 2 also presents the time variations in the relative circumferential stress $\kappa_\theta^{(n)}$, $n = 2, 3$, for the three-layer cylinder (8) at the contact between the second and third layers (that is, at $r = 0.9$), in the second, softer, layer ($\kappa_\theta^{(2)}$ is the dotted curve *2*) and in the third one ($\kappa_\theta^{(3)}$ is the dashed curve *3*). The thin solid curve *4* shows the time variation in the value $\kappa_\theta^{(1)}$ at the point $r = 0.9$ for homogeneous cylinder ($N = 1$) at the same $f(\tau)$, $r_0, \nu_0^{(1)}, \gamma_v^{(1)}, \gamma_s^{(1)}$.

Figure 2b presents analogous results for the case of linear elastic materials, when all relaxation kernels are zero and other initial data for each graph are the same as the corresponding graph in Fig. 2a. Curves *2* and *3* in Figs. 2a and 2b demonstrate a significant jump in the circumferential stresses at contact of homogeneous components in the case of three-layer material. A similar situation was also observed at $r = 0.6$.

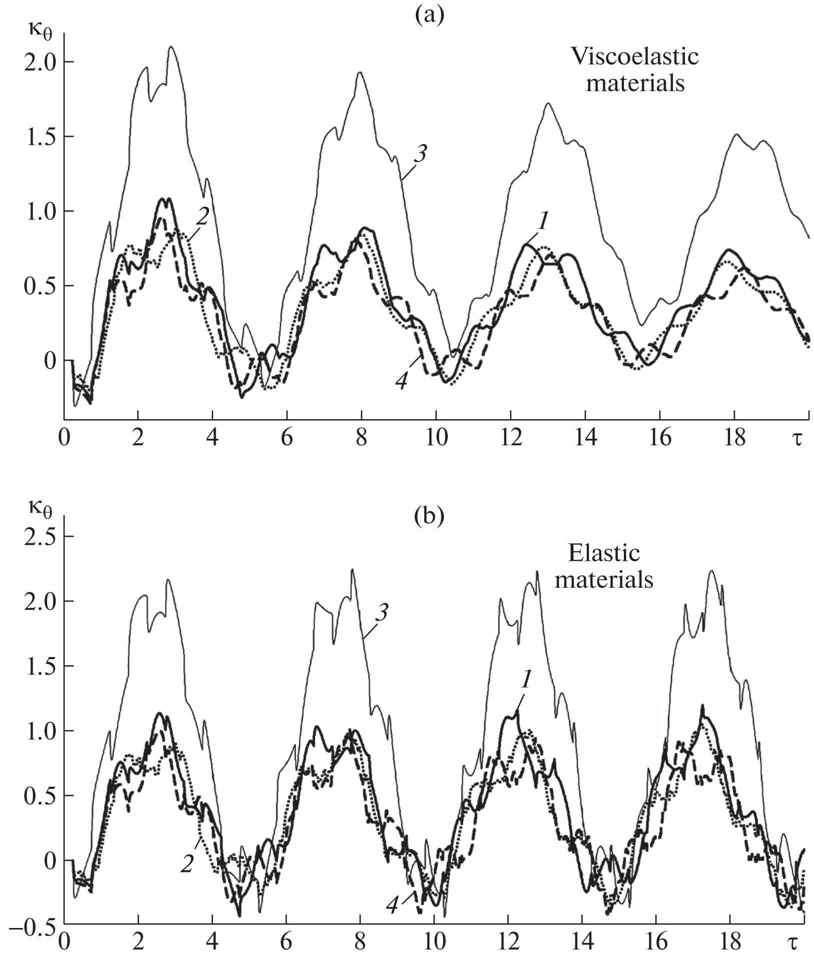

Fig. 3. Time variation of relative circumferential stress at $r = 0.75$: (*1*) an FGM cylinder with quadratic law, (*2*) a three-layer, (*3*) a homogeneous cylinder, (*4*) an FGM cylinder with piecewise-linear law.

Figures 3a and 3b present the time dependences of the relative circumferential stress at the point $r = r_b = 0.75$, respectively, for viscoelastic and elastic materials. In Fig. 3a the thick solid curve *1* corresponds to the FGM with a quadratic law of variation in the properties (6), the dotted curve *2* corresponds to the three-layer material (8), and the thin solid curve *3* stands for the homogeneous material. The dashed curve *4* was obtained for the FGM with a piecewise-linear variation in the density and instantaneous shear modulus (7). In Fig. 3b each graph is plotted under the condition that all relaxation kernels are zero and other initial data are the same as for the corresponding graph in Fig. 3a. As we see, the graphs of the time dependence of the circumferential stresses at the considered point are in general similar to each other, but substantially differ from the corresponding graph in the case of homogeneous material.

In the case of viscoelastic materials, Figs. 4a and 4b show the time variation in the relative circumferential stress at the boundaries $r = r_0 = 0.5$ and $r = 1$, respectively. Curves *1*–*4* are plotted at the same initial data as the corresponding curves in Fig. 3a.

By the results of our studies, partially presented in Figs. 2 and 3, we can make a conclusion that the FGMs of considered types under stepwise loading are in general advantageous against the three-layer composite and the homogeneous material with respect to the level of extending circumferential stresses at characteristic points in the cylinder cross section. Although at the cylinder boundaries (Fig. 4) the level of stresses in the homogeneous material is considerably lower, the considered FGM have the highest stiffness exactly at these boundaries.

Figures 2 and 3 also demonstrate the effect of viscoelastic material properties on the wave process. Viscosity has no time to manifest itself at relatively short times, but, then, the wave processes gradually

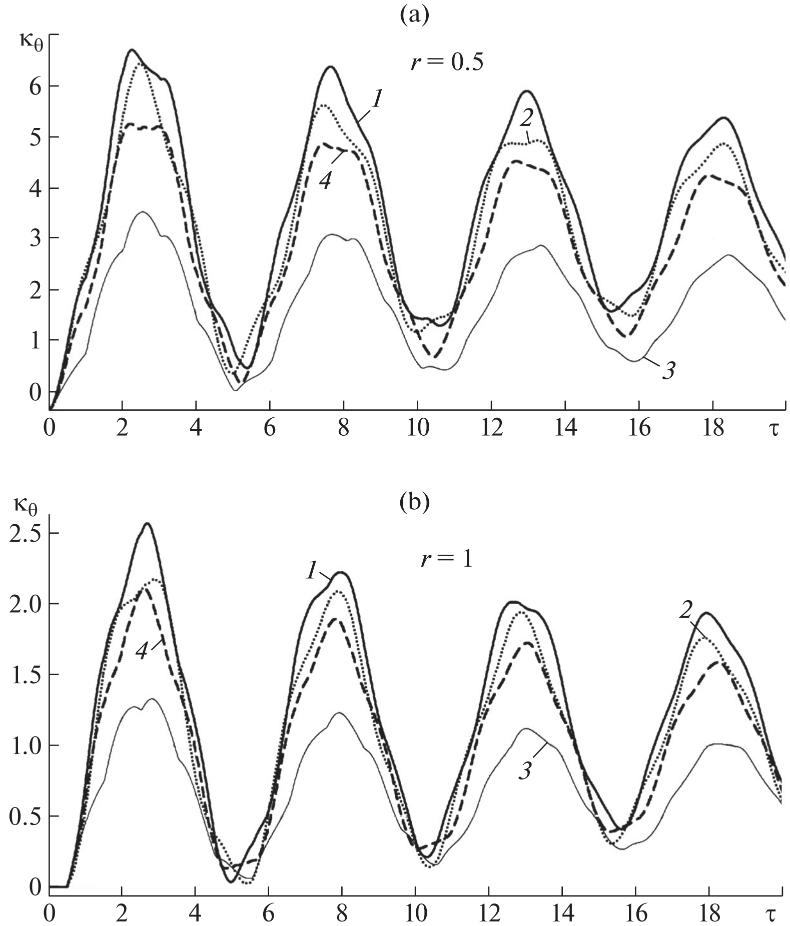

Fig. 4. Time variation in relative circumferential stress at $r =$ (a) 0.5 and (b) 1: (*1*) cylinder made of FGM with quadratic law, (*2*) three-layer cylinder, (*3*) homogeneous cylinder, and (*4*) cylinder made of FGM with piecewise-linear law.

decay in viscoelastic materials, unlike the elastic ones. Under the chosen character of external load, the stresses tend with time to the values determined from the solutions to static problems of the theory of elasticity in which the material constants are long-term moduli. This is confirmed by the calculations and agrees with the theoretical results of work [27].

CONCLUSIONS

For a continuous function of external load, we confirmed the justification of using the method for approximating an FGM by a layered structure in unsteady problems of the considered type. The proposed approach firstly allowed simulating the propagation of unsteady waves in viscoelastic FGMs with nonmonotonic dependence of their physicomechanical parameters on the coordinates. We investigated unsteady dynamic processes in the cross section of a hollow infinite cylinder made of viscoelastic FGM under nonmonotonic variation in its properties in the radial direction. We compared the characteristics of the transient wave processes at different variants of such variation, as well as juxtapose them against the corresponding characteristics for the three-layer and homogeneous cylinders. We demonstrated the advantage of the FGM against the three-layer composite, as well as against the homogeneous material, subjected to unsteady radial loading of the cylinder cavity. We revealed the effect of the viscosity of materials on the unsteady process.

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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