Minimizing a Sensitivity Function as Boundary-Value Problem in Terminal Control

Elena Khoroshilova
Lomonosov Moscow State University
Faculty of Computational Mathematics and Cybernetics
Leninskiye Gory, 1-52, 119991 Moscow, Russia
Email: khorelena@gmail.com

Abstract—A problem of terminal control with linear dynamics on a finite time interval is considered. The right-hand end of trajectory is defined implicitly as a solution of boundary-value problem. The problem is reduced to finding a saddle point of the Lagrange function, and linear dynamics is regarded as an equality-type constraint. Dual extraproximal iterative method for solving the problem was proposed, and convergence of the method to problem solution in all components was proved.

I. INTRODUCTION

We formulate a controlled dynamic model with a boundary-value problem of minimizing a sensitivity function under constraints [1,2,3]. Solution of boundary-value problem implicitly defines a terminal condition for the dynamic model. In the model, a unique trajectory corresponds to each control taken from a bounded set. The problem is to select the control such that the corresponding trajectory takes an object from an arbitrary initial state to the terminal state. In this paper, the dynamic model is treated as a problem of stabilization, and the terminal state of the object is interpreted as a state of equilibrium.

II. FINITE-DIMENSIONAL BOUNDARY-VALUE EQUILIBRIUM PROBLEM

A. The problem formulation

We consider the problem of minimizing the sensitivity function as a finite-dimensional problem on the right-hand end of time interval. The boundary-value problem is a system of two problems. One of them is a parametric convex programming problem, where the vector \( y \in Y \subset R^n \) acts as a parameter and is a right-hand part of functional constraints. Regarding this parameter, we construct the optimal value function that will be called the sensitivity function, due to the alleged regularity constraints (such as Slater condition).

Construction of the sensitivity function leads us to the formulation of the second problem, which is a problem of minimizing the sensitivity function on the set \( Y \) of admissible values of \( y \). Together, both problems generate a finite-dimensional system, the solution of which will implicitly define the boundary-value condition of the controlled dynamics:

\[
\varphi(y) = f(x^*) = \text{Min}\{f(x) \mid g(x) \leq y, \ x \in X \subset R^n\},
\]

where \( f(x) \) is a scalar convex function; \( Y = \{y \in R^n_+ \mid G(y) \leq d, \ d \in R^m_+\} \); \( g(x), G(y) \) are vector functions, each component of which is also convex; \( d \) is a fixed vector; \( R^n_+ \) is a positive orthant; \( X \subset R^n \), \( Y \subset R^n_+ \) are convex closed sets.

B. The Lagrange function and new problem formulation

We introduce the Lagrange function for problem (1)

\[
L(x, y^*; p) = f(x) + \langle p, g(x) - y^* \rangle.
\]

This function is defined for all \( x \in X \subset R^n \), \( p \in R^n_+ \). Saddle point \((x^*, p^*)\) of this function satisfies saddle-point inequalities

\[
f(x^*) + \langle p, g(x^*) - y^* \rangle \leq f(x^*) + \langle p^*, g(x^*) - y^* \rangle \leq f(x) + \langle p^*, g(x) - y^* \rangle
\]

for all \( x \in X \), \( p \in R^n_+ \). We rewrite system (3) in the form

\[
x^* \in \text{Argmin}\{f(x) + \langle p^*, g(x) - y^* \mid x \in X\}, \ y^* \in R^n_+ , \ \langle p - p^* , g(x^*) - y^* \rangle \leq 0 \]

for all \( x \in X \), \( p \in R^n_+ \).

Until now, the vector \( y^* \in R^n_+ \) was arbitrary in our reasoning. Now we will choose this vector so that it minimizes the sensitivity function and is the solution of (2). This solution satisfies the necessary and sufficient condition for the minimum

\[
\langle \nabla \varphi(y^*), y - y^* \rangle \geq 0, \ y \in Y,
\]

where \( \nabla \varphi(y^*) \) is subgradient of sensitivity function, \( Y = \{y \geq 0 \mid G(y) \leq d\} \). We have \( \nabla \varphi(y^*) = -p^* \), then the last system of inequalities can be represented as

\[
x^* \in \text{Argmin}\{f(x) + \langle p^*, g(x) - y^* \mid x \in X\}, \ y^* \in Y, \ \langle p - p^* , g(x^*) - y^* \rangle \leq 0, \ p \in R^n_+ , \ \langle p^*, y - y^* \rangle \leq 0, \ y \in Y.
\]

It follows that the solution of (4) satisfies (1),(2), and vice versa. In turn, the vectors triplet \((x^*, y^*; p^*)\) is a solution of extreme problems:

\[
x^* \in \text{Argmin}\{f(x) \mid g(x) \leq y^*, \ x \in X\}, \ y^* \in Y, \ p^* \in R^n_+.
\]
The form of the proximal operator. The system then takes the form

\[ p^* \in \text{Argmax}\{ (p, g(x^*) - y^*) \mid p \in \mathbb{R}^m_+ \}, \]

\[ y^* \in \text{Argmax}\{ (p^*, y) \mid y \in Y \}. \]  

(5)

We emphasize that the system (4) or (5) is a necessary and sufficient condition for the solution to problem (1),(2).

Rewriting variational inequalities from (4) in the form of operator equations, we present the problem as a system

\[ x^* \in \text{Argmin}\{ f(x) + (p^*, g(x) - y^*)) \mid x \in X \}, \]

\[ p^* = \pi_+ (p^* + \alpha (g(x^*) - y^*)) \],

\[ y^* = \pi_Y (y^* + \alpha p^*), \alpha > 0. \]

It is reasonable to regularize it and write out in the equivalent form of the proximal operator. The system then takes the form

\[ x^* \in \text{Argmin}\left\{ \frac{1}{2} \| x - x^* \|^2 + \alpha (f(x) + (p^*, g(x) - y^*)) \right\} \mid x \in X \}, \]

\[ p^* = \pi_+ (p^* + \alpha (g(x^*) - y^*)) \],

\[ y^* = \pi_Y (y^* + \alpha p^*). \]  

(6)

C. The dual extraproximal method

We apply a dual form of extraproximal method to solve (6):

\[ p^k = \pi_+ (p^k + \alpha (g(x^k) - y^k)), \]

\[ y^{k+1} = \pi_Y (y^k + \alpha p^k), \]

\[ x^{k+1} = \text{Argmin}\left\{ \frac{1}{2} \| x - x^k \|^2 + \alpha (f(x) + (p^k, g(x) - y^k)) \right\} \mid x \in X \}, \]

\[ p^{k+1} = \pi_+ (p^k + \alpha (g(x^{k+1}) - y^{k+1})). \]

(7)

The theorem of convergence of (7) to the solution was proved.

**Theorem 1.** If a solution of equilibrium problem (1),(2) exists, functions \( f(x), g(x) \) are convex, function \( g(x) \) is subject to Lipschitz condition, \( X, Y \) are convex closed sets then sequence \( (x^k, y^k; p^k) \) of dual extraproximal method (7) with parameter \( \alpha \) satisfying the condition \( 0 < \alpha < \min\{1/(2|g|), 1/2\} \) converges monotonically in norm to one of solutions to the problem, i.e., \( (x^k, y^k; p^k) \to (x^*, y^*; p^*) \) as \( k \to \infty \) for all \( (x^0, y^0; p^0) \).

III. TERMINAL CONTROL BY BOUNDARY-VALUE PROBLEM OF SENSITIVITY FUNCTION MINIMIZATION

A. The problem formulation

We formulate a dynamic model of terminal control with boundary-value problem of minimizing the sensitivity function under constraints

\[ x^*_1 \in \text{Argmin}\{ f(x_1) \mid g(x_1) \leq y^*_1, x_1 \in X_1 \}, \]

\[ p^*_1 \in \text{Argmax}\{ (p_1, g(x^*_1) - y^*_1) \mid p_1 \geq 0 \}, \]

\[ y^*_1 \in \text{Argmax}\{ (p^*_1, y) \mid y \in Y \}, \]

\[ \frac{d}{dt} x(t) = D(t) x(t) + B(t) u(t), \quad t_0 \leq t \leq t_1, \]

\[ x(t_0) = x_0, \quad x(t_1) = x^*_1 \in X_1 \subset \mathbb{R}^n, \quad u(\cdot) \in U, \]  

where the set of admissible controls is supposed to be integrally bounded:

\[ U = \{ u(\cdot) \in L^2_{\{t_0, t_1\}} \mid \| u(\cdot) \|^2_{L^2} \leq C \}. \]

In linear differential system, for any control \( u(\cdot) \in U \) there exists a unique trajectory \( x(\cdot) \), which belongs to linear variety of absolutely continuous functions \( AC^0[t_0, t_1] \).

Here \( X_1 \) is the attainability set, \( Y = \{ y \geq 0 \mid \| g(y) \| \leq d, \}

\( d \in \mathbb{R}^m_+ \} \) is a convex closed set in space of parameters, \( f(x_1) \) stands for a terminal convex differentiable function, \( D(t), B(t) \) are continuous matrices, \( x(t), u(t) \) are trajectory and control in the differential system, \( x_0, x^*_1 \in \mathbb{R}^n \) are initial and terminal conditions of the system. Below we consider the case \( g(x_1) = A_1 x_1 - y^*_1 \). Under the conditions of the problem, the right-hand end \( x^*(t_1) = x^*_1 \) is a solution of terminal problem and, at the same time, an element of the attainability set. Vector \( p_1 \) is a dual variable in finite-dimensional terminal problem, \( p^*_1 \) is a vector of Lagrange multipliers, pair \( (p^*_1, x^*_1) \) is a saddle point for finite-dimensional terminal problem.

B. The Lagrange function for terminal control problem

The system (8) can be regarded as an analogue of a convex programming problem formulated in functional space. Because this system satisfies the Slater regularity condition, the Lagrange function for this problem will always have a saddle point. We write out the dynamic Lagrangian for (8):

\[ L(p_1, \psi(\cdot); y, x_1, x(\cdot), u(\cdot)) = f(x_1) + \langle p_1, A_1 x_1 - y \rangle \]

\[ + \int_{t_0}^{t_1} \psi(t), D(t) x(t) + B(t) u(t) - \frac{d}{dt} x(t) dt \]

defined for all \( (p_1, \psi(\cdot)) \in \mathbb{R}^m_+ \times \Psi_2[t_0, t_1], \quad (y, x_1, x(\cdot), u(\cdot)) \in Y \times X_1 \times AC^0[t_0, t_1] \times U \). Here \( \Psi_2[t_0, t_1] \) is a linear variety of absolutely continuous functions from the conjugate space. Note that \( y \in \mathbb{R}^m_+ \) is variable, not just a parameter.

In the case of regular constraints, the Lagrange function always has a saddle point \( (p^*_1, \psi^*(\cdot); y^*_1, x^*_1, x^*(\cdot), u^*(\cdot)) \), which satisfies the saddle-point inequalities

\[ L(p_1, \psi(\cdot); y^*_1, x^*_1, x^*(\cdot), u^*(\cdot)) \leq \]

\[ \leq L(p^*_1, \psi^*(\cdot); y^*_1, x^*_1, x^*(\cdot), u^*(\cdot)) \]
C. Method for solving

This system is equivalent to the original problem.

Duality extraproximal iterative method \([6,10]\) has the form:

To solve system (9), we use a dynamic analogue of simple \(\psi\) iteration method:

\[
\begin{align*}
\psi^k(t) &= \psi^k(0) + \alpha(D(t)x^k(t) + B(t)u^k(t) - \frac{dt}{dt} x^k(t)), \\
y_1^{k+1} &= \pi_Y(y_1^k + \alpha p_1^k), \\
p_1^{k+1} &= \pi_+(p_1^k + \alpha(A_1 x_1^k - y_1^k)), \\
\psi^{k+1}(t) &= \psi^k(t) + \alpha(D(t)x^k(t) + B(t)u^k(t) - \frac{dt}{dt} x^k(t)), \quad \alpha > 0.
\end{align*}
\]

This system is equivalent to the original problem.

C. Method for solving

To solve system (9), we use a dynamic analogue of simple iteration method:

\[
\begin{align*}
(x_{k+1}^1(t_1), x_{k+1}^2(t_1), u_{k+1}^1(t_1)) &\in \\
\text{Argmin} \left\{ f(x(t)) + \langle p_1^k, A_1 x(t_1) - y_1^k \rangle \right. \\
+ \int_{t_0}^{t_1} \langle \psi^k(t), D(t)x(t) + B(t)u(t) - \frac{dt}{dt} x(t) \rangle dt \right. \\
- \langle \pi_Y(y_1^k + \alpha p_1^k), x_{k+1}^1(t_1) - y_1^k \rangle, \\
+ \int_{t_0}^{t_1} \langle \psi^k(t), D(t)x(t) + B(t)u(t) - \frac{dt}{dt} x(t) \rangle dt \right\}, \\
\end{align*}
\]

where the minimum is sought in all the variables \((x(t_1), x(t), u(t_1)) \in X_1 \times AC^n[0,t_1] \times U.\)

Theorem 2 (on convergence of the method). Suppose that the solution \((p_1^k, \psi^k(.); y_1^k, x_1^k, u^k(.))\) of equilibrium problem (8) exists, the boundary value problem is convex and regular (Lagrangian function has saddle points), \(Y_1 \subset \mathbb{R}^{+}\), \(Y_1\) is a closed convex set. Then:

1) the dual method sequence \(\langle p_1^k, \psi^k(.); y_1^k, x_1^k, u^k(.), u^k(.) \rangle\) with the parameter \(\alpha\), satisfying condition \(0 < \alpha < \min(1/(2||A||)), 1/2\) has a subsequence \((p_1^k, \psi^k(.); y_1^k, x_1^k, u^k(.), u^k(.))\), which converges to the solution \(k \to \infty\), including the weak convergence in controls, the strong convergence in trajectories and conjugate trajectories, as well as in all terminal variables; 2) the sequence

\[
\left| x_{1}^{k}(t_1) - x_{1}^{0}(t_1) \right|^2 + \left| y_{1}^{k} - y_{1}^{0} \right|^2 + \left| p_{1}^{k} - p_{1}^{0} \right|^2 \\
+ \left| x_{1}^{k}(t_1) - x_{1}^{0}(t_1) \right|^2 + \left| u_{1}^{k} - u_{1}^{0} \right|^2 + \left| \psi_{1}^{k} - \psi_{1}^{0} \right|^2
\]

is monotone decreasing for all initial values \((p_1^0, \psi^0(.); y_1^0, x_0(t_1), u_0(\cdot))\).

IV. CONCLUSION

The paper deals with the dynamic equilibrium terminal problem which contains two components: a controlled linear dynamics and boundary-value problem in the form of equilibrium extremal problem. To solve this problem the new approach based on the idea of duality was proposed. This approach allows to construct saddle-point methods for solving this kind of equilibrium dynamic problems. Convergence of the method in all components of the primal and dual solutions was proved.

ACKNOWLEDGMENT

The author would like to thank RFBR (project 15-01-06045)

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