AUTOMATED CORRESPONDENCE ANALYSIS FOR THE BINARY EXTENSIONS OF THE LOGIC OF PARADOX

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Abstract. B. Kooi and A. Tamminga present a correspondence analysis for extensions of G. Priest's logic of paradox. Each unary or binary extension is characterizable by a special operator and analyzable via a sound and complete natural deduction system. The present paper develops a sound and complete proof searching technique for the binary extensions of the logic of paradox.

§1. The logic of paradox and correspondence analysis. In (Kooi & Tamminga, 2012), the authors present a uniform method (entitled as *correspondence analysis* following the notion of correspondence theory from modal logic[1]) to set up natural deduction systems for the unary or binary truth-functional extensions of the logic of paradox (LP) (Asenjo, 1954, 1966; Priest 1979, 1984, 2002). They extensively use the fact that LP is not functionally complete. The method has been extended to the strong three-valued logic K_3 (Kleene, 1938, 1952) in (Tamminga, 2014) and to the relevant logic FDE (Anderson & Belnap, 1975, 1992) in (Tamminga, 2016; Petrukhin, 2016).

The authors put the idea of this method as follows:

> "We show that for every single entry E in a truth table f for a unary or a binary operator there is an inference scheme Π/ϕ such that E is an entry in f if and only if Π/ϕ is valid according to f. As a consequence, each truth table for a unary (or binary) operator can be characterized in terms of three (or nine) inference schemes. Moreover, adding the inference schemes that characterize a truth table f as derivation rules to a natural deduction system for LP yields a natural deduction system which is sound and complete with respect to the semantics that also contains, next to LP's truth-tables for $\neg$, $\vee$, and $\vee$, the truth table f." (Kooi & Tamminga, 2012, p. 721)

Now let us briefly introduce LP. We stick to the notation in (Kooi & Tamminga, 2012). The language $\mathcal{L}$ of LP is a standard propositional language with a countable set $\mathcal{P}$ of propositional variables $\{p_1, \ldots, p_n, \ldots\}$ and $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). For the purpose of this paper, $\mathcal{L}$ does not

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[1]See (Sahlqvist, 1975; van Benthem, 1976, 2001) for details about modal correspondence theory.

include $\rightarrow$ (implication). Nevertheless, one may define it in a standard way: $\phi \rightarrow \psi := \neg\phi \vee \psi$. The notions of both a well-formed formula and a valuation v from $\mathcal{P}$ to the set $\{1, i, 0\}$ of truth-values "true", "both", and "false" are as usual. A literal is a (negation of a) propositional variable. LP-connectives are defined by the following truth tables:

	$f_\neg$
1	0
i	i
0	1

$f_\vee$	1	i	0
1	1	1	1
i	1	i	i
0	1	i	0

$f_\wedge$	1	i	0
1	1	i	0
i	i	i	0
0	0	0	0

LP-consequence relation ($\Pi \models \phi$) is defined as follows: if $v(\psi) \neq 0$, for all $\psi \in \Pi$, then $v(\phi) \neq 0$, for each valuation v. LP-tautology is a formula ϕ iff $v(\phi) \neq 0$, for each valuation v. Note that one obtains the strong logic K_3 (Kleene, 1938, 1952), if one defines $\Pi \models \phi$ as follows: if $v(\psi) = 1$, for all $\psi \in \Pi$, then $v(\phi) = 1$, for each valuation v. In other words, LP is K_3 with two designated values and K_3 is LP with one designated value. The idea to consider K_3 with two designated values *as a paraconsistent logic* first appeared in (Asenjo, 1954, 1966). Later under the name of LP it was studied in (Priest, 1979, 1984, 2002). Note also that Resher (1969) proved that the class of LP-tautologies coincides with the class of classical ones[2]. However, LP-consequence relation differs from a classical one: for example, $\phi \wedge (\neg\phi \vee \psi) \not\models \psi$ in LP.

Let $\mathcal{L}_{(\sim)m(\circ)n}$ be an extension of $\mathcal{L}$ by unary $\sim_1, \ldots, \sim_m$ and binary $\circ_1, \ldots, \circ_n$ operators. Let us denote a logic built in $\mathcal{L}_{(\sim)m(\circ)n}$ by $LP_{(\sim)m(\circ)n}$. Thus, truth tables of $LP_{(\sim)m(\circ)n}$-connectives are $f_\neg$, $f_\wedge$, $f_\vee$, $f_{\sim_1}, \ldots, f_{\sim_m}$, $f_{\circ_1}, \ldots, f_{\circ_n}$. In a particular case, LP might be extended only by binary operators $\circ_1, \ldots, \circ_n$. So we obtain a logic $LP_{(\circ)n}$ in a language $\mathcal{L}_{(\circ)n}$.

The definition of a *single entry correspondence* is as follows:

Definition 1.1 (Kooi & Tamminga, 2012). Let $\Pi \cup \{\phi\} \subseteq \mathcal{L}_{(\sim)m(\circ)n}$. Let $x, y, z \in \{0, i, 1\}$. Let E be a truth table entry of the type $f_\sim(x) = y$ or $f_\circ(x, y) = z$. Then the truth table entry E is characterized by an inference scheme Π/ϕ, if

$$E \text{ if and only if } \Pi \models \phi.$$

Kooi and Tamminga introduce inference schemes which characterize all possible entries in truth-tables $f_\sim$ and $f_\circ$. In the paper, we confine ourselves to the case of $f_\circ$ only (see theorem 2.3 in section 2).

The paper is organised as follows. The rest of the section 1 is devoted to the interesting extentions of LP. The 2^{nd} section is about natural deduction systems for LP and its binary extensions. Proof search procedure is described in the 3^{rd} section. Algo-proof examples are in the section 4. Soundness, completenes and termination are shown in the section 5. The 6^{th} section contains concluding remarks and future work.

Implicational extensions of *LP*. In this section, we will try to clarify possible interpretations of $\circ$. It seems to be quite natural to consider $\circ$ as an implication. Of course, it is not the only one possible interpretation of $\circ$. One may understand $\circ$ as an equivalence, the Sheffer's stroke or any other binary connective. However, in this paper, we stick to the interpretation of $\circ$ as a

[2] See also (Martin, 1975) and (Epstein, 1990).

conditional operator. Kooi and Tamminga point out to some well-known logics resulted from adding binary operators to *LP*: RM_3 (Anderson & Belnap, 1975) and J_3 (D'Ottaviano & da Costa, 1970; Epstein & D'Ottaviano, 2000). We may also add the following logics mentioned in the literature: *PCont* (which occurs under different names in the literature), *LFI*1 & *LFI*2 (Carnielli, Marcos & Amo, 2000), and $LP_{\Rightarrow}$ (Thomas, 2013). Moreover, we add logics with *LP*'s connections and implications from Heyting's (1930) G_3 and Sette's (1973) P_1. At last, we discuss a class of logics with so called a natural implication (Tomova, 2012).

$f_{\to_1}$	1	i	0
1	1	i	0
i	1	1	0
0	1	1	1

$f_{\to_2}$	1	i	0
1	1	i	0
i	1	i	0
0	1	1	1

$f_{\to_3}$	1	i	0
1	1	0	0
i	1	i	0
0	1	1	1

$f_{\to_4}$	1	i	0
1	1	0	0
i	1	1	0
0	1	1	1

$f_{\to_5}$	1	i	0
1	1	1	0
i	1	1	0
0	1	1	1

$f_{\to_6}$	1	i	0
1	1	1	0
i	1	i	0
0	1	1	1

Heyting's implication $\to_1$. Although Heyting's logic G_3 (Heyting, 1930; Gödel, 1932; Jaśkowski, 1936) does not have the same concept of consequence relation as *LP*, G_3's implication $\to_1$ has one nice property: it verifies modus ponens with one and with two designated values. Recall that *LP*'s own implication defined as $\neg\phi \vee \psi$ fails to verify modus ponens. Notice that Batens (1980) considers K_3's extension by $\to_1$ (G_3's conjunction and disjunction are the same as K_3's and *LP*'s ones, but G_3's negation differs from K_3's and *LP*'s ones).

Jaśkowski's implication $\to_2$. Jaśkowski's 1948 paper is well-known as one of the first works dedicated to paraconsistency. Jaśkowski suggested several paraconsistent logics, and one of them has $\to_2$ as a conditional operator. Later this implication appeared in (Asenjo & Tamburino, 1975; Battens, 1980; Rozonoer, 1983a, 1983b; Avron, 1986), where *LP* with $\to_2$ is considered. Following (Rozonoer, 1983a, 1983b), we call this logic *PCont*. Besides, $\to_2$ appeared in D'Ottaviano & da Costa's (1970) J_3, Batens' (1989) *CLuNs* and Carnielli, Marcos & Amo's (2000) *LFI*1.

Sobociński's implication $\to_3$. In 1952, Sobociński presented $\to_3$ as an implication which is free from so called paradoxes of the classical conditional. Conjunction and disjunction in Sobociński's logic are defined via $\neg$ and $\to_3$; consequence relation is the same as in *LP*. It seems to be natural to replace Sobociński's conjunction and disjunction by more familiar *LP*'s ones. Such a logic (under the name of RM_3) is mentioned in (Anderson & Belnap, 1975).

Rescher's implication $\to_4$. This implication came from Rescher's logic (Rescher, 1969) which, unlike *LP*, has the only one designated value. Moreover, it is not an extension of K_3. However, $\to_4$ verifies modus ponens with one and with two designated values, likewise $\to_1$. Recently Thomas (2013) proposed *LP*'s extension by $\to_4$ without referring to (Rescher, 1969).

Sette's implication $\to_5$. Implication of Sette's (1973) paraconsistent logic P_1 is another fine candidate to be an extension of *LP*. P_1 has two designated values and its truth tables for negation, conjunction, and disjunction differ from *LP*'s ones. Notice that P_1 was analysed in (da Costa, & Alves, 1981; Mortenson, 1989; Popov, 1999). Besides, $\to_5$ is an implication of the logic P_2 (Marcos, 2005).

Carnielli, Marcos & Amo's implication $\to_6$. Besides abovementioned *LFI*1, Carnielli, Marcos & Amo (2000) consider also the logic *LFI*2 which consists of *LP*'s negation, an unary operator of inconsistency, conjunction and disjunction (which differ from *LP*'s ones), and an implication $\to_6$.

Tomova's natural implications. We have considered six interesting implications from various logics which can be added to *LP*. Now let us describe the special class of implications which can be added to *LP*.

Following Tomova (2012, p. 175), we call an implication $\to$ *natural*, iff the following conditions are fulfilled ($V_3 = \{1, i, 0\}$ and D is a set of designated values):

1. **C**-extending, i.e. restrictions to the subset $\{0, 1\}$ of V_3 coincide with the classical implication.
2. If $x \to y \in D$ and $x \in D$, then $y \in D$, i.e. matrices for implication need to be normal in the sense of Łukasiewicz-Tarski (1930) (they verify modus ponens).
3. Let $x \leq y$, then $x \to y \in D$;
4. $x \to y \in V_3$, in other cases.

Tomova (2012) showed that in three-valued logics with two designated values there are 24 natural implications:

$f_\to$	1	i	0
1	1	b	0
i	a	a	0
0	1	a	1

where $a \in \{1, i\}$ and $b \in \{1, i, 0\}$. Note these implications are not natural in the sense of (Avron, 1991). Note also among these implications are those which have already been mentioned in the literature. For example, all implications $\to_i$ ($i \in \{1, \ldots, 6\}$) discussed above are natural.

Later on following (Rescher, 1969), Tomova (2015a, 2015b) extended the class of natural implications by making the distinction between a stronger and a weaker versions of modus ponens. The stronger version is designation-preserving and the weaker one is tautology-preserving. As follows from (Tomova, 2015b), if we replace the condition (2) in the definition of natural implication by the weaker version of modus ponens (if $\models x \to y$ and $\models x$, then $\models y$) then we obtain 16 new natural implications (if $D = \{1, i\}$):

$f_\to$	1	i	0
1	1	i	0
i	a	a	a
0	1	1	1

$f_\to$	1	i	0
1	1	i	0
i	a	a	i
0	1	i	1

$f_\to$	1	i	0
1	1	b	0
i	1	1	1
0	1	1	1

$f_\to$	1	i	0
1	1	0	0
i	a	a	a
0	1	1	1

$f_\to$	1	i	0
1	1	1	0
i	1	1	1
0	1	a	1

One may add any of these implications to LP and set up a proof system for this extension of LP via correspondence analysis.

§2. Natural deduction systems for LP and its extensions. Kooi & Tamminga introduce the following rules of ND_{LP}[3]. We divide them into elimination and introduction rules.

Elimination rules:

$$(\wedge E_1)\ \frac{\phi \wedge \psi}{\phi} \qquad (\wedge E_2)\ \frac{\phi \wedge \psi}{\psi} \qquad (\vee E)\ \frac{\phi \vee \psi \quad \begin{matrix}[\phi] \\ \chi\end{matrix} \quad \begin{matrix}[\psi] \\ \chi\end{matrix}}{\chi}$$

$$(\neg\neg E)\ \frac{\neg\neg\phi}{\phi} \qquad (\neg\vee E)\ \frac{\neg(\phi \vee \psi)}{\neg\phi \wedge \neg\psi} \qquad (\neg\wedge E)\ \frac{\neg(\phi \wedge \psi)}{\neg\phi \vee \neg\psi}$$

Introduction rules:

$$(\wedge I)\ \frac{\phi \quad \psi}{\phi \wedge \psi} \qquad (\vee I_1)\ \frac{\phi}{\phi \vee \psi} \qquad (\vee I_2)\ \frac{\psi}{\phi \vee \psi} \qquad (\neg\neg I)\ \frac{\phi}{\neg\neg\phi}$$

$$(\neg\vee I)\ \frac{\neg\phi \wedge \neg\psi}{\neg(\phi \vee \psi)} \qquad (\neg\wedge I)\ \frac{\neg\phi \vee \neg\psi}{\neg(\phi \wedge \psi)} \qquad (EM)\ \frac{}{\phi \vee \neg\phi}$$

There are following reasons to consider $(\neg\vee E)$ and $(\neg\wedge E)$ ($(\neg\vee I)$ and $(\neg\wedge I)$) as elimination (introduction) rules. For example, $(\neg\vee E)$ might be replaced with $(\neg\vee E_1)\ \frac{\neg(\phi \vee \psi)}{\neg\phi}$ and $(\neg\vee E_2)\ \frac{\neg(\phi \vee \psi)}{\neg\psi}$; and $(\neg\wedge I)$ might be replaced with $(\neg\wedge I_1)\ \frac{\neg\phi}{\neg(\phi \wedge \psi)}$ and $(\neg\wedge I_2)\ \frac{\neg\psi}{\neg(\phi \wedge \psi)}$. We, again, recall that we stick to (Kooi & Tamminga, 2012).

In contradistinction to (Kooi & Tamminga, 2012), we define a natural deduction derivation in a linear ("Fitch-style") format.[4]

Definition 2.1. A derivation in ND_{LP} of a formula ϕ from a set of assumptions Π is a finite non-empty sequence of formulae with the following conditions:

- Each formula is a assumption or follows from the previous formulae via a ND_{LP}-rule;
- By applying $(\vee E)$ each formula starting from the assumption ϕ until a formula χ, inclusively, as well as each formula starting from the assumption ψ until a formula χ, inclusively, is *discarded* from the derivation.

Definition 2.2. A proof in ND_{LP} is a derivation from the empty set of assumptions.

Natural deduction systems $ND_{LP_{(\circ)n}}$ for extensions of LP. To obtain natural deduction systems for extensions of LP, the authors introduce inference schemes which characterize all possible entries in truth-tables for $f_\circ$ [Kooi & Tamminga 2012, p. 722-723].

Theorem 2.3 (Kooi & Tamminga, 2012). *Let $\phi, \psi, \chi \in \mathcal{L}_{(\circ)n}$. Then:*

[3] This system was first formulated in (Priest, 2002). An alternative natural deduction system for LP can de found in (Roy, 2006).

[4] This definition is a standard one: see, for example (Copi, Cohen, McMahon, 2011, p. 366).

$$f_\circ(0,0) = \begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \phi \vee \psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\phi \vee \psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \phi \vee \psi \end{cases}$$

$$f_\circ(0,i) = \begin{cases} 0 & \textit{iff} \quad \psi \wedge \neg\psi, \phi \circ \psi \models \phi \\ i & \textit{iff} \quad \psi \wedge \neg\psi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \phi \\ 1 & \textit{iff} \quad \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \phi \end{cases}$$

$$f_\circ(0,1) = \begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \phi \vee \neg\psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\phi \vee \neg\psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \phi \vee \neg\psi \end{cases}$$

$$f_\circ(i,0) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \phi \circ \psi \models \psi \\ i & \textit{iff} \quad \phi \wedge \neg\phi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \psi \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \neg(\phi \circ \psi) \models \psi \end{cases}$$

$$f_\circ(i,i) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi, \phi \circ \psi \models \chi \\ i & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi \models (\phi \circ \psi) \wedge \neg(\phi \circ \psi) \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \chi \end{cases}$$

$$f_\circ(i,1) = \begin{cases} 0 & \textit{iff} \quad \phi \wedge \neg\phi, \phi \circ \psi \models \neg\psi \\ i & \textit{iff} \quad \phi \wedge \neg\phi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\psi \\ 1 & \textit{iff} \quad \phi \wedge \neg\phi, \neg(\phi \circ \psi) \models \neg\psi \end{cases}$$

$$f_\circ(1,0) = \begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \neg\phi \vee \psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\neg\phi \vee \psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \neg\phi \vee \psi \end{cases}$$

$$f_\circ(1,i) = \begin{cases} 0 & \textit{iff} \quad \psi \wedge \neg\psi, \phi \circ \psi \models \neg\phi \\ i & \textit{iff} \quad \psi \wedge \neg\psi \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee \neg\phi \\ 1 & \textit{iff} \quad \psi \wedge \neg\psi, \neg(\phi \circ \psi) \models \neg\phi \end{cases}$$

$$f_\circ(1,1) = \begin{cases} 0 & \textit{iff} \quad \phi \circ \psi \models \neg\phi \vee \neg\psi \\ i & \textit{iff} \quad \models ((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\neg\phi \vee \neg\psi) \\ 1 & \textit{iff} \quad \neg(\phi \circ \psi) \models \neg\phi \vee \neg\psi \end{cases}$$

In (Kooi & Tamminga, 2012), ND_{LP} and $ND_{LP_{(\circ)n}}$ are shown to be sound and complete.

Example.

A proof of $\phi \circ \psi$ in the system with $R_\circ(0,0,1)$, $R_\circ(1,0,1)$, $R_\circ(1,1,1)$, $R_\circ(i,0,1)$, $R_\circ(i,1,1)$, $R_\circ(i,i,1)$, and $R_\circ(1,i,1)$.[5]

(1)	$(\phi \circ \psi) \vee \neg(\phi \circ \psi)$	(EM)
(2)	$\neg(\phi \circ \psi)$	assumption
(3)	$\phi \vee \psi$	$R_\circ(0,0,1)$: 2
(4)	$\neg\phi \vee \psi$	$R_\circ(1,0,1)$: 2
(5)	$\neg\phi \vee \neg\psi$	$R_\circ(1,1,1)$: 2
(6)	ϕ	assumption
(7)	$\neg\phi$	assumption
(8)	$\phi \wedge \neg\phi$	$(\wedge I)$: 6, 7
(9)	ψ	$R_\circ(i,0,1)$: 8, 2
(10)	$\neg\psi$	$R_\circ(i,1,1)$: 8, 2

[5]Discarded formulae are put in square brackets, after the analysis of a proper inference step.

(11)	$\psi \wedge \neg\psi$	$(\wedge I)$: 9, 10
(12)	$\phi \circ \psi$	$R_\circ(i, i, 1)$: 11, 8, 2
(13)	ψ	assumption
(14)	$\neg\phi$	assumption
(15)	$\phi \wedge \neg\phi$	$(\wedge I)$: 6, 14
(16)	$\neg\psi$	$R_\circ(i, 1, 1)$: 15, 2
(17)	$\psi \wedge \neg\psi$	$(\wedge I)$: 16, 13
(18)	$\phi \circ \psi$	$R_\circ(i, i, 1)$: 17, 15, 2
(19)	$\neg\psi$	assumption
(20)	$\psi \wedge \neg\psi$	$(\wedge I)$: 13, 19
(21)	$\neg\phi$	$R_\circ(1, i, 1)$: 20, 2
(22)	$\phi \wedge \neg\phi$	$(\wedge I)$: 6, 21
(23)	$\phi \circ \psi$	$R_\circ(i, i, 1)$: 21, 20, 2
(24)	$\phi \circ \psi$	$(\vee E)$: 19, 14, 5, [19-23], [14-18]
(25)	$\phi \circ \psi$	$(\vee E)$: 13, 7, 4, [13-24], [7-12]
(26)	ψ	assumption
(27)	$\neg\phi$	assumption
(28)	$\phi \wedge \neg\phi$	$(\wedge I)$: 6, 27
(29)	$\neg\psi$	$R_\circ(i, 1, 1)$: 28, 2
(30)	$\psi \wedge \neg\psi$	$(\wedge I)$: 29, 26
(31)	$\phi \circ \psi$	$R_\circ(i, i, 1)$: 30, 28, 2
(32)	$\neg\psi$	assumption
(33)	$\psi \wedge \neg\psi$	$(\wedge I)$: 32, 26
(34)	$\neg\phi$	$R_\circ(1, i, 1)$: 33, 2
(35)	$\phi \wedge \neg\phi$	$(\wedge I)$: 34, 6
(36)	$\phi \circ \psi$	$R_\circ(i, i, 1)$: 35, 33, 2
(37)	$\phi \circ \psi$	$(\vee E)$: 27, 32, 5, [27-31], [32-36]
(38)	$\phi \circ \psi$	$(\vee E)$: 6, 26, 3, [6-25], [26-37]
(39)	$\phi \circ \psi$	assumption
(40)	$\phi \circ \psi$	$(\vee E)$: 2, 39, 1, [2-38], [39]

§3. Proof search for ND_{LP} and $ND_{LP_{(\circ)n}}$. To the best of our knowledge, there are no papers concerning *natural* proof searching for *LP*. In the paper, we follow an original approach to proof searching for natural deduction systems in classical and non-classical logics (Bolotov, Basukoski, Grigoriev & Shangin, 2006), (Bolotov, Bocharov, Gorchakov & Shangin, 2009). In treating *LP*-connectives $\neg$, $\wedge$ and $\vee$, we use proof searching for the paraconsistent logic *PCont* (Bolotov & Shangin, 2012). Therefore, in proof searching for each *binary* extension of *LP* we are left to deal with the derivation rules which are the characterizing inference schemes for $f_{\circ_1}, \ldots, f_{\circ_n}$.

Dealing with all the systems in question, we keep in mind that we always search for a proof in a particular system.

The proof searching procedure is a *goal-directed* one and generates two sequences: $list\_proof$ and $list\_goals$. The first sequence contains either a derivation (if it can be found) or a counterexample (if a derivation can't be found). The second sequence stacks goals and starts with the $initial\_goal$ (the statement we desire to prove). The proof searching procedures described below define an *algo-derivation* for ND_{LP} and $ND_{LP_{(\circ)n}}$ (we abbreviate it as ALG_{LP} and $ALG_{LP_{(\circ)n}}$). At each step of a proof search, we choose a specific goal (the $current\_goal$), which we aim to reach. Note a goal is always a formula.

Suppose we are tasked to find an algo-derivation of ϕ from Π in some $ND_{LP_{(\circ)n}}$. The notation "$\Gamma \vdash \Delta$", where $\Gamma \subseteq \Pi$ and ϕ is the first constituent of Δ (i.e. ϕ is the $initial\_goal$), should be read as follows: Γ and Δ are the $list\_proof$ and the $list\_goals$, correspondingly, of an algo-derivation of ϕ from Π in some $ND_{LP_{(\circ)n}}$ *at the current step of a proof search.* In particular, "$\Gamma, \psi \vdash \Delta, \chi$" is supposed to mean that ψ is a constituent of Γ and χ is the last constituent of Δ (i.e. χ is the $current\_goal$).

Definition 3.1. Let $\Gamma, \psi \vdash \Delta, \psi$. A $current\_goal$ ψ is said to be *reached* if ψ isn't discarded from the $list\_proof$.

The idea of the $current\_goal$ ψ's reachability is that it occurs (as an assumption or being inferred) in the $list\_proof$. However, it mustn't be discarded for sometimes the same formula reappears as the $current\_goal$. While proof searching we check the reachability of the $current\_goal$. If successful we apply the appropriate introduction rule. If unsuccessful, both $list\_proof$ and $list\_goals$ are updating.

Definition 3.2. $\Gamma \vdash \Delta$ is said to *lead* to $\Gamma' \vdash \Delta'$ if an algorithm starts at a moment with $\Gamma \vdash \Delta$ and stops at a moment with $\Gamma' \vdash \Delta'$.

Proof searching procedures. In describing proof searching procedures we follow (Bolotov & Shangin, 2012, pp. 8–10) with considerable modifications.

Procedure 1 (P1). The elimination rules are applied following P1. The $list\_proof$ is added with a conclusion of an elimination rule if the $list\_proof$ contains premise(s) of this rule. P1 is supplemented with marks. Once a rule is applied, both its premise(s) is/are marked to avoid infinite applications of the same rule to the same formula. If the conclusion of the rule is discarded from the $list\_proof$ then its premises are unmarked and another application of this rule to the same formula is possible.

Procedure 2 (P2). P2 checks whether the $current\_goal$ is reached. A reached goal is deleted from the $list\_goals$, and the previous goal becomes the current one.

Procedure 3 (P3). P3 launches if P1 stops and one doesn't reach the $current\_goal$. P3 consists of two subprocedures: P3.1 and P.3.2. P3.1 deals with the $current\_goal$ in the $list\_goals$ whereas P3.2 deals with special compound formulae in the $list\_proof$.

Procedure 3.1 (P3.1). P3.1 analyses the $current\_goal$. If one doesn't reach the $current\_goal$ then a new current_goal is added to the $list\_goals$ and some formulae are, possibly, added to the $list\_proof$ as follows:

P3.1.1. If $\Gamma \vdash \Delta, \phi \wedge \psi$ and $\phi \wedge \psi$ isn't reached then $\Gamma \vdash \Delta, \phi, \psi$ (if a conjunctive formula isn't reached we search for both of its conjuncts *one by one* starting from the right one).

P3.1.2.1. If $\Gamma \vdash \Delta, \phi \vee \psi$ and $\phi \vee \psi$ isn't reached then $\Gamma \vdash \Delta, \phi$ (if a disjunctive formula isn't reached we search for its left disjunct).

P3.1.2.2. If $\Gamma \vdash \Delta, \phi$ and ϕ isn't reached then $\Gamma \vdash \Delta, \psi$ (if the left disjunct isn't reached we delete the left one from the *list_goals* and search for its right one).[6]

P3.1.2.3. If $\Gamma \vdash \Delta, \psi$ and ψ isn't reached then $\Gamma^*, (\phi \vee \psi) \vee \neg(\phi \vee \psi), \neg(\phi \vee \psi) \vdash \Delta, \phi \vee \psi, \phi \vee \psi$, where $\Gamma^* = \Gamma \cup \{\neg(\phi \vee \psi)\}$ (if no disjunct is reached we delete the right one and search for the disjunctive formula by refutation, i.e. adding $(\phi \vee \psi) \vee \neg(\phi \vee \psi)$ with the mark by the rule (EM) and an assumption $\neg(\phi \vee \psi)$ in the *list_proof* and setting $\phi \vee \psi$ as a *new current_goal*).[7]

P3.1.3. If $\Gamma \vdash \Delta, \neg(\phi \wedge \psi)$ and $\neg(\phi \wedge \psi)$ isn't reached then $\Gamma \vdash \Delta, \neg\phi \vee \neg\psi$ (if $\neg(\phi \wedge \psi)$ isn't reached we search for $\neg\phi \vee \neg\psi$).

P3.1.4. If $\Gamma \vdash \Delta, \neg(\phi \vee \psi)$ and $\neg(\phi \vee \psi)$ isn't reached then $\Gamma \vdash \Delta, \neg\phi \wedge \neg\psi$ (if $\neg(\phi \vee \psi)$ isn't reached we search for $\neg\phi \wedge \neg\psi$).

P3.1.5. If $\Gamma \vdash \Delta, \neg\neg\phi$ and $\neg\neg\phi$ isn't reached then $\Gamma \vdash \Delta, \neg\neg\phi, \phi$ (if $\neg\neg\phi$ isn't reached we search for ϕ).

P3.1.6. If $\Gamma \vdash \Delta, \psi$, where ψ is a literal or $\phi \circ \psi$ or $\neg(\phi \circ \psi)$, and ψ isn't reached then $\Gamma^*, \psi \vee \neg\psi, \neg\psi \vdash \Delta, \psi, \psi$, where $\Gamma^* = \Gamma \cup \{\neg\psi\}$ (literals and formulae $\phi \circ \psi$, $\neg(\phi \circ \psi)$ are treated in the same way as P3.1.2.3 treats a formula $\neg(\phi \vee \psi)$).[8]

P3.1.7. If $\Gamma \vdash \Delta, \psi \vee \neg\psi$ and $\psi \vee \neg\psi$ isn't reached then $\Gamma, \psi \vee \neg\psi \vdash \Delta, \psi \vee \neg\psi$ (if $\psi \vee \neg\psi$ isn't reached we add it in the *list_proof* by (EM)).

Procedure 3.2 (P3.2). P3.2 launches if P3.1 stops and one doesn't reach the *current_goal*. In particular, P3.2 deals with *some* compound formulae in the *list_proof* trying to update the *list_goals* with new goals or the *list_proof* with new formulae. These new goals or formulae, roughly, are useful either in proving the *initial_goal* or in extracting a counterexample.

P3.2 is supplemented with marks. First, it deals with only those compound formulae in the *list_proof* which aren't marked by P1 (i.e. with those which the elimination rules haven't been applied to). Second, P3.2 has its own marking. Once it is applied, the compound formula is marked to avoid infinite applications of P3.2 to the same formula. If the result(s) of an application of P3.2 to the formula is/are discarded from the *list_proof* then the formula is unmarked and another application of P3.2 to it is possible.

[6]Note $\Gamma \vdash \Delta, \phi$ and $\Gamma \vdash \Delta, \psi$ differ with respect to the *current_goal* only. It means all the formulae from both the *list_proof* and the *list_goals* which have been inferred under P3.1.2.1 are deleted before launching P3.1.2.2. Note also that a term "deleted formula" shouldn't be confused with a term "discarded formula". The latter is a part of a derivation (hence, of a proof search) whereas the former is a part of a proof search only.

[7]Despite a new *current_goal* is the same formula $\phi \vee \psi$ it's not the same constituent in the *list_goals*. Now we are to reach $\phi \vee \psi$ with the presence of an assumption $\phi \vee \psi$ in order to reach then $\phi \vee \psi$ without this assumption.

[8]We don't unite P3.1.2.3 and P3.1.6 in one procedure in order to highlight the well-known problem of natural deduction with disjunction (Prawitz, 1965), (D'Agostino, 1990).

P3.2.1. If $\Gamma, \phi \vee \psi \vdash \chi$ and χ isn't reached then $\Gamma^*, \phi \vee \psi, \phi \vdash \Delta, \chi, \chi$, where $\Gamma^* = \Gamma \cup \{\phi\}$ (if a disjunctive formula is unmarked and χ isn't reached we add an assumption ϕ in the *list_proof* and set χ as a *new current_goal*).

P3.2.1.1. If $\Gamma^*, \phi \vee \psi, \phi \vdash \Delta, \chi, \chi$ and χ isn't reached then $\Gamma^*, \phi \vee \psi, \phi, \psi \vee \neg\psi, \psi \vdash \Delta, \chi, \chi, \chi$ (if χ isn't reached we add $\psi \vee \neg\psi$ with the mark by the rule (EM) and one more assumption ψ in the *list_proof* and set χ as a *new current_goal*).

P3.2.1.2. If $\Gamma^*, \phi \vee \psi, \phi, \psi \vee \neg\psi, \psi \vdash \Delta, \chi, \chi, \chi$ leads $\Gamma^*, \phi \vee \psi, \phi, \psi \vee \neg\psi, \psi, \chi \vdash \Delta, \chi, \chi, \chi$ then $\Gamma^{**}, \phi \vee \psi, \phi, \psi \vee \neg\psi, \neg\psi \vdash \Delta, \chi, \chi, \chi$, where $\Gamma^{**} = \Gamma \cup \{\neg\psi\}$ (if χ is reached we discard an assumption ψ, add an assumption $\neg\psi$ in the *list_proof*, delete the *current_goal* χ and set χ as a *new current_goal*).[9]

P3.2.1.3. If $\Gamma^{**}, \phi \vee \psi, \phi, \psi \vee \neg\psi, \neg\psi \vdash \Delta, \chi, \chi, \chi$ leads to $\Gamma^{**}, \phi \vee \psi, \phi, \psi \vee \neg\psi, \neg\psi, \chi \vdash \Delta, \chi, \chi, \chi$ then $\Gamma^*, \phi \vee \psi, \psi \vdash \Delta, \chi, \chi$, where $\Gamma^* = \Gamma \cup \{\psi\}$ (the *current_goal* χ is inferred from an assumption ϕ so we add an assumption ψ in the *list_proof* and set χ as a *new current_goal*).

P3.2.1.4-P3.2.1.5. are in the same fashion as P3.2.1.1-P3.2.1.2.

P3.2.1.4. If $\Gamma^*, \phi \vee \psi, \psi \vdash \Delta, \chi, \chi$ and χ isn't reached then $\Gamma^*, \phi \vee \psi, \psi, \phi \vee \neg\phi, \phi \vdash \Delta, \chi, \chi, \chi$ (if χ isn't reached we add $\phi \vee \neg\phi$ with the mark by the rule (EM) and one more assumption ϕ in the *list_proof* and set χ as a *new current_goal*).

P3.2.1.5. If $\Gamma^*, \phi \vee \psi, \psi, \phi \vee \neg\phi, \phi \vdash \Delta, \chi, \chi, \chi$ leads to $\Gamma^*, \phi \vee \psi, \psi, \phi \vee \neg\phi, \phi, \chi \vdash \Delta, \chi, \chi, \chi$ then $\Gamma^*, \phi \vee \psi, \psi, \phi \vee \neg\phi, \neg\phi, \vdash \Delta, \chi, \chi$, where $\Gamma^{**} = \Gamma \cup \{\neg\phi\}$ (if χ is reached we discard an assumption ϕ, add an assumption $\neg\phi$ in the *list_proof*, delete the *current_goal* χ and set χ as a *new current_goal*).

P3.2.2.1-P3.2.2.10 govern cases, where $\phi \circ \psi$ is in the *list_proof*. We apply them depending on the rules for an operator $\circ$ in a particular system. Note P3.2.2.1-P3.2.2.10 don't govern one premise rules $R_\circ(0,0,0)$, $R_\circ(0,1,0)$, $R_\circ(1,0,0)$, and $R_\circ(1,1,0)$. They are governed by P1.

P3.2.2.1. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi \vdash \Delta, \chi, \psi \wedge \neg\psi$ (if the system in question has one of the two premises rules $R_\circ(0,i,0)$ and $R_\circ(1,i,0)$ or one of the one premise rules $R_\circ(0,i,i)$ and $R_\circ(1,i,i)$).

P3.2.2.2. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi \vdash \Delta, \chi, \phi \wedge \neg\phi$ (if the system in question has one of the two premises rules $R_\circ(i,0,0)$ or $R_\circ(i,1,0)$ or one of the one premise rules $R_\circ(i,0,i)$ or $R_\circ(i,1,i)$).

P3.2.2.3. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi \vdash \Delta, \chi, \phi \wedge \neg\phi, \psi \wedge \neg\psi$ (if the system in question has a three premises rule $R_\circ(i,i,0)$ or a two premises rule $R_\circ(i,i,i)$).

P3.2.2.4. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi, \phi \vee \neg\phi, \psi \vee \neg\psi \vdash \Delta, \chi$ (if the system in question has neither i-rules, nor 0-rules, i.e. it has 1-rules only).

P3.2.2.5. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi, ((\phi \circ \psi) \wedge \neg(\phi \circ \psi) \vee (\phi \vee \psi) \vdash \Delta, \chi$ (if the system in question has a zero premise rule $R_\circ(0,0,i)$).

P3.2.2.6. If $\Gamma, \phi \circ \psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi \circ \psi, ((\phi \circ \psi) \wedge \neg(\phi \circ \psi) \vee (\phi \vee \neg\psi) \vdash \Delta, \chi$ (if the system in question has a zero premise rule $R_\circ(0,1,i)$).

[9] We infer the *current_goal* χ from an assumption ψ and then try to infer the new *current_goal* χ from an assumption $\neg\psi$.

P3.2.2.7. If $\Gamma, \phi\circ\psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi\circ\psi, ((\phi\circ\psi)\wedge\neg(\phi\circ\psi)\vee(\neg\phi\vee\psi) \vdash \Delta, \chi$ (if the system in question has a zero premise rule $R_\circ(1, 0, i)$).

P3.2.2.8. If $\Gamma, \phi\circ\psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi\circ\psi, ((\phi\circ\psi)\wedge\neg(\phi\circ\psi)\vee(\neg\phi\vee\neg\psi) \vdash \Delta, \chi$ (if the system in question has a zero premise rule $R_\circ(1, 1, i)$).

The last two procedures apply only after the previous ones have been applied.

P3.2.2.9. If $\phi \notin \Gamma, \neg\phi \notin \Gamma$, $\Gamma, \phi\circ\psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi\circ\psi, \phi\vee\neg\phi \vdash \Delta, \chi$ (we add an unmarked $\phi\vee\neg\phi$ to the $list\_proof$ by (EM) if the previous procedures fail).

P3.2.2.10. If $\psi \notin \Gamma, \neg\psi \notin \Gamma$, $\Gamma, \phi\circ\psi \vdash \Delta, \chi$ and χ isn't reached then $\Gamma, \phi\circ\psi, \psi\vee\neg\psi \vdash \Delta, \chi$ (we add an unmarked $\psi\vee\neg\psi$ to the $list\_proof$ by (EM) if the previous procedures fail).

Procedure 4 (P4). P4 governs applications of the introduction rules. The $list\_proof$ is added with a conclusion of an introduction rule if the $list\_proof$ contains premise(s) of this rule following P3.1. In particular, P.3.1.1 governs $(\wedge I)$, P.3.1.2.1 governs $(\vee I_1)$, P.3.1.2.2 governs $(\vee I_2)$, P.3.1.3 governs $(\neg\vee I)$, P.3.1.4 governs $(\neg\wedge I)$, P.3.1.5 governs $(\neg\neg I)$, P.3.1.2.3, P.3.1.6 and P.3.1.7 govern (EM). Suppose we search for a goal $\phi\wedge\psi$. By P.3.1.1 we search for both ϕ and ψ starting from ψ. If we are successful in searching for them we apply $(\wedge I)$ to reach $\phi\wedge\psi$. Similar considerations (sometimes more subtle) hold for the rest of goals.

P4 is supplemented with marks. As in case of the elimination rules, once a rule is applied, both its premise(s) and a conclusion are marked to avoid infinite applications of the same rule to the same formula. Another application of this rule to the same formula is possible only if the conclusion of the rule is discarded from the $list\_proof$. P4, additionally, marks conclusions of the introduction rules to avoid applications of P1 and P3 to them (i.e. no elimination rules are applied to these formulae and they can't be sources for new goals).

Proof searching algorithm $ALG_{LP_{(\circ)n}}$. Here we give an informal presentation. A flowchart of the algorithm is in Figure 1 below.

The algorithm starts searching for a derivation of ϕ from Γ by adding all the formulae from Γ (if any) to the $list\_proof$ and setting ϕ as the $initial\_goal$ in the $list\_goals$. This is the box *Input* below. Then P2 starts up and checks the reachability of the $current\_goal$ (the box *P2*).

If it is reached the algorithm goes to the box *Is the current_goal the initial_goal*. In case the answer is YES we have the box *The algo-derivation of ϕ from Γ; stop.* NO-answer leads to the box *P4*, where the introduction rules are applicable. Then the algorithm returns to the box *P2*.

If it is not reached we go to the box *Are the elimination rules applicable.* If YES we go to the box *P1*, where the elimination rules are applicable, and then return to the box *P2*. If NO we go to the box *Is the current_goal analyzable*?

If we can analyse the $current\_goal$ we go to the box *P3.1*, where the algorithm deals with the current goal in the $list\_goals$, and then to the box *P2*. If we can't analyse the $current\_goal$ we go to the box *Are there unmarked formulae in the list_proof*? NO-answer leads to the box *Counterexample extraction; stop.* YES-answer leads to the box *P3.2*, where the algorithm deals with compound formulae in the $list\_proof$, and then to the box *P2*.

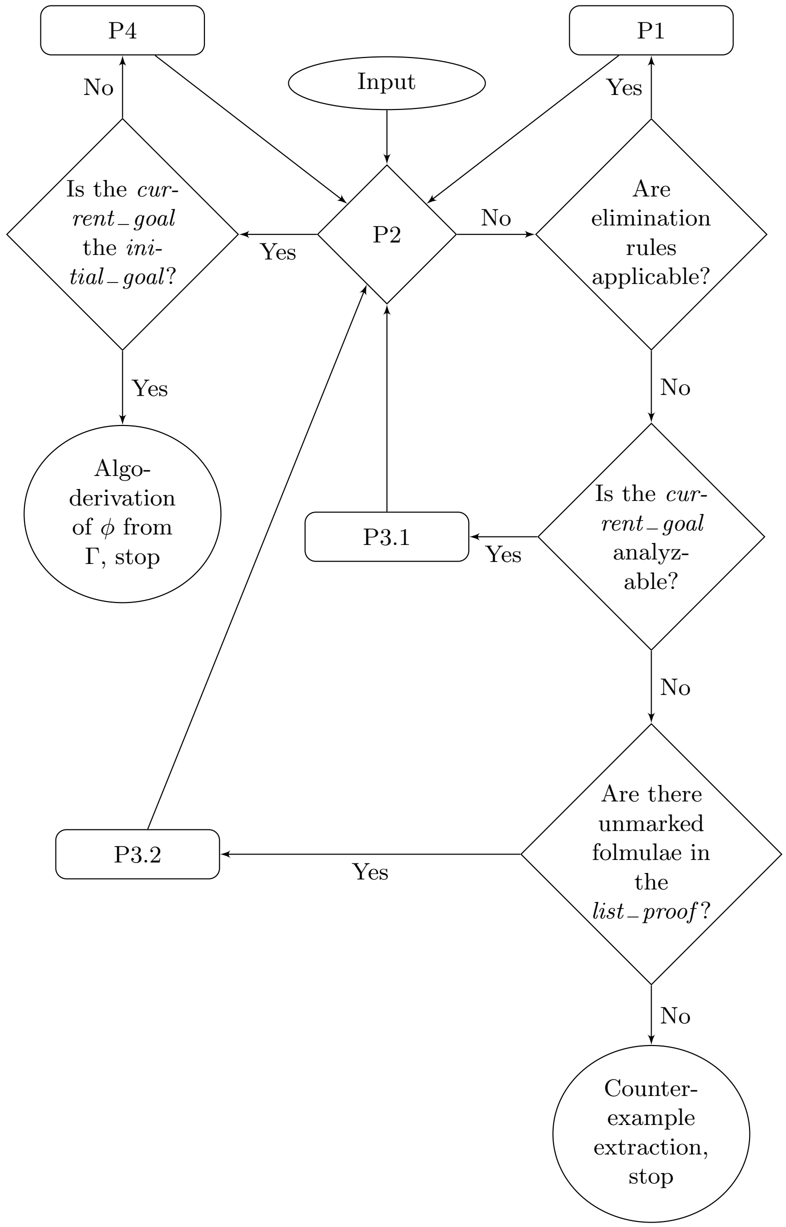

FIGURE 1. A flowchart of the algorithm.

§4. Algo-Proof Examples.

Example 1. We give a *sketch* of the algo-proof of a formula $p \circ \neg\neg p$ in the system $[R_\circ(0,0,1),\ R_\circ(0,i,0),\ R_\circ(0,1,0),\ R_\circ(i,0,0),\ R_\circ(i,i,1),\ R_\circ(i,1,0),\ R_\circ(1,0,0),\ R_\circ(1,i,0),\ R_\circ(1,1,1)]$. We start out by setting $p \circ \neg\neg p$ as the *initial_goal* and proceed to analyse it. By P3.1.6, we add both $(p \circ \neg\neg p) \vee$

$\neg(p \circ \neg\neg p)$ and $\neg(p \circ \neg\neg p)$ to the *list_proof* (the former by (EM) and the latter as an assumption) while set $p \circ \neg\neg p$ as the *current_goal*. P1 is launched with inferring both $p \vee \neg\neg p$ and $\neg p \vee \neg\neg\neg p$ (by $R_\circ(0,0,1)$ and $R_\circ(1,1,1)$, respectfully).

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2

The *current_goal* $p \circ \neg\neg p$ is not reached so we proceed to analyse it via P3.2.2.3 for the system has $R_\circ(i,i,i)$. Two goals are added, $\neg\neg p \wedge \neg\neg\neg p$ and $p \wedge \neg p$. P3.1.1 applies to the *current_goal* $p \wedge \neg p$ and adds new goals $\neg p$ and p with p being the *current_goal*. Then, by P3.1.6, both $p \vee \neg p$ and $\neg p$ are added to the *list_proof* while p is set to be the *current_goal*.
Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$, $\neg\neg p \wedge \neg\neg\neg p$, $p \wedge \neg p$, $\neg p$, p, p.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2
(5)	$p \vee \neg p$	(EM)
(6)	$\neg p$	assumption

The *current_goal* p is not reached so we apply P3.2.1 to $p \vee \neg\neg p$, mark it and add p as an assumption. Now the *current_goal* p is reached. Then we add $\neg\neg p$ as an assumption and reach the *current_goal* p via $(\neg\neg E)$. So we infer the *current_goal* p from both disjuncts of $p \vee \neg\neg p$. Then $(\vee E)$ is applied with 7^{th} and $8\text{-}9^{\text{th}}$ formulae being discarded from the *list_proof* and setting p as a new *current_goal*.

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$, $\neg\neg p \wedge \neg\neg\neg p$, $p \wedge \neg p$, $\neg p$, p.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2
(5)	$p \vee \neg p$	(EM)
(6)	$\neg p$	assumption
(7)	p	assumption
(8)	$\neg\neg p$	assumption
(9)	p	$(\neg\neg I)$: 8
(10)	p	$(\vee E)$: 3, 7, 9, [7], [8-9]

The *current_goal* p is reached at the moment and we add p as an assumption to infer p by $(\vee E)$.

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$, $\neg\neg p \wedge \neg\neg\neg p$, $p \wedge \neg p$, $\neg p$.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2
(5)	$p \vee \neg p$	(EM)
(6)	$\neg p$	assumption
(7)	p	assumption
(8)	$\neg\neg p$	assumption
(9)	p	$(\neg\neg I)$: 8
(10)	p	$(\vee E)$: 3, 7, 9, [7], [8-9]
(11)	p	assumption
(12)	p	$(\vee E)$: 5, 10, 11, [6-10], [11]

Now $\neg p$ is the *current_goal* and we infer it analogously. Note $p \vee \neg\neg p$ is marked now, and below we use $\neg p \vee \neg\neg\neg p$ in the same way we have used $p \vee \neg\neg p$ above. So, we present this part of the proof without a detailed description. Note on the 16^{th} and 17^{th} steps we apply P3.2.1 to the 4^{th} formula.

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$, $\neg\neg p \wedge \neg\neg\neg p$.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2
(5)	$p \vee \neg p$	(EM)
(6)	$\neg p$	assumption
(7)	p	assumption
(8)	$\neg\neg p$	assumption
(9)	p	$(\neg\neg I)$: 8
(10)	p	$(\vee E)$: 3, 7, 9, [7], [8-9]
(11)	p	assumption
(12)	p	$(\vee E)$: 5, 10, 11, [6-10], [11]
(13)	$\neg\neg p \vee \neg p$	(EM)
(14)	$\neg\neg p$	assumption
(15)	p	$(\neg\neg E)$: 14
(16)	$\neg p$	assumption
(17)	$\neg\neg\neg p$	assumption
(18)	$\neg p$	$(\neg\neg E)$: 17
(19)	$\neg p$	$(\vee E)$: 4, 16, 18, [16], [17-18]
(20)	$\neg p$	assumption
(21)	$\neg p$	$(\vee E)$: 13, 19, 20, [14-19], [20]
(22)	$p \wedge \neg p$	$(\wedge I)$: 12, 21

As previously, P3.1.1 applies to the *current_goal* $\neg\neg p \wedge \neg\neg\neg p$ and adds both $\neg\neg\neg p$ and $\neg\neg p$ to the *list_goals* with the latter being the *current_goal*. P3.1.5 applies to it and add p as the *current_goal*. This goal is reached for a formula p is not discarded from the *list_goals* (step 12). So, $(\neg\neg I)$ is applied to it to infer $\neg\neg p$ with the *current_goal* $\neg\neg p$ being reached. Then $\neg\neg\neg p$ is set to be the *current_goal*. We reason analogously in reaching it and, therefore, skip a detailed presentation. On 25$^{\text{th}}$ step the *current_goal* $p \circ \neg\neg p$ is reached via $R_\circ(i, i, 1)$. At last, the initial goal $p \circ \neg\neg p$ is reached by $(\vee E)$, and we have successfully proven the desired formula in the system.

The algo-proof finally looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0, 0, 1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1, 1, 1)$: 2
(5)	$p \vee \neg p$	(EM)
(6)	$\neg p$	assumption
(7)	p	assumption
(8)	$\neg\neg p$	assumption
(9)	p	$(\neg\neg I)$: 8
(10)	p	$(\vee E)$: 3, 7, 9, [7], [8-9]
(11)	p	assumption
(12)	p	$(\vee E)$: 5, 10, 11, [6-10], [11]
(13)	$\neg\neg p \vee \neg p$	(EM)
(14)	$\neg\neg p$	assumption
(15)	p	$(\neg\neg E)$: 14
(16)	$\neg p$	assumption
(17)	$\neg\neg\neg p$	assumption
(18)	$\neg p$	$(\neg\neg E)$: 17
(19)	$\neg p$	$(\vee E)$: 4, 16, 18, [16], [17-18]
(20)	$\neg p$	assumption
(21)	$\neg p$	$(\vee E)$: 13, 19, 20, [14-19], [20]
(22)	$p \wedge \neg p$	$(\wedge I)$: 12, 21
(23)	$\neg\neg p$	$(\neg\neg I)$: 12
(24)	$\neg\neg\neg p$	$(\neg\neg I)$: 21
(25)	$\neg\neg p \wedge \neg\neg\neg p$	$(\wedge I)$: 23, 24
(26)	$p \circ \neg\neg p$	$R_\circ(i, i, 1)$: 22, 25, 2
(27)	$p \circ \neg\neg p$	assumption
(28)	$p \circ \neg\neg p$	$(\vee E)$: 1, 26, 27, [2-26], [27]

Example 2. We want to prove a formula $p \circ \neg\neg p$ in the system $[R_\circ(0, 0, 1)$, $R_\circ(0, i, 0)$, $R_\circ(0, 1, 0)$, $R_\circ(i, 0, 0)$, $R_\circ(i, i, 0)$, $R_\circ(i, 1, 0)$, $R_\circ(1, 0, 0)$, $R_\circ(1, i, 0)$, $R_\circ(1, 1, 1)]$. Note this system differs from the one in Example 1 only with respect to $R_\circ(i, i, 0)$. We set $p \circ \neg\neg p$ as the initial goal. By P3.1.6, we add both $(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$ and $\neg(p \circ \neg\neg p)$ to the *list_proof* (the former by (EM) and the latter

as an assumption) while setting $p \circ \neg\neg p$ as the current goal. P1 is launched with inferring both $p \vee \neg\neg p$ and $\neg p \vee \neg\neg\neg p$ (by $R_\circ(0,0,1)$ and $R_\circ(1,1,1)$, respectfully).

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$.

The algo-proof now looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2

The *current_goal* $p \circ \neg\neg p$ is not reached. Note the system has neither $R_\circ(i,i,i)$, nor $R_\circ(i,i,1)$ and, therefore, we can't apply P3.2.2.3 or its $\neg\circ$-analog to the *current_goal*. So we apply P3.2.1 to $p \vee \neg\neg p$, mark it and add p as an assumption. By P3.2.1.1, both $\neg\neg p \vee \neg\neg\neg p$ and $\neg\neg p$ are added to the *list_proof* (the former by (EM) and the latter as an assumption). Then $(\neg\neg E)$ is applied to $\neg\neg p$ to infer p. For the *current_goal* isn't reached we apply P3.2.1 to $\neg p \vee \neg\neg\neg p$, mark it and add $\neg p$ as an assumption. By P3.2.1.1, both $\neg\neg\neg p \vee \neg\neg\neg\neg p$ and $\neg\neg\neg p$ are added to the *list_proof* (the former by (EM) and the latter as an assumption). Then $(\neg\neg E)$ is applied to $\neg\neg\neg p$ to infer $\neg p$. The *current_goal* $p \circ \neg\neg p$ is, still, not reached while all compound formulae in the *list_proof* are marked. (Let us remind the reader 1^{st}, 6^{th}, and 10^{th} formulae can't be a source for new goals. See P3.2.1.1.) We stop by finding both p and $\neg p$ in the *list_proof*. From this fact we extract a valuation v such that $v(p) = i$. So, $v(p \circ \neg\neg p) = 0$ in the logic in question.

Now *list_goals* is as follows: $p \circ \neg\neg p$, $p \circ \neg\neg p$.

The algo-proof finally looks as follows:

(1)	$(p \circ \neg\neg p) \vee \neg(p \circ \neg\neg p)$	(EM)
(2)	$\neg(p \circ \neg\neg p)$	assumption
(3)	$p \vee \neg\neg p$	$R_\circ(0,0,1)$: 2
(4)	$\neg p \vee \neg\neg\neg p$	$R_\circ(1,1,1)$: 2
(5)	p	assumption
(6)	$\neg\neg p \vee \neg\neg\neg p$	(EM)
(7)	$\neg\neg p$	assumption
(8)	p	$(\neg\neg E)$: 7
(9)	$\neg p$	assumption
(10)	$\neg\neg\neg p \vee \neg\neg\neg\neg p$	(EM)
(11)	$\neg\neg\neg p$	assumption
(12)	$\neg p$	$(\neg\neg E)$: 11

§5. Soundness, completeness and termination.

THEOREM 5.1 (Termination of the algorithm). *The algorithm halts on any input.*

PROOF. P1 is finite because the number of formulae in the *list_proof* is finite at each start of P1. Once an elimination rule is applied both its conclusion and premise(s) are marked to prevent infinite applications of the same rule to the same formula. This formula may be unmarked if the conclusion of this application is discarded from the $list\_goals$. It means the $list\_proof$ has changed since this application, and, again, the number of formulae in the updated $list\_proof$ is finite.

P2 is finite for it matches the $current\_goal$ with the formulae in the $list\_proof$ which is finite by the argument above.

P3 consists of two subprocedures, and we start out with P.3.1. First, in P3.1.1, P3.1.2.1, P3.1.2.2 and P.1.3.5 the new $current\_goal$ is of *less* degree as the previous one. (The degree of a formula is as usual defined to be the number of connectives in it.) We add P.1.3.7 to this group, where no new $current\_goal$ appears. Second, P3.1.2.3, P3.1.6, P3.1.8.1, P3.1.8.3, P3.1.8.6 and P3.1.8.7 analyse the $current\_goal$ *without increasing* its degree.

The third group contains the other subprocedures which *indirectly* decrease the degree of the $current\_goal$. For example, in P3.1.4 the degree of the new $current\_goal$ $\neg\phi \wedge \neg\psi$ is more than the degree of the previous one $\neg(\phi \vee \psi)$. However, at the next step P3.1.1 applies to $\neg\phi \wedge \neg\psi$. One by one, formulae $\neg\psi$ and $\neg\phi$ become the new $current\_goals$ with the degrees of both formulae being less than the degree of $\neg(\phi \vee \psi)$.

P3.2 is finite because the number of formulae in the $list\_proof$ is finite at each start of P3.2. Once P3.2 is applied both the formula in the $list\_goal$ and the $current\_goal$ are marked to prevent infinite applications of this subprocedure to the same formula. This formula may be unmarked if this goal is reached, the elimination rule in question is applied and the conclusion of this application is discarded from the $list\_proof$. It means the $list\_proof$ has changed since this application, and, again, the number of formulae in the updated $list\_proof$ is finite. This concludes the argument that P3 is finite.

P4 is finite for applications of the introduction rules are determined by the $list\_goals$ which is finite by the argument above. The domino effect of P4 results in reaching the previous goal(s) after reaching the $current\_goal$. For example, if we prove both conjuncts then we prove the conjunction immediately. □

THEOREM 5.2 (Soundness of the algorithm). *The algorithm is sound.*

PROOF. By theorem 3.3. in (Kooi & Tamminga, 2012), each system is sound. Any algo-derivation is a derivation in one of the systems. Therefore, the algorithm is sound. □

To prove completeness we need two lemmata. We use a technique from and, for the reason of space, refer the reader to (Bolotov & Shangin, 2012) in some cases.

LEMMA 5.3. *A truth-value assignment ξ of a formula in a model is inductively defined as follows*:

1.	$\xi(\neg\neg\phi)$:		
1.1.	If $\xi(\neg\neg\phi) = 1$	then	$\xi(\phi) = 1$.
1.2.	If $\xi(\neg\neg\phi) = i$	then	$\xi(\phi) = i$.
2.	$\xi(\phi \wedge \psi)$:		
2.1.	If $\xi(\phi \wedge \psi) = 1$	then	$\xi(\phi) = 1$ and $\xi(\psi) = 1$.
2.2.	If $\xi(\phi \wedge \psi) = i$	then	
		2.2.1.	$\xi(\phi) = i$ and $\xi(\psi) = 1$, or

		2.2.2.	$\xi(\phi) = i$ and $\xi(\psi) = i$, or
		2.2.3.	$\xi(\phi) = i$ and $\xi(\psi) = 1$.
3.	$\xi(\phi \vee \psi)$:		
3.1.	If $\xi(\phi \vee \psi) = 1$	then	
		3.1.1.	$\xi(\phi) = 1$, $\xi(\psi) = 0$; or
		3.1.2.	$\xi(\phi) = 1$, $\xi(\psi) = i$; or
		3.1.3.	$\xi(\phi) = 1$, $\xi(\psi) = 1$; or
		3.1.4.	$\xi(\phi) = 0$, $\xi(\psi) = 1$; or
		3.1.5.	$\xi(\phi) = i$, $\xi(\psi) = 1$.
3.2.	If $\xi(\phi \vee \psi) = i$	then	
		3.2.1.	$\xi(\phi) = i$ and $\xi(\psi) = 0$, or
		3.2.2.	$\xi(\psi) = i$ and $\xi(\psi) = i$, or
		3.2.3.	$\xi(\psi) = 0$ and $\xi(\psi) = i$.
4.	$\xi(\phi \circ \psi)$:		
4.1	If $\xi(\phi \circ \psi) = 1$	then	
		4.1.1.	$\xi(\phi) = 1$ and $\xi(\psi) = 1$, or
		4.1.2.	$\xi(\phi) = 1$ and $\xi(\psi) = i$, or
		4.1.3.	$\xi(\phi) = 1$ and $\xi(\psi) = 0$, or
		4.1.4.	$\xi(\phi) = i$ and $\xi(\psi) = 1$, or
		4.1.5.	$\xi(\phi) = i$ and $\xi(\psi) = i$, or
		4.1.6.	$\xi(\phi) = i$ and $\xi(\psi) = 0$, or
		4.1.7.	$\xi(\phi) = 0$ and $\xi(\psi) = 1$, or
		4.1.8.	$\xi(\phi) = 0$ and $\xi(\psi) = i$, or
		4.1.9.	$\xi(\phi) = 0$ and $\xi(\psi) = 0$.
4.2	If $\xi(\phi \circ \psi) = i$	then	
		4.2.1.	$\xi(\phi) = 1$ and $\xi(\psi) = 1$, or
		4.2.2.	$\xi(\phi) = 1$ and $\xi(\psi) = i$, or
		4.2.3.	$\xi(\phi) = 1$ and $\xi(\psi) = 0$, or
		4.2.4.	$\xi(\phi) = i$ and $\xi(\psi) = 1$, or
		4.2.5.	$\xi(\phi) = i$ and $\xi(\psi) = i$, or
		4.2.6.	$\xi(\phi) = i$ and $\xi(\psi) = 0$, or
		4.2.7.	$\xi(\phi) = 0$ and $\xi(\psi) = 1$, or
		4.2.8.	$\xi(\phi) = 0$ and $\xi(\psi) = i$, or
		4.2.9.	$\xi(\phi) = 0$ and $\xi(\psi) = 0$.
5.	$\xi(\neg(\phi \wedge \psi))$:		
5.1.	If $\xi(\neg(\phi \wedge \psi)) = 1$	then	$\xi(\neg\phi \vee \neg\psi) = 1$.
5.2.	If $\xi(\neg(\phi \wedge \psi)) = i$	then	$\xi(\neg\phi \vee \neg\psi) = i$.
6.	$\xi(\neg(\phi \vee \psi))$:		
6.1.	If $\xi(\neg(\phi \vee \psi)) = 1$	then	$\xi(\neg\phi \wedge \neg\psi) = 1$.
6.2.	If $\xi(\neg(\phi \vee \psi)) = i$	then	$\xi(\neg\phi \wedge \neg\psi) = i$.
7.	$\xi(\neg(\phi \circ \psi))$:		
7.1	If $\xi(\neg(\phi \circ \psi)) = 1$	then	
		7.1.1.	$\xi(\phi) = 1$ and $\xi(\psi) = 1$, or
		7.1.2.	$\xi(\phi) = 1$ and $\xi(\psi) = i$, or
		7.1.3.	$\xi(\phi) = 1$ and $\xi(\psi) = 0$, or
		7.1.4.	$\xi(\phi) = i$ and $\xi(\psi) = 1$, or
		7.1.5.	$\xi(\phi) = i$ and $\xi(\psi) = i$, or
		7.1.6.	$\xi(\phi) = i$ and $\xi(\psi) = 0$, or

7.1.7. $\xi(\phi) = 0$ and $\xi(\psi) = 1$, or
7.1.8. $\xi(\phi) = 0$ and $\xi(\psi) = i$, or
7.1.9. $\xi(\phi) = 0$ and $\xi(\psi) = 0$.

7.2 If $\xi(\neg(\phi \circ \psi)) = i$ then
7.2.1. $\xi(\phi) = 1$ and $\xi(\psi) = 1$, or
7.2.2. $\xi(\phi) = 1$ and $\xi(\psi) = i$, or
7.2.3. $\xi(\phi) = 1$ and $\xi(\psi) = 0$, or
7.2.4. $\xi(\phi) = i$ and $\xi(\psi) = 1$, or
7.2.5. $\xi(\phi) = i$ and $\xi(\psi) = i$, or
7.2.6. $\xi(\phi) = i$ and $\xi(\psi) = 0$, or
7.2.7. $\xi(\phi) = 0$ and $\xi(\psi) = 1$, or
7.2.8. $\xi(\phi) = 0$ and $\xi(\psi) = i$, or
7.2.9. $\xi(\phi) = 0$ and $\xi(\psi) = 0$.

PROOF. From the matrix definitions of $LP_{(\circ)n}$-connectives. □

This definition is easily extended to sets of formulae. For example, $\xi(\Gamma) \neq 0$ iff $\xi(\phi) = 1$ or $\xi(\phi) = i$, for each ϕ from Γ.

LEMMA 5.4. *If the algorithm with a task to find a derivation of α from Γ in some system $ND_{LP_{(\circ)n}}$ stops without finding the derivation of α from Γ in this system then the list_proof contains a set Σ, $\Gamma \subseteq \Sigma$, of non-discarded formulae such that $\xi(\Gamma) \neq 0$ and $\xi(\alpha) = 0$.*

PROOF. We, first, show that $\xi(\alpha) = 0$, by Procedure 3.1, which analyzes the *current_goal* in the *list_goals*. The number of cases depends on the type of the *current_goal*.

1. If a is a literal, or $\phi \vee \psi$ or $\phi \circ \psi$ or $\neg(\phi \circ \psi)$ then P3.1.2 or P3.1.6 is launched by adding $\neg\alpha$ to the *list_proof*. It is easy to see that if $\xi(\neg\alpha) = 1$ then $\xi(\alpha) = 0$.
2. If a is $\phi \wedge \psi$ then P3.1.1 is launched with ψ and ϕ being the *current_goal*, one by one. It is easy to see that if $\xi(\psi) = 0$ or $\xi(\phi) = 0$ then $\xi(\phi \wedge \psi) = 0$.
3. If a is $\neg(\phi \wedge \psi)$ then P3.1.3 is launched with $\neg\phi \vee \neg\psi$ being the *current_goal*. It is easy to see that if $\xi(\neg\phi \vee \neg\psi) = 0$ then $\xi(\neg(\phi \wedge \psi)) = 0$.
4. If a is $\neg(\phi \vee \psi)$ then P3.1.4 is launched with $\neg\phi \wedge \neg\psi$ being the *current_goal*. It is easy to see that if $\xi(\neg\phi \wedge \neg\psi) = 0$ then $\xi(\neg(\phi \vee \psi)) = 0$.
5. If a is $\neg\neg\phi$ then P3.1.5 is launched with ϕ being the *current_goal*. It is easy to see that if $\xi(\phi) = 0$ then $\xi(\neg\neg\phi) = 0$.

We, second, show that $\xi(\Gamma) \neq 0$, that is, a set Γ is a model set. The number of cases depends on the type of a formula in Γ. See (Bolotov & Shangin, 2012) for the proofs of cases 1-2 and 5-6. Here we give a new proof of case 3 for it is extensively used in proving cases 4 and 7 below.

Case 1. If $\neg\neg\phi \in \Sigma$ then
1.1 $\phi \in \Sigma$; or
1.2 $\phi \in \Sigma$, $\neg\phi \in \Sigma$.

Case 2. If $\phi \wedge \psi \in \Sigma$ then
2.1. $\phi \in \Sigma$, $\psi \in \Sigma$; or
2.2. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$; or
2.3. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$; or
2.4. $\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$.

Case 3. If $\phi \vee \psi \in \Sigma$ then
3.1.1. $\phi \in \Sigma, \psi \notin \Sigma, \neg\phi \notin \Sigma, \neg\psi \in \Sigma$; or
3.1.2. $\phi \in \Sigma, \psi \in \Sigma, \neg\phi \notin \Sigma, \neg\psi \in \Sigma$; or
3.1.3. $\phi \in \Sigma, \psi \in \Sigma, \neg\phi \notin \Sigma, \neg\psi \notin \Sigma$; or
3.1.4. $\phi \notin \Sigma, \psi \in \Sigma, \neg\phi \in \Sigma, \neg\psi \notin \Sigma$; or
3.1.5. $\phi \in \Sigma, \psi \in \Sigma, \neg\phi \in \Sigma, \neg\psi \notin \Sigma$; or
3.1.6. $\phi \in \Sigma, \psi \notin \Sigma, \neg\phi \in \Sigma, \neg\psi \in \Sigma$; or
3.1.7. $\phi \in \Sigma, \psi \in \Sigma, \neg\phi \in \Sigma, \neg\psi \in \Sigma$; or
3.1.8. $\phi \notin \Sigma, \psi \in \Sigma, \neg\phi \in \Sigma, \neg\psi \in \Sigma$.

If $\phi \vee \psi \in \Sigma$ then P3.2.1 is applied and there are four variants: (1) $\phi \in \Sigma$ and $\psi \in \Sigma$ (P3.2.1.1), or (2) $\phi \in \Sigma$ and $\neg\psi \in \Sigma$ (P3.2.1.2), or (3) $\phi \in \Sigma$ and $\psi \in \Sigma$ (P3.2.1.4), or (4) $\neg\phi \in \Sigma$ and $\psi \in \Sigma$ (P3.2.1.5).

Depending on if $\neg\phi \in \Sigma$ or $\neg\psi \in \Sigma$, variants (1), (3) cover cases 3.1.2, 3.1.3, 3.1.5 and 3.1.7.

By the *corresponding* cases of Lemma 5.3, $\xi(\phi \vee \psi) = 1$ or $\xi(\phi \vee \psi) = i$.

Depending on if $\neg\phi \in \Sigma$ or $\psi \in \Sigma$, variant (2) covers cases 3.1.1, 3.1.2, 3.1.6 and 3.1.7. By the *corresponding* cases of Lemma 5.3, $\xi(\phi \vee \psi) = 1$ or $\xi(\phi \vee \psi) = i$.

Depending on if $\phi \in \Sigma$ or $\neg\psi \in \Sigma$, variant (4) covers cases 3.1.4, 3.1.6, 3.1.7 and 3.1.8. By the *corresponding* cases of Lemma 5.3, $\xi(\phi \vee \psi) = 1$ or $\xi(\phi \vee \psi) = i$.

Case 4. If $\phi \circ \psi \in \Sigma$ then
4.1. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
4.2. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
4.3. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$, or
4.4. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
4.5. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
4.6. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$, or
4.7. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
4.8. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
4.9. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$.

Note these conditions are syntactical counterparts of the corresponding conditions of Lemma 5.3.

We will use the following notation. By a (x, y)-*cluster*, where $x, y \in \{0, i, 1\}$, we mean a set of $\circ$-rules with $\xi(\phi) = x$ and $\xi(\psi) = y$. For example, the (0, 0)-cluster is $\{R_\circ(0,0,0),\ R_\circ(0,0,i),\ R_\circ(0,0,1)\}$. By a *x-rule* we mean a $\circ$-rule with $\xi(\phi \circ \psi) = x$. $R_\circ(0,0,i)$ is an example of an i-rule.

We divide subcases into groups and prove this case for an arbitrary cluster from each group depending on the type of a 0-rule in it. Group 1 consists of the (0, 0)-cluster, the (0, 1)-cluster, the (1, 0)-cluster and the (1, 1)-cluster. Group 2 consists of the (0, i)-cluster, the (1, i)-cluster, the (i, 0)-cluster and the (i, 1)-cluster. The (i, i)-cluster forms group 3.

We start with group 1 and choose the (0, 0)-cluster. The analogous argument holds if we would choose the (0, 1)-cluster, or the (1, 0)-cluster, or the (1, 1)-cluster.

4.1. A system has $R_\circ(0,0,0)$.

By $R_\circ(0,0,0)$, $\phi \vee \psi \in \Sigma$. By case 3 of this Lemma, there are four variants: (1) $\phi \in \Sigma$ and $\psi \in \Sigma$, or (2) $\phi \in \Sigma$ and $\neg\psi \in \Sigma$, or (3) $\phi \in \Sigma$ and $\psi \in \Sigma$, or (4) $\neg\phi \in \Sigma$ and $\psi \in \Sigma$. We show variants (1) and (3), and the others are treated analogously.

If $\phi \in \Sigma$ and $\psi \in \Sigma$ then there are four subvariants depending on if $\neg\phi \in \Sigma$ or $\neg\psi \in \Sigma$. We show that in each of this subvariant, the system in question has some i-rule or 1-rule. By *correspondence analysis*, it means $\xi(\phi \circ \psi) = 1$ or $\xi(\phi \circ \psi) = i$.

4.1.1: $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$ and $\neg\psi \in \Sigma$. Let us consider the (i, i)-cluster. By *correspondence analysis*, the system must have one and only one rule from the (i, i)-cluster. Suppose the system has $R_\circ(i, i, 0)$. Therefore, an arbitrary $\chi \in \Sigma$ and this fact contradicts the condition of the Lemma that the $current\_goal$ isn't reached. So, the system hasn't $R_\circ(i, i, 0)$ and it has either $R_\circ(i, i, 1)$ or $R_\circ(i, i, i)$. By cases 4.1.5 and 4.2.5 of Lemma 5.3, it means $\xi(\phi \circ \psi) = 1$ or $\xi(\phi \circ \psi) = i$. Anyway, $\xi(\phi \circ \psi) \neq 0$.

4.1.2: $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$ and $\neg\psi \in \Sigma$. Let us consider the (1, i)-cluster. By *correspondence analysis*, the system must have one and only one rule from the (1, i)-cluster. Suppose the system has $R_\circ(1, i, 0)$. Therefore, $\neg\phi \in \Sigma$ and this fact contradicts the condition of the Lemma that $\neg\phi \notin \Sigma$. So, the system hasn't $R_\circ(1, i, 0)$ and it has either $R_\circ(1, i, 1)$ or $R_\circ(1, i, i)$. By cases 4.1.2 and 4.2.2 of Lemma 5.3, it means $\xi(\phi \circ \psi) = 1$ or $\xi(\phi \circ \psi) = i$. Anyway, $\xi(\phi \circ \psi) \neq 0$.

4.1.3: $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$ and $\neg\psi \notin \Sigma$. We treat it analogously to 4.1.2.

4.1.4: $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$ and $\neg\psi \notin \Sigma$. We treat it analogously to 4.1.2.

We conclude the argument concerning $R_\circ(0, 0, 0)$. The analogous argument is easily applicable to the other rules, where $\phi \circ \psi$ is the only premise, $R_\circ(0, 1, 0)$, $R_\circ(1, 0, 0)$ and $R_\circ(1, 1, 0)$.

4.2. Suppose the system has $R_\circ(0, 0, i)$.

If $\phi \circ \psi \in \Sigma$ then, by P3.2.2.5, $((\phi \circ \psi) \wedge \neg(\phi \circ \psi)) \vee (\phi \vee \psi) \in \Sigma$. By case 3 above, there are four variants:

(1) $(\phi \circ \psi) \wedge \neg(\phi \circ \psi) \in \Sigma$ and $\phi \vee \psi \in \Sigma$, or

(2) $(\phi \circ \psi) \wedge \neg(\phi \circ \psi) \in \Sigma$ and $\neg(\phi \vee \psi) \in \Sigma$, or

(3) $\phi \vee \psi \in \Sigma$ and $(\phi \circ \psi) \wedge \neg(\phi \circ \psi) \in \Sigma$, or

(4) $\phi \vee \psi \in \Sigma$ and $\neg((\phi \circ \psi) \wedge \neg(\phi \circ \psi))) \in \Sigma$.

Let us consider variant (2), first. $\neg(\phi \vee \psi) \in \Sigma$ implies $\neg\phi \in \Sigma$ and $\neg\psi \in \Sigma$. Depending on if $\phi \in \Sigma$ or $\psi \in \Sigma$, there are four subvariants:

(3.1) $\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\phi \in \Sigma$, $\neg\psi \in \Sigma$. By case 4.2.5 of Lemma 5.3, it means $\xi(\phi \circ \psi) = \xi(\neg(\phi \circ \psi)) = i$.

(3.2) $\phi \in \Sigma$, $\psi \notin \Sigma$, $\neg\phi \in \Sigma$, $\neg\psi \in \Sigma$. By case 4.2.6 of Lemma 5.3, it means $\xi(\phi \circ \psi) = \xi(\neg(\phi \circ \psi)) = i$.

(3.3) $\phi \notin \Sigma$, $\psi \notin \Sigma$, $\neg\phi \in \Sigma$, $\neg\psi \in \Sigma$. By case 4.2.9 of Lemma 5.3, it means $\xi(\phi \circ \psi) = \xi(\neg(\phi \circ \psi)) = i$.

(3.4) $\phi \notin \Sigma$, $\psi \in \Sigma$, $\neg\phi \in \Sigma$, $\neg\psi \in \Sigma$. By case 4.2.8 of Lemma 5.3, it means $\xi(\phi \circ \psi) = \xi(\neg(\phi \circ \psi)) = i$.

For a proof of variants (1), (3) and (4) see case 3 above.

4.3. A system has $R_\circ(0, 0, 1)$.

The cases below depend on the possible o-rules of a system.

4.3.1. The system has neither i-rules, nor 0-rules, i.e. it has 1-rules only. In this case, none of P3.2.2.1-P3.2.2.8 is applicable. So, by P3.2.2.9-P3.2.2.10, both $\phi \vee \neg\phi \in \Sigma$ and $\psi \vee \neg\psi \in \Sigma$ via (EM). See case 3.

In the cases below, the system has at least one i-rule or 0-rule.

4.3.2. The system has one of the rules $R_\circ(0,1,0)$, $R_\circ(1,0,0)$ and $R_\circ(1,1,0)$. See case 4.1, where we treat the remaining rule of this kind $R_\circ(0,0,0)$.

4.3.3. The system has one of the rules $R_\circ(0,i,0)$, $R_\circ(i,0,0)$, $R_\circ(i,1,0)$, $R_\circ(1,i,0)$. Suppose it has $R_\circ(0,i,0)$. The other cases are treated analogously.

By P3.2.2.1, there are two subvariants:

4.3.3.1. $\phi \circ \psi \in \Sigma$ and $\psi \wedge \neg\psi \in \Sigma$;

4.3.3.2. $\phi \circ \psi \in \Sigma$ and $\psi \wedge \neg\psi \notin \Sigma$.

If $\phi \circ \psi \in \Sigma$ and $\psi \wedge \neg\psi \in \Sigma$ then $\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, by $R_\circ(0,i,0)$, $(\wedge E_1)$, $(\wedge E_2)$. If $\neg\phi \in \Sigma$ then $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$. See case 4.1.2.

In order to prove 4.3.3.2 we reason as follows. By 3.1.1, if the *current_goal* is $\psi \wedge \neg\psi$ then the algorithm sets one by one $\neg\psi$ and ψ as the *current_goal*. So, by P3.1.6, we have (I) $\neg\psi \in \Sigma$ and $\psi \notin \Sigma$ or (II) $\neg\neg\psi \in \Sigma$ and $\neg\psi \notin \Sigma$. In both cases, P3.2.2.9 is applied to $\phi \circ \psi$ and, therefore, $\phi \vee \neg\phi \in \Sigma$. See case 3.

4.3.4. The system has the rule $R_\circ(i,i,0)$. By the condition, the *current_goal* isn't reached. So, $\phi \wedge \neg\phi \notin \Sigma$ or $\psi \wedge \neg\psi \notin \Sigma$. See 4.3.3.

4.3.5. The system has one of the rules $R_\circ(0,1,i)$, $R_\circ(1,0,i)$ and $R_\circ(1,1,i)$. See case 4.2, where we treat the remaining rule of this kind $R_\circ(0,0,i)$.

4.3.6. The system has one of the rules $R_\circ(0,i,i)$, $R_\circ(i,0,i)$, $R_\circ(i,1,i)$, $R_\circ(1,i,i)$. Suppose it has $R_\circ(0,i,i)$. The other cases are treated analogously.

By P3.2.2.1, there are two subvariants:

4.3.6.1. $\phi \circ \psi \in \Sigma$ and $\psi \wedge \neg\psi \in \Sigma$;

4.3.6.2. $\phi \circ \psi \in \Sigma$ and $\psi \wedge \neg\psi \notin \Sigma$.

If $\psi \wedge \neg\psi \in \Sigma$ then $\psi \in \Sigma$, $\neg\psi \in \Sigma$, by $(\wedge E_1)$, $(\wedge E_2)$. Then P3.2.2.9 is applied to $\phi \circ \psi$ and, therefore, $\phi \wedge \neg\phi \in \Sigma$. See case 3.

For a proof of 4.3.6.2 see case 4.3.3.2.

4.3.7. The system has the rule $R_\circ(i,i,i)$. P3.2.2.3, ($\phi \wedge \neg\phi \in \Sigma$ or $\phi \wedge \neg\phi \notin \Sigma$) and ($\psi \wedge \neg\psi \in \Sigma$ or $\psi \wedge \neg\psi \notin \Sigma$). Note $\phi \vee \neg\phi \in \Sigma$ or $\psi \vee \neg\psi \in \Sigma$, by P3.2.2.9-P3.2.2.10. See case 3.

This concludes a proof for group 1. Let us recall group 1 consists of the (0, 0)-cluster, the (0, 1)-cluster, the (1, 0)-cluster and the (1, 1)-cluster.

A proof for group 2 is analogous one (especially note case 4.3.3). This group consists of the (0, i)-cluster, the (1, i)-cluster, the (i, 0)-cluster and the (i, 1)-cluster.

A proof for group 3 is analogous one (especially note case 4.3.4). This group consists of the (i, i)-cluster.

This concludes a proof of case 4.

Case 5. If $\neg(\phi \wedge \psi) \in \Sigma$ then $\neg\phi \vee \neg\psi \in \Sigma$.

Case 6. If $\neg(\phi \vee \psi) \in \Sigma$ then $\neg\phi \wedge \neg\psi \in \Sigma$.

Case 7. If $\neg(\phi \circ \psi) \in \Sigma$ then
7.1. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
7.2. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
7.3. $\phi \in \Sigma$, $\neg\phi \notin \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$, or
7.4. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
7.5. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
7.6. $\phi \in \Sigma$, $\neg\phi \in \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$, or
7.7. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \notin \Sigma$, or
7.8. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \in \Sigma$, $\neg\psi \in \Sigma$, or
7.9. $\phi \notin \Sigma$, $\neg\phi \in \Sigma$, $\psi \notin \Sigma$, $\neg\psi \in \Sigma$.

A proof of case 7 is analogous to the proof of case 4. □

THEOREM 5.5 (Completeness of the algorithm). *The algorithm is complete.*

PROOF. The contraposition of Lemma 5.4 yields us the proof of this theorem. □

§6. Concluding remarks and future work. In the paper, we propose a proof searching procedure for the natural deduction calculi for the binary extensions of the logic of paradox in (Kooi & Tamminga, 2012). We show that the algorithm is finite, sound and complete.

We believe the procedure will be useful in solving a problem posed in (Kooi & Tamminga 2012, p. 729): which of the nine derivation rules that characterize a truth table $f_\circ$ are necessary and sufficient for which axioms and derivation rules in axiomatizations of $f_\circ$. On the other hand, we hope to apply this procedure to both the other extensions of *LP* and to extensions of the other functionally incomplete logics.

Implementation of this algorithm is another task for the future work. We also plan to investigate derivable rules for the systems in question with respect to making proof searching more efficient. It is of much importance to find rules which are derivable in a maximum number of the systems. Last, not least, some conventional natural deduction rules (for example, the $\circ$-introduction rule in the form of deduction theorem: if $\phi \vdash \psi$ then $\vdash \phi \circ \psi$) need to be studied.

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