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Quantum Hydrodynamic Representation of the Exchange Interaction in The Theory of Description of Magnetically Ordered Media

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Abstract—Ferromagnetics, multiferroics, and other magnetically ordered materials are described by various models of magnetization evolution of the medium. In this work, we develop a multi-particle quantum hydrodynamics method for such systems. We derive the evolution equation for macroscopic magnetization, corresponding to the non-dissipative version of the Landau-Lifshitz equation for spin- $\frac{1}{2}$ particles, using the Heisenberg Hamiltonian. It is shown that the well-known form of the exchange interaction contribution in the Landau-Lifshitz equation arises at the third order in the interaction radius. The possibilities for systematic generalization of the obtained result are discussed when considering the fifth order or particles with a large spin.

Keywords: ferromagnetism, quantum hydrodynamics, Heisenberg Hamiltonian, Landau-Lifshitz equation

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INTRODUCTION

The tradition of deriving equations for the collective dynamics of many-particle systems traces back to the work of *N.N. Bogoliubov*, “Problems of the Dynamic Theory in Statistical Physics” [1], where a method for constructing chains of kinetic equations and subsequent closure procedures was proposed. First and foremost, Bogoliubov’s chain method allows for obtaining the Boltzmann kinetic equation for weakly interacting gases, where atomic interactions are taken into account through the collision integral. Additionally, in [1], the derivation of the Vlasov kinetic equation for plasma is presented, where long-range interactions between charged particles are accounted for via self-consistent electric fields. This result was obtained in the quasi-static approximation, leaving open the question of deriving the complete system of Maxwell-Vlasov equations, which has recently been addressed in works [2–4].

The approach proposed by *N.N. Bogoliubov* has undergone significant development and has served as an example for constructing chains of equations based on alternative methods for describing physical systems.

Prominent examples include the derivation of the Landau-Silin kinetic equation for Fermi liquids by *A.S. Kondratiev* (see, for example, [5]), and the diagrammatic method by *M.V. Keldysh*, explored in the works [6, 7].

An important issue in hydrodynamic and kinetic theories, within the context of classical physics, is the need to construct smooth material fields (scalar density field, vector current field, pressure field, temperature field, etc.) based on the dynamics of point particles. This is addressed by *Y.L. Klimontovich*, who considers it as a superposition of delta functions [8, 9].

One of the primary methods of smoothing (constructing smooth fields) is the method of averaging over the distribution function [10]. However, this approach is valid for stochastic systems where the particle system “forgets” the initial conditions after some time τ_0 , which is associated with partial neglect of cause-and-effect relationships in the dynamics of individual particles. A more general approach to constructing smooth material fields while preserving cause-and-effect relationships in the dynamics of interacting particles can be achieved through explicit spatial averaging, specifically over a physically infinitesimal volume. Usually, in the literature, this operation either remains implicit or is replaced by averaging over the distribution function. However,

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an analytical operator that provides explicit averaging over a physically infinitesimal volume for a specific quantity Δ was proposed by *M. Drofa* and *L.S. Kuzmenkov* in their paper [11]. This approach allowed for formulating classical mechanics of many-particle systems in terms of the evolution of material fields. The closure of the system of equations, under certain restrictions on the set of material fields, led to the derivation of hydrodynamic equations in a deterministic interpretation while preserving cause-and-effect relationships in the particle evolution. Considering material fields in a six-dimensional phase space of coordinates and momenta enables the non-probabilistic derivation of kinetic equations and the corresponding interpretation of the “distribution function”. These results have led to the need for implementing a similar approach within the framework of quantum mechanics [12, 13].

The many-particle wave function contains information about a quantum system of particles, and its evolution satisfies the non-stationary *Schrödinger equation*. This evolution can be reformulated in terms of material fields, analogous to those introduced in classical mechanics. Let us focus on the definition of the simplest material field in quantum mechanics, which is the scalar field of concentration (particle number density):

$$n(\mathbf{r}, t) = \sum_{S=s_1, \dots, s_N} \int dR \times \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \Psi_S^\dagger(R, t) \Psi_S(R, t), \quad (1)$$

where $dR = \prod_{i=1}^N d\mathbf{r}_i$ is the volume element in the $3N$ -dimensional configuration space. Here, N is the number of particles, and $\Psi(R, t)$ is the wave function of the particle system. Equation (1) explicitly includes summation over spin variables (below, for the sake of formula compactness, the summation over spin is omitted $\sum_{S=s_1, \dots, s_N}$).

The representation of quantum mechanics as the evolution of material fields, whose dynamics follow the evolution of the many-particle wave function, is a general concept. However, along this path, it has been possible to construct hydrodynamic models for a variety of physical systems. Spin-polarized plasma-like media can be described by a system of equations known as quantum hydrodynamics, which consists of equations for the evolution of concentration, velocity fields, and spin density.

Alternatively, spin-polarized plasma-like media can be described using quantum hydrodynamics with separate spin evolutions. Another example is the derivation of quantum hydrodynamics equations for ultracold quantum gases. This includes both bosonic

systems (atoms with spin 0 or 1) and fermionic systems (spin-1/2). The dynamics of spin in such systems is studied, although the mechanisms of spin polarization are only partially considered. One example is the direct interaction of an external magnetic field with the spins of particles. However, in magnetically ordered materials, spontaneous magnetization can occur. One of the simplest mechanisms for its emergence in ferromagnetic and antiferromagnetic materials is the exchange interaction, which can be expressed at the microscopic level in the form of the Heisenberg Hamiltonian: $\hat{H}_H = -J\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$, here, $\hat{\mathbf{S}}_1$ and $\hat{\mathbf{S}}_2$ represent the spins of two interacting particles, and J is the exchange integral associated with the overlap of wave functions.

On the other hand, in the Landau–Lifshitz equation for the evolution of the magnetization \mathbf{M} of a medium, there is a contribution from the exchange interaction, which can be phenomenologically derived from symmetry considerations [14]: $\partial_t \mathbf{M} = -\frac{g|e|}{2m_e c} \times \alpha[\mathbf{M}, \Delta \mathbf{M}]$, here, g is the gyromagnetic ratio for the ferromagnetic material, e is the charge of the electron, m_e is the electron mass, and c is the speed of light.

The symmetric tensor $\hat{\alpha}$ is chosen in a diagonal form, which corresponds to cubic crystals. In the literature, an estimation of the parameter α is presented in terms of the Curie temperature T_c , lattice constant a , and magnetization magnitude M : $\alpha \sim T_c/(aM^2)$ [14].

In this equation, the right-hand side is due to the slow variation of the magnetization vector in a medium with non-uniform magnetic moment density. In fact, the Heisenberg Hamiltonian has already been used to derive the model of a Bose–Einstein condensate for spin-1 atoms [15], where it is shown that this Hamiltonian does not contribute to the evolution equation of the magnetic moment and the nematic tensor.

The standard theory of a Bose–Einstein condensate is based on the Gross–Pitaevskii equation, which arises in the first order of the interaction radius for both spinless particles and particles with spin. Therefore, in the work [15], the spin evolution equation is considered in the first order of the interaction radius. The spin-dependent short-range interaction, taken in the form of the Heisenberg Hamiltonian, is examined in the third order of the interaction radius in this study. This approach allowed for reproducing the contribution of the inhomogeneous magnetization distribution in the sample.

The following method of quantum hydrodynamics exhibits similarities to many methods used to derive hydrodynamic or kinetic equations in quantum physics (see, for example, the Wigner function method for deriving the quantum generalization of the Vlasov

equation in the book *V.P. silin* [16]). The main difference of the “quantum hydrodynamics method” lies in the systematic development of the field representation method of classical mechanics for quantum particle systems. The derivation of quantum hydrodynamics equations can also be considered as a generalization of the works by *Madelung* and *Takabayasi* to systems with a large number of particles. In the case of multi-particle systems, there is a need to project the multidimensional configuration space onto the three-dimensional physical space, whereas for a single particle, these spaces are equivalent to each other.

The simplest function that describes the dynamics of a particle ensemble is the particle number concentration. In classical physics, its definition has a clear physical meaning as the number of particles per unit volume. By considering identical volume elements at different points in space, one can obtain a scalar field of concentration. In quantum physics, where the wave function has a probabilistic interpretation, the particle concentration can be identified with the total probability density of all particles in the system located in the vicinity of a chosen point in space and is given by: (1).

The formula (1) can be rewritten in the following form:

$$n(\mathbf{r}, t) = \sum_{i=1}^N \int \Psi^\dagger(\mathbf{r}_1, \dots, \mathbf{r}_{i-1}, \mathbf{r}, \mathbf{r}_{i+1}, \dots, \mathbf{r}_N, t), \\ \Psi(\mathbf{r}_1, \dots, \mathbf{r}_{i-1}, \mathbf{r}, \mathbf{r}_{i+1}, \dots, \mathbf{r}_N, t) \\ \times d\mathbf{r}_1 \dots d\mathbf{r}_{i-1} d\mathbf{r}_{i+1} \dots d\mathbf{r}_N, \quad (2)$$

here, the coordinate of the i -th particle in the wave function is represented by the coordinate without an index, \mathbf{r} . After implicit integration over the coordinates of all particles except for the selected one, we obtain the probability density of the i -th particle projected into physical space as $n(\mathbf{r}, t) = \sum_{i=1}^N \rho_i(\mathbf{r}, t)$, where $\rho_i(\mathbf{r}, t)$ implicitly depends on the state of other particles in the system through the common wave function $\Psi(\mathbf{r}_1, \dots, \mathbf{r}_N)$.

The evolution of the concentration is determined by the evolution of the wave function, and its calculation requires an explicit form of the Hamiltonian of the particle system. In this work, we will consider a simple model Hamiltonian necessary for solving the problem of constructing a hydrodynamic model of ferromagnets. The key element of the Hamiltonian is the exchange Coulomb interaction, which, for particles with spin 1/2, is described by the Heisenberg Hamiltonian $\hat{H}_H = -J\hat{\mathbf{S}}_1 \cdot \hat{\mathbf{S}}_2$ (see, for example, [17, 18]).

This article is structured as follows. Section II provides the foundations of the quantum hydrodynamics method. Section III discusses the derivation

of the equation of spin/magnetization evolution in a medium under the influence of exchange interaction. Section IV provides a brief discussion of the obtained results.

1. FUNDAMENTAL PRINCIPLES OF QUANTUM HYDRODYNAMICS METHOD

Let us write the non-stationary Schrödinger equation in the coordinate representation with a model Hamiltonian that enables examining one of the fundamental properties of the magnetically ordered systems under investigation:

$$i\hbar\partial_t\Psi = \left(\sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m} - \frac{1}{2} \sum_{i,j=1, j \neq i}^N U(|\mathbf{r}_i - \mathbf{r}_j|) \hat{\mathbf{S}}_i \hat{\mathbf{S}}_j \right), \quad (3)$$

here, $\mathbf{p}_i = -i\hbar\nabla_i$ is the momentum operator of the i -th particle, \hbar is the reduced Planck constant, m is the mass of the particle, N is the total number of particles, $\Psi = \Psi(R, t)$ is the wave function of the particle system, $R = \{\mathbf{r}_1, \dots, \mathbf{r}_N\}$ is the set of coordinates of the N particles, forming a vector in the $3N$ -dimensional configuration space, $\hat{\mathbf{S}}_i$ is the spin operator of the i -th particle.

The first (second) term on the right-hand side of equation (3) represents the sum of kinetic energies (the sum of potential energies of pairwise interactions), where the potential energy is chosen in the form of the Heisenberg Hamiltonian. It is assumed that all the particles under consideration belong to the same type, hence the mass m does not include the particle index. In fact, we are restricting ourselves to particles with spin 1/2, as for particles with larger spins, a more general form of the exchange interaction should be considered. However, the results obtained below are part of a more general model that arises for larger spins. Therefore, we do not specify the form of the spin operator $\hat{\mathbf{S}}_i$.

The Heisenberg Hamiltonian arises as a result of comparing the energy values of the exchange Coulomb interaction for different mutual orientations of the spins of two interacting particles. From the perspective of further applications, the Heisenberg Hamiltonian characterizes an effective short-range spin-spin interaction. This makes it possible for us to follow the method of expansion in terms of the interaction radius, as presented in the works [15, 19, 20], which corresponds to an expansion in a small dimensionless parameter formed by the ratio of the action radius of the potential r_0 (in our case, the region of overlap of the wave functions of the valence electrons of interacting atoms or ions, leading to

exchange interaction) to the characteristic scale of the system's inhomogeneity, which can be chosen as the wavelength, soliton width, or diameter of a vortex structure.

The study of the evolution of particle concentration (1) obviously leads to the continuity equation $\partial_t n + \nabla \cdot \mathbf{j} = 0$, in which an explicit definition of the particle flux (momentum density) emerges as $\mathbf{j} = n\mathbf{v}$ (see [12, 15, 19], where the method of introducing the velocity field \mathbf{v} into the equations of quantum hydrodynamics is also described).

The evolution of the particle flux leads to the Euler equation:

$$m\partial_t j^\alpha + m\partial_\beta \Pi^{\alpha\beta} = F_{int}^\alpha, \quad (4)$$

here, the indices correspond to the projections of three-dimensional vectors. The quantity $\Pi^{\alpha\beta}$ represents the tensor of momentum flux, which is extensively discussed in other works (see [15, 19, 20]). The quantity F_{int}^α represents the force density resulting from particle interactions, which will be discussed further after considering the spin density. For this particular Hamiltonian, the interaction force density can be expressed in terms of the two-particle function.

2. EQUATION OF SPIN DENSITY EVOLUTION

Changes in magnetization can occur due to the movement of spin-polarized particles over macroscopic distances (spin current of quasifree particles), local displacements of particles, or changes in the spin projection of immobile particles (as in the Ising model for stationary atoms located at the nodes of the crystalline lattice). In particular, in the Ising model, the particle concentration is constant and the velocity field is zero.

Let us examine the evolution of the spin density vector field without restrictions on the nature of particle motion, using the selected model Hamiltonian (3). The definition of spin density in quantum hydrodynamics takes the following form [15]:

$$\mathbf{S}(\mathbf{r}, t) = \int dR \sum_i \delta(\mathbf{r} - \mathbf{r}_i) \Psi^\dagger(R, t) \hat{\mathbf{S}}_i \Psi(R, t). \quad (5)$$

Let us consider the time derivative of the function (5) to determine the equation of spin density evolution. The time derivatives of multi-particle wave functions can be expressed in terms of the Hamiltonian according to equation (3). By separating the contributions from the kinematic terms and the interaction terms,

we obtain the equation of evolution for spin density in the following form:

$$\partial_t S^\alpha + \partial_\beta J^{\alpha\beta} = N_{int}^\alpha, \quad (6)$$

here, $J^{\alpha\beta}$ represents the spin current density, and its microscopic definition can be found in the works [15, 21]. The equation of spin evolution (6) also includes the torque term:

$$N_{int}^\alpha = \varepsilon^{\alpha\beta\gamma} \int d\mathbf{r}' U(|\mathbf{r} - \mathbf{r}'|) S_2^{\beta\gamma}(\mathbf{r}, \mathbf{r}', t), \quad (7)$$

here

$$S_2^{\beta\gamma}(\mathbf{r}, \mathbf{r}', t) = \int dR \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \delta(\mathbf{r}' - \mathbf{r}_j) \times \Psi^\dagger(R, t) \hat{S}_i^\beta \hat{S}_j^\gamma \Psi(R, t) \quad (8)$$

represents a two-particle/two-coordinate spin function. In most cases, the literature uses the concept of the two-particle function. Consequently, the concentration and spin density are referred to as one-particle functions, despite the fact that they describe many-particle systems. The term “two-coordinate function” is also encountered in the literature since the function describing the evolution of a many-particle system depends on two coordinates, \mathbf{r} and \mathbf{r}' .

During the derivation of the spin evolution equation (6), the commutation relations for spin operators were utilized: $[\hat{S}_i^\alpha, \hat{S}_j^\beta] = i\hbar\delta_{ij}\varepsilon^{\alpha\beta\gamma}\hat{S}_i^\gamma$.

It should be noted that in the self-consistent field approximation, which is not generally applicable to the short-range/contact interactions considered here, the two-coordinate spin function has a multiplicative form (8): $S_2^{\beta\gamma}(\mathbf{r}, \mathbf{r}', t) = S^\beta(\mathbf{r}, t) S^\gamma(\mathbf{r}', t)$. In this formula, the structure of the considered two-coordinate function is emphasized. It should also be noted that the application of the radius expansion in the Euler equation (4) in the main approximation leads to an expression that is obtained later and bears external similarity to the multiplicative approximation.

The main focus of the article is on the spin evolution equation, so let us present the explicit form of the kinematic part of this equation, following the work [21],

$$n(\partial_t + \mathbf{v} \cdot \nabla) s^\alpha - \frac{\hbar}{2m} \varepsilon^{\alpha\mu\nu} \partial_\beta (n s^\mu \partial^\beta s^\nu) + \mathfrak{S}^\alpha = N_{int}^\alpha, \quad (9)$$

here $\mathbf{s}(\mathbf{r}, t) = \mathbf{S}(\mathbf{r}, t)/n$ is the specific density of spin, and \mathfrak{S} is the thermal component of the spin current. The quantity \mathfrak{S} in equation (9) is analogous to pressure. In the case of degenerate fermionic systems,

where the distribution of particles among quantum states is governed by the Pauli principle, the spin current \mathfrak{S} takes a non-zero value, which can be referred to as the Fermi spin current. An example of a systematic analysis of the thermal component of spin current can be found in the work [21], although it is not taken into account in most studies (see, for example, [22–24]). The presented expression arises when introducing the velocity field into the equations of quantum hydrodynamics [21]. The term proportional to the Planck constant reflects the contribution of the Bohm potential to the spin evolution.

Let us consider the main steps of the radius-of-interaction expansion method presented in the works [19, 20]. For this, let us express the torque field density (7) without using auxiliary two-coordinate functions:

$$N_{int}^\alpha = \varepsilon^{\alpha\beta\gamma} \int dR \sum_{i,j \neq i} \delta(\mathbf{r} - \mathbf{r}_i) \times U(|\mathbf{r}_i - \mathbf{r}_j|) \Psi^\dagger(R, t) \hat{S}_i^\beta \hat{S}_j^\gamma \Psi(R, t), \quad (10)$$

where we explicitly highlight the contribution of the coordinates of the i -th and j -th particles in the argument of the wave function and the $3N$ -dimensional differential: $\Psi(R, t) = \Psi(\mathbf{r}_1, \dots, \mathbf{r}_i, \dots, \mathbf{r}_j, \dots, \mathbf{r}_N, t)$ and $dR = d\mathbf{r}_i d\mathbf{r}_j dR_{N-2}$. Let us introduce the coordinates of the center of mass and the relative motion for the i -th and j -th particles $\mathbf{r}_{ij} = \mathbf{r}_i - \mathbf{r}_j$, $\mathbf{R}_{ij} = (\mathbf{r}_i + \mathbf{r}_j)/2$, and make the corresponding substitution in formula (10). Moreover, using the symmetry of the integrand expression in formula (10), the torque density can be rewritten in the following form:

$$N_{int}^\alpha = \frac{1}{2} \varepsilon^{\alpha\beta\gamma} \int dR \sum_{i,j} (\delta(\mathbf{r} - \mathbf{r}_i) - \delta(\mathbf{r} - \mathbf{r}_j)) \times U(|\mathbf{r}_i - \mathbf{r}_j|) \Psi^\dagger(R, t) \hat{S}_i^\beta \hat{S}_j^\gamma \Psi(R, t), \quad (11)$$

where the symmetry/antisymmetry of the wave function under the interchange of coordinates is used. The key property of the potential $U(r_{ij})$ is that it rapidly decays as the relative distance between the particles r_{ij} increases. Because of this, the entire integrand expression tends to zero for large values of the relative distance between the particles r_{ij} . Therefore, we can expand the integrand expression in a Taylor series around the relative distance vector between the particles, \mathbf{r}_{ij} . The expansion of the difference of delta functions yields odd powers of the projections of the relative distance r_{ij}^α . The lowest order of the expansion that yields a non-zero contribution occurs when we take the zeroth-order expansion of one wave function and the first-order expansion of the second wave function. As a result, the integral over the relative

motion coordinates of the two particles separates and gives us an interaction constant $g_M \sim \int d\xi \xi^2 U(\xi)$.

The procedure described above allows for taking into account the small range of the potential. In further calculations, we treat the multi-particle wave function as a product of single-particle wave functions:

$$n(\partial_t + \mathbf{v} \cdot \nabla) \mathbf{s} - \frac{\hbar}{2m} \partial^\beta [n \mathbf{s}, \partial^\beta \mathbf{s}] = g_M [n \mathbf{s}, \Delta(n \mathbf{s})], \quad (12)$$

here

$$g_M \equiv \frac{1}{6} \int d\xi \xi^2 U(\xi). \quad (13)$$

2.1. Interaction Force Density

The field of interaction force density, appearing in the Euler equation, is expressed in terms of the two-coordinate spin function (8):

$$F_{int}^\alpha = \int d\mathbf{r}' (\nabla^\alpha U(|\mathbf{r} - \mathbf{r}'|)) S_2^{\beta\beta}(\mathbf{r}, \mathbf{r}', t). \quad (14)$$

Similarly to the result obtained for the torque acting on spin density, one can calculate the interaction force density by considering the contribution from the expansion of the force density in a series around the interaction radius, focusing on the leading non-zero term.

This corresponds to the first order in the expansion around the interaction radius, and leads to the following expression for the force density: $F_{int}^\alpha = g_0 S^\beta \partial^\alpha S^\beta$, here, $g_0 = \int U(\xi) d\xi$ is the interaction constant different from (13).

2.2. On the Structure of Dispersion Dependence of Magnons

Let us investigate the dispersion dependence of spin waves within the proposed model. We will consider an unbounded medium of neutral atoms whose spins/magnetic moments are polarized in a single direction (we choose the z -axis parallel to this direction) and analyze small perturbations around the equilibrium state, characterized by equilibrium values of concentration n_0 and specific spin density $\mathbf{s}_0 = s_0 \mathbf{e}_z$. The velocity field in the equilibrium state is zero. The dipole–dipole interaction, as evident from the presented model, is assumed to be small compared to the exchange interaction. This approximation holds only in specific cases, but we aim to investigate the chosen model scenario. The approximation mentioned above is valid only in certain cases, but we are focused on examining the chosen model case. Let us represent dynamic quantities as the sum of equilibrium values and small perturbations, for example,

$\mathbf{s} = \mathbf{s}_0 + \delta\mathbf{s}$, in the spin evolution equation, neglecting terms that are nonlinear in small perturbations. As a result, we obtain a linearized spin evolution equation:

$$n_0 \partial_t \delta\mathbf{s} - \frac{\hbar}{2m} n_0 [\mathbf{s}_0, \Delta \delta\mathbf{s}] - g_M n_0^2 [\mathbf{s}_0, \Delta \delta\mathbf{s}] = 0. \quad (15)$$

Note that the obtained equation is closed within the considered approximation.

Let us investigate small perturbations in the form of plane monochromatic waves $|\delta\mathbf{s}| \sim \exp(-i\omega t + i\mathbf{k}\mathbf{r})$, which propagate in an arbitrary direction \mathbf{k} . Their dispersion relation is obtained in the following form:

$$\omega = s_0 k^2 \left| n_0 g_M + \frac{\hbar}{2m} \right|. \quad (16)$$

For the considered ferromagnets, the constant $g_M > 0$ is positive, so we can omit the modulus sign. The presented model, including the chosen form of the equilibrium state, does not take into account the periodic arrangement of atoms in the lattice. Therefore, we obtain the long-wavelength limit of the magnon dispersion relation in a crystal. Let us compare the obtained result with the formulas arising for a chain of classical magnetic moments (see, for example, [17]), where the following expression appears: $\omega = 2Js_0(1 - \cos(ka))$, transitioning to the long-wavelength limit $\omega = Js_0 a^2 k^2$, here, a is the lattice period of the crystal given by $a = n_0^{-1/3}$, and for the magnetically ordered systems under consideration $J > 0$. From the above formulas, it can be seen that the lattice dispersion relation is expressed in terms of the exchange integral, whereas the proposed model leads to an integral involving the exchange integral. A more comprehensive comparison can be obtained by approximately calculating the interaction constant (13) for model forms of short-range interaction/exchange integral potentials.

Before presenting the result of the interaction constant estimation, it is worth noting that the long-wavelength dispersion relation (16) contains a quantum contribution $\sim \hbar$ compared to the classical chain of magnetic moments. As a first example, let us consider the exchange integral in the form of the Heaviside function, where $U(\xi) = J$ for $\xi \leq r_0$ and $U(\xi) = 0$ for $\xi > r_0$. Taking the lattice period as the effective range of the potential, we have $r_0 = a$, resulting in $g_M = 2\pi J a^5/15$. The factor $a^3 = n_0^{-1}$ that appears here leads to the reduction of the corresponding coefficient in the dispersion relation.

This gives the following semiclassical limit of the formula (16): $\omega = 2\pi s_0 k^2 J a^2/15$. The extra factor of

$2\pi/15$ can be compensated by an appropriate choice of the potential range r_0 .

Let us consider a second model example of an effective potential in the form of the delta function on a sphere of radius a : $U(\xi) = J a \delta(\xi - a)$. This leads to the following expression for the interaction constant: $g_M = 2\pi J a^5/3$, and the frequencies are given by $\omega = 2\pi s_0 k^2 J a^2/3$. These examples demonstrate a direct connection between the obtained frequency expression and the dispersion relation for a chain of classical magnetic moments.

Based on the analysis of the interaction constant, let us focus on the corrections that arise in the fifth order of the interaction radius in the spin evolution equation. The corresponding expansion in equation (11) leads to an additional interaction constant $g_{2M} \sim \int d\xi \xi^4 U(\xi)$. For model potentials, this constant can be expressed in terms of the exchange integral J or the interaction constant g_M (from equation 13), specifically $g_{2M} \sim a^2 g_M \sim n_0^{-2/3} g_M$.

The functional dependence can be expressed in terms of derivatives of the amplitudes or phases of the wave function. In the first case, which corresponds to the aforementioned result, one can expect corrections involving higher-order spatial derivatives. The second case leads to contributions from the tensor of kinetic pressure or higher-rank kinetic functions. Such examples are realized in certain types of fermionic systems [20]. When considering particles with large spins, additional terms arise in the Hamiltonian (3) describing additional effective spin interactions. This interaction is proportional to higher powers of the scalar product of the spin operators of the interacting particle pair. It contributes to the additional spin density of macroscopic functions and the corresponding generalization of the spin evolution equation. For small deviations from the equilibrium ferromagnetic state, the additional functions can be approximated in terms of the spin density, and the additional terms take a form that coincides with the right-hand side of the spin evolution equation (12). Let us consider the influence of spin dynamics on sound waves described by the Euler equation, taking into account the spin-independent short-range interaction. We will estimate the contribution of the effective spin-spin interaction associated with the Heisenberg exchange interaction in the Euler equation. The corresponding force density takes the form $\mathbf{F} \sim \nabla(n\mathbf{s})^2/2$. In the linear approximation, the spin density can be represented as $\mathbf{F} \sim n_0 \mathbf{s}_0 \nabla(n_0 \delta\mathbf{s} + \mathbf{s}_0 \delta n)$. It should be noted that the spin evolution equation leads to the expression $\mathbf{s}_0 \delta\mathbf{s} = s_0 \delta s_z = 0$. In the end, the additional term in the force density is nonzero and is expressed in terms of the concentration perturbation: $\mathbf{F} = g_0 n_0 s_0^2 \nabla \delta n$, which leads to a change in

the square of the sound velocity, compared to the analogous spinless system, by $\Delta v^2 = g_0 s_0^2 n_0 / m$.

The interaction constant g_0 can also be expressed in terms of the effective value of the exchange integral J when using a model dependence of the exchange integral on distance. As before, let us consider the example of the exchange integral in the form of the Heaviside function: $U(\xi) = J$ for $\xi \leq r_0 = a$ and $U(\xi) = 0$ for $\xi > r_0$. This gives the representation of the interaction constant as $g_0 = 4\pi r_0^3 J / 3 = 4\pi J / 3n_0$, and the change in the square of the sound velocity can be written as $\Delta v^2 = 4\pi J s_0^2 / 3m$.

Let us briefly mention ferrofluids and note that the proposed model, despite its hydrodynamic form, is developed to describe magnetically ordered media in various aggregate states where exchange Coulomb interaction plays a crucial role in spin dynamics.

CONCLUSIONS

The study demonstrates that the quantum hydrodynamic approach allows for deriving the evolution equation for the magnetic moment in magnetically ordered media, where the dynamic properties of the magnetic moment are governed by exchange interaction. Such an approach includes the consideration of multiferroics, although the magnetoelectric effect that distinguishes them from other magnetically ordered media will be separately addressed. At the microscopic level, for fermions with spin 1/2, the exchange interaction is described by the Heisenberg Hamiltonian. The paper demonstrates the transition from the microscopic model to the macroscopic one. It shows the emergence of a set of interaction constants and their approximate representation in terms of an effective value of the exchange integral.

DATA AVAILABILITY

The question of data availability is not applicable to this article, as this purely theoretical research did not involve the creation or analysis of new data.

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CONFLICT OF INTEREST

The authors declare that they have no conflicts of interest.

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