J. Phys. A: Math. Gen. **35** (2002) L439–L445 PII: S0305-4470(02)36186-9

LETTER TO THE EDITOR

Action–angle coordinates for time-dependent completely integrable Hamiltonian systems

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Received 24 April 2002
Published 12 July 2002
Online at stacks.iop.org/JPhysA/35/L439

Abstract

A time-dependent completely integrable Hamiltonian system is proved to admit the action–angle coordinates around any instantly compact regular invariant manifold. Written relative to these coordinates, its Hamiltonian and first integrals are functions only of action coordinates.

PACS numbers: 45.20.Jj, 02.30.Ik

1. Introduction

A time-dependent Hamiltonian system of m degrees of freedom is called a completely integrable system (CIS), if it admits m independent first integrals in involution. In order to provide this with action–angle coordinates, we use the fact that a time-dependent CIS of m degrees of freedom can be extended to an autonomous Hamiltonian system of $m + 1$ degrees of freedom where time is regarded as a dynamic variable [2, 3, 7]. We show that it is an autonomous CIS; however, the classical theorem [1, 5] on action–angle coordinates cannot be applied to this CIS since its invariant manifolds are never compact because of the time axis. Generalizing this theorem, we first prove that there is a system of action–angle coordinates in an open neighbourhood U of a regular invariant manifold M of an autonomous CIS if Hamiltonian vector fields of first integrals on U are complete and the foliation of U by invariant manifolds is trivial. If M is compact, these conditions always hold [5]. Afterwards, we show that, if a regular connected invariant manifold of a time-dependent CIS is compact at each instant, it is diffeomorphic to the product of the time axis $\mathbb{R}$ and an m-dimensional torus T^m, and it admits an open neighbourhood equipped with the time-dependent action–angle coordinates $(I_i; t, \phi^i)$, $i = 1, \ldots, m$, where t is the Cartesian coordinate on $\mathbb{R}$ and ϕ^i are cyclic coordinates on T^m. Written with respect to these coordinates, a Hamiltonian and the first integrals of a time-dependent CIS are functions only of action coordinates I_i.

For instance, there are action–angle coordinates $(\overline{I}_i; \overline{\phi}^i)$ such that a Hamiltonian of a time-dependent CIS vanishes. They are particular initial data coordinates, constant along the trajectories of a Hamiltonian system. Furthermore, given an arbitrary smooth function $\mathcal{H}$ on $\mathbb{R}^m$, there exist action–angle coordinates $(I_i; \phi^i)$, obtained by the relevant time-dependent canonical transformations of $(\overline{I}_i; \overline{\phi}^i)$, such that a Hamiltonian of a time-dependent CIS with respect to these coordinates equals $\mathcal{H}(I_i)$. Thus, time-dependent action–angle coordinates provide a solution to the problem of representing a Hamiltonian of a time-dependent CIS in terms of first integrals [4, 6]. However, this representation need not hold with respect to any coordinate system because a Hamiltonian fails to be a scalar under time-dependent canonical transformations.

2. Time-dependent completely integrable Hamiltonian systems

Recall that the configuration space of a time-dependent mechanical system is a fibre bundle $Q \to \mathbb{R}$ over the time axis $\mathbb{R}$ equipped with the bundle coordinates (t, q^k), $k = 1, \ldots, m$. The corresponding momentum phase space is the vertical cotangent bundle V^*Q of $Q \to \mathbb{R}$ endowed with holonomic coordinates $(t, q^k, p_k = \dot{q}_k)$ [8–10]. The cotangent bundle T^*Q, coordinated by $(q^\lambda, p_\lambda) = (t, q^k, p_0, p_k)$, is the homogeneous momentum phase space of time-dependent mechanics. It is provided with the canonical Liouville form $\Xi = p_\lambda \, \mathrm{d}q^\lambda$, the canonical symplectic form $\Omega = \mathrm{d}p_\lambda \wedge \mathrm{d}q^\lambda$, and the corresponding Poisson bracket

$$\{f, f'\}_T = \partial^\lambda f \partial_\lambda f' - \partial_\lambda f \partial^\lambda f' \qquad f, f' \in C^\infty(T^*Q). \tag{1}$$

There is the one-dimensional trivial affine bundle

$$\zeta : T^*Q \to V^*Q. \tag{2}$$

Given its global section h, one can equip T^*Q with the global fibre coordinate $r = p_0 - h$. The fibre bundle (2) provides the vertical cotangent bundle V^*Q with the canonical Poisson structure $\{\,,\,\}_V$ such that

$$\zeta^*\{f, f'\}_V = \{\zeta^* f, \zeta^* f'\}_T \qquad \forall f, f' \in C^\infty(V^*Q) \tag{3}$$

$$\{f, f'\}_V = \partial^k f \partial_k f' - \partial_k f \partial^k f'. \tag{4}$$

A Hamiltonian of time-dependent mechanics is defined as a global section

$$h : V^*Q \to T^*Q \qquad p_0 \circ h = -\mathcal{H}(t, q^j, p_j)$$

of the affine bundle ζ (2) [8, 9]. It yields the pull-back Hamiltonian form

$$H = h^*\Xi = p_k \, \mathrm{d}q^k - \mathcal{H} \, \mathrm{d}t \tag{5}$$

on V^*Q. Then there exists a unique vector field γ_H on V^*Q such that

$$\gamma_H \rfloor \mathrm{d}t = 1 \qquad \gamma_H \rfloor \mathrm{d}H = 0$$
$$\gamma_H = \partial_t + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k. \tag{6}$$

Its trajectories obey the Hamilton equation

$$\dot{q}^k = \partial^k \mathcal{H} \qquad \dot{p}_k = -\partial_k \mathcal{H}. \tag{7}$$

The first integral of the Hamilton equation (7) is a smooth real function F on V^*Q whose Lie derivative

$$\mathbf{L}_{\gamma_H} F = \gamma_H \rfloor \mathrm{d}F = \partial_t F + \{\mathcal{H}, F\}_V$$

along the vector field γ_H (6) vanishes, i.e. F is constant on trajectories of γ_H. A time-dependent Hamiltonian system (V^*Q, H) is said to be completely integrable if the Hamilton equation (7) admits m first integrals F_k which are in involution with respect to the Poisson bracket $\{\,,\,\}_V$ (4), and whose differentials $\mathrm{d}F_k$ are linearly independent almost everywhere (i.e. the set of points where this condition fails is nowhere dense). One can associate this CIS with an autonomous CIS on T^*Q as follows.

Let us consider the pull-back ζ^*H of the Hamiltonian form H (5) onto the cotangent bundle T^*Q. It is readily observed that

$$\mathcal{H}^* = \partial_t \rfloor (\Xi - \zeta^* h^* \Xi) = p_0 + \mathcal{H} \tag{8}$$

is a function on T^*Q. Let us regard $\mathcal{H}^*$ as a Hamiltonian of an autonomous Hamiltonian system on the symplectic manifold (T^*Q, Ω) [10]. Its Hamiltonian vector field

$$\gamma_T = \partial_t - \partial_t \mathcal{H} \partial^0 + \partial^k \mathcal{H} \partial_k - \partial_k \mathcal{H} \partial^k \tag{9}$$

is projected onto the vector field γ_H (6) on V^*Q so that

$$\zeta^*(\mathbf{L}_{\gamma_H} f) = \{\mathcal{H}^*, \zeta^* f\}_T \qquad \forall f \in C^\infty(V^*Q).$$

An immediate consequence of this relation is the following.

Theorem 1. *(i) Given a time-dependent CIS $(\mathcal{H}; F_k)$ on V^*Q, the Hamiltonian system $(\mathcal{H}^*, \zeta^* F_k)$ on T^*Q is a CIS. (ii) Let N be a connected regular invariant manifold of $(\mathcal{H}; F_k)$. Then $h(N) \subset T^*Q$ is a connected regular invariant manifold of the autonomous CIS $(\mathcal{H}^*, \zeta^* F_k)$.*

Hereafter, we assume that the vector field γ_H (6) is complete. In this case, the Hamilton equation (7) admits a unique global solution through each point of the momentum phase space V^*Q, and trajectories of γ_H define a trivial fibre bundle $V^*Q \to V_t^*Q$ over any fibre V_t^*Q of $V^*Q \to \mathbb{R}$. Without loss of generality, we choose the fibre $i_0 : V_0^*Q \to V^*Q$ at $t = 0$. Since N is an invariant manifold, the fibration

$$\xi : V^*Q \to V_0^*Q \tag{10}$$

also yields the fibration of N onto $N_0 = N \cap V_0^*Q$ such that $N \cong \mathbb{R} \times N_0$ is a trivial bundle.

3. Time-dependent action–angle coordinates

Let us introduce the action–angle coordinates around an invariant manifold N of a time-dependent CIS on V^*Q using the action–angle coordinates around the invariant manifold $h(N)$ of the autonomous CIS on T^*Q in theorem 1. Since N and, consequently, $h(N)$ are non-compact, we first prove the following.

Theorem 2. *Let M be a connected invariant manifold of an autonomous CIS $\{F_\lambda\}$, $\lambda = 1, \ldots, n$, on a symplectic manifold (Z, Ω_Z). Let U be an open neighbourhood of M such that: (i) the differentials $\mathrm{d}F_\lambda$ are independent everywhere on U, (ii) the Hamiltonian vector fields ϑ_λ of the first integrals F_λ on U are complete and (iii) the submersion $\times F_\lambda : U \to \mathbb{R}^n$ is a trivial bundle of invariant manifolds over a domain $V' \subset \mathbb{R}^n$. Then U is isomorphic to the symplectic annulus*

$$W' = V' \times (\mathbb{R}^{n-m} \times T^m) \tag{11}$$

provided with the action–angle coordinates

$$(I_1, \ldots, I_n; x^1, \ldots, x^{n-m}; \phi^1, \ldots, \phi^m) \tag{12}$$

such that the symplectic form on W' reads

$$\Omega_Z = \mathrm{d}I_a \wedge \mathrm{d}x^a + \mathrm{d}I_i \wedge \mathrm{d}\phi^i$$

and the first integrals F_λ depend only on the action coordinates I_α.

Proof. In accordance with the well-known theorem [1], the invariant manifold M is diffeomorphic to the product $\mathbb{R}^{n-m} \times T^m$, which is the group space of the quotient $G = \mathbb{R}^n/\mathbb{Z}^m$ of the group $\mathbb{R}^n$ generated by Hamiltonian vector fields ϑ_λ of first integrals F_λ on M. Namely, M is provided with the group space coordinates $(y^\lambda) = (s^a, \varphi^i)$ where φ^i are linear functions of parameters s^λ along integral curves of the Hamiltonian vector fields ϑ_λ on U. Let (J_λ) be coordinates on V' which are values of first integrals F_λ. Let us choose a trivialization of the fibre bundle $U \to V$ seen as a principal bundle with the structure group G. We fix its global section χ. Since parameters s^λ are given up to a shift, let us provide each fibre M_J, $J \in V$, with the group space coordinates (y^λ) centred at the point $\chi(J)$. Then $(J_\lambda; y^\lambda)$ are bundle coordinates on the annulus W' (11). Since M_J are Lagrangian manifolds, the symplectic form Ω_Z on W' is given relative to the bundle coordinates $(J_\lambda; y^\lambda)$ by

$$\Omega_Z = \Omega^{\alpha\beta}\, \mathrm{d}J_\alpha \wedge \mathrm{d}J_\beta + \Omega^\alpha_\beta\, \mathrm{d}J_\alpha \wedge \mathrm{d}y^\beta. \tag{13}$$

By the very definition of coordinates (y^λ), the Hamiltonian vector fields ϑ_λ of first integrals take the coordinate form $\vartheta_\lambda = \vartheta^\alpha_\lambda(J_\mu)\partial_\alpha$. Moreover, since the cyclic group S^1 cannot act transitively on $\mathbb{R}$, we have

$$\vartheta_a = \partial_a + \vartheta^i_a(J_\lambda)\partial_i \qquad \vartheta_i = \vartheta^k_i(J_\lambda)\partial_k. \tag{14}$$

The Hamiltonian vector fields ϑ_λ obey the relations

$$\vartheta_\lambda \rfloor \Omega_Z = -\mathrm{d}J_\lambda \qquad \Omega^\alpha_\beta \vartheta^\beta_\lambda = \delta^\alpha_\lambda. \tag{15}$$

It follows that Ω^α_β is a non-degenerate matrix and $\vartheta^\alpha_\lambda = (\Omega^{-1})^\alpha_\lambda$, i.e. the functions Ω^α_β depend only on coordinates J_λ. A substitution of (14) into (15) results in the equalities

$$\Omega^a_b = \delta^a_b \qquad \vartheta^\lambda_a \Omega^i_\lambda = 0 \tag{16}$$

$$\vartheta^k_i \Omega^j_k = \delta^j_i \qquad \vartheta^k_i \Omega^a_k = 0. \tag{17}$$

The first of the equalities (17) shows that the matrix Ω^j_k is non-degenerate, and so is the matrix ϑ^k_i. The second one gives $\Omega^a_k = 0$. By virtue of the well-known Künneth formula for the de Rham cohomology of a product of manifolds, the closed form Ω_Z (13) on W' (11) is exact, i.e. $\Omega_Z = \mathrm{d}\Xi$ where Ξ reads

$$\Xi = \Xi^\alpha(J_\lambda, y^\lambda)\, \mathrm{d}J_\alpha + \Xi_i(J_\lambda)\, \mathrm{d}\varphi^i + \partial_\alpha \Phi(J_\lambda, y^\lambda)\, \mathrm{d}y^\alpha$$

where Φ is a function on W'. Taken up to an exact form, Ξ is brought into the form

$$\Xi = \Xi'^\alpha(J_\lambda, y^\lambda)\, \mathrm{d}J_\alpha + \Xi_i(J_\lambda)\, \mathrm{d}\varphi^i. \tag{18}$$

Owing to the fact that components of $\mathrm{d}\Xi = \Omega_Z$ are independent of y^λ and obey the equalities (16) and (17), we obtain the following.

(i) $\Omega^a_i = -\partial_i \Xi'^a + \partial^a \Xi_i = 0$. It follows that $\partial_i \Xi'^a$ is independent of φ^i, i.e. Ξ'^a is affine in φ^i and, consequently, is independent of φ^i since φ^i are cyclic coordinates. Hence, $\partial^a \Xi_i = 0$, i.e. Ξ_i is a function only of coordinates J_j.
(ii) $\Omega^k_i = -\partial_i \Xi'^k + \partial^k \Xi_i$. Similarly to item (i), one shows that Ξ'^k is independent of φ^i and $\Omega^k_i = \partial^k \Xi_i$, i.e. $\partial^k \Xi_i$ is a non-degenerate matrix.
(iii) $\Omega^a_b = -\partial_b \Xi'^a = \delta^a_b$. Hence, $\Xi'^a = -s^a + D^a(J_\lambda)$.
(iv) $\Omega^i_b = -\partial_b \Xi'^i$, i.e. Ξ'^i is affine in s^a.

In view of items (i)–(iv), the Liouville form Ξ (18) reads

$$\Xi = x^a\, \mathrm{d}J_a + \left[D^i(J_\lambda) + B^i_a(J_\lambda)s^a\right] \mathrm{d}J_i + \Xi_i(J_j)\, \mathrm{d}\varphi^i$$

where we put

$$x^a = -\Xi'^a = s^a - D^a(J_\lambda). \tag{19}$$

Since the matrix $\partial^k \Xi_i$ is non-degenerate, one can introduce new coordinates $I_i = \Xi_i(J_j)$, $I_a = J_a$. Then we have

$$\Xi = -x^a \, \mathrm{d}I_a + \left[D'^i(I_\lambda) + B'^i_a(I_\lambda)s^a\right] \mathrm{d}I_i + I_i \, \mathrm{d}\varphi^i.$$

Finally, put

$$\phi^i = \varphi^i - \left[D'^i(I_\lambda) + B'^i_a(I_\lambda)s^a\right] \tag{20}$$

in order to obtain the desired action–angle coordinates

$$I_a = J_a \qquad I_j(J_j) \qquad x^a = s^a + S^a(J_\lambda) \qquad \phi^i = \varphi^i + S^i(J_\lambda, s^b).$$

These are bundle coordinates on $U \to V'$ where the coordinate shifts (19) and (20) correspond to a choice of another trivialization of $U \to V'$.

Of course, the action–angle coordinates (12) are by no means unique. For instance, let $\mathcal{F}_a$, $a = 1, \ldots, n - m$ be an arbitrary smooth function on $\mathbb{R}^m$. Let us consider the canonical coordinate transformation

$$I'_a = I_a - \mathcal{F}_a(I_j) \qquad I'_k = I_k \qquad x'^a = x^a \qquad \phi'^i = \phi^i + x^a \partial^i \mathcal{F}_a(I_j). \tag{21}$$

Then $(I'_a, I'_k; x'^a, \phi'^k)$ are action–angle coordinates on the symplectic annulus which differ from W' (11) in another trivialization. □

Now, we apply theorem 2 to the CISs in theorem 1.

Theorem 3. *Let N be a connected regular invariant manifold of a time-dependent CIS on V^*Q, and let the image N_0 of its projection ξ (10) be compact. Then the invariant manifold $h(N)$ of the autonomous CIS on T^*Q has an open neighbourhood U obeying the condition of theorem 2.*

Proof. (i) We first show that functions $i_0^* F_k$ make up a CIS on the symplectic leaf (V_0^*Q, Ω_0) and N_0 is its invariant manifold without critical points (i.e. where first integrals fail to be dependent). Clearly, the functions $i_0^* F_k$ are in involution, and N_0 is their connected invariant manifold. Let us show that the set of critical points of $\{i_0^* F_k\}$ is nowhere dense in V_0^*Q and N_0 has none of these points. Let V_0^*Q be equipped with some coordinates $(\overline{q}^k, \overline{p}_k)$. Then the trivial bundle ξ (10) is provided with the bundle coordinates $(t, \overline{q}^k, \overline{p}_k)$ which play a role of the initial date coordinates on the momentum phase space V^*Q. Written with respect to these coordinates, the first integrals F_k become time-independent. It follows that

$$\mathrm{d}F_k(y) = \mathrm{d}i_0^* F_k(\xi(y)) \tag{22}$$

for any point $y \in V^*Q$. In particular, if $y_0 \in V_0^*Q$ is a critical point of $\{i_0^* F_k\}$, then the trajectory $\xi^{-1}(y_0)$ is a critical set for the first integrals $\{F_k\}$. The desired statement at once follows from this result.

(ii) Since N_0 obeys the condition in item (i), there is an open neighbourhood of N_0 in V_0^*Q isomorphic to $V \times N_0$ where $V \subset \mathbb{R}^m$ is a domain, and $\{v\} \times N_0$, $v \in V$, are also invariant manifolds in V_0^*Q [5]. Then

$$W = \xi^{-1}(V \times N_0) \cong V \times N \tag{23}$$

is an open neighbourhood in V^*Q of the invariant manifold N foliated by invariant manifolds $\xi^{-1}(\{v\} \times N_0)$, $v \in V$, of the time-dependent CIS on V^*Q. By virtue of the equality (22), the first integrals $\{F_k\}$ have no critical points in W. For any real number $r \in (-\varepsilon, \varepsilon)$, let us consider a section

$$h_r : V^*Q \to T^*Q \qquad p_0 \circ h_r = -\mathcal{H}(t, q^j, p_j) + r$$

of the affine bundle ζ (2). Then the images $h_r(W)$ of W (23) make up an open neighbourhood U of $h(N)$ in T^*Q. Because $\zeta(U) = W$, the pull-backs $\zeta^* F_k$ of first integrals F_k are free from critical points in U, and so is the function $\mathcal{H}^*$ (8). Since the coordinate $r = p_0 - h$ provides a trivialization of the affine bundle ζ, the open neighbourhood U of $h(N)$ is diffeomorphic to the product

$$(-\varepsilon, \varepsilon) \times h(W) \cong (-\varepsilon, \varepsilon) \times V \times h(N)$$

which is a trivialization of the fibration

$$\mathcal{H}^* \times (\times \zeta^* F_k) : U \to (-\varepsilon, \varepsilon) \times V.$$

(iii) It remains to prove that the Hamiltonian vector fields of $\mathcal{H}^*$ and $\zeta^* F_k$ on U are complete. It is readily observed that the Hamiltonian vector field γ_T (9) of $\mathcal{H}^*$ is tangent to the manifolds $h_r(W)$, and is the image $\gamma_T = Th_r \circ \gamma_H \circ \zeta$ of the vector field γ_H (6). The latter is complete on W, and so is γ_T on U. Similarly, the Hamiltonian vector field

$$\gamma_k = -\partial_t F_k \partial^0 + \partial^i F_k \partial_i - \partial_i F_k \partial^i$$

of the function $\zeta^* F_k$ on T^*Q with respect to the Poisson bracket $\{,\}_T$ (1) is tangent to the manifolds $h_r(W)$, and is the image $\gamma_k = Th_r \circ \vartheta_k \circ \zeta$ of the Hamiltonian vector field ϑ_k of the first integral F_k on W with respect to the Poisson bracket $\{,\}_V$ (4). The vector fields ϑ_k on W are vertical relative to the fibration $W \to \mathbb{R}$, and are tangent to compact manifolds. Therefore, they are complete, and so are the vector fields γ_k on U. Thus, U is the desired open neighbourhood of the invariant manifold $h(N)$. □

In accordance with theorem 2, the open neighbourhood U of the invariant manifold $h(N)$ of the autonomous CIS in theorem 3 is isomorphic to the symplectic annulus

$$W' = V' \times (\mathbb{R} \times T^m) \qquad V' = (-\varepsilon, \varepsilon) \times V \tag{24}$$

provided with the action–angle coordinates $(I_0, \ldots, I_m; t, \phi^1, \ldots, \phi^m)$ such that the symplectic form on W' reads

$$\Omega = \mathrm{d}I_0 \wedge \mathrm{d}t + \mathrm{d}I_k \wedge \mathrm{d}\phi^k.$$

From the construction in theorem 2, $I_0 = J_0 = \mathcal{H}^*$ and the corresponding generalized angle coordinate is $x^0 = t$, while the first integrals $J_k = \zeta^* F_k$ depend only on the action coordinates I_i.

Since the action coordinates I_i are independent of the coordinate J_0, the symplectic annulus W' (24) inherits the fibration

$$W' \xrightarrow{\zeta} W'' = V \times (\mathbb{R} \times T^m). \tag{25}$$

From the relation similar to (3), the product W'' (25), coordinated by $(I_i; t, \phi^i)$, is provided with the Poisson structure

$$\{f, f'\}_W = \partial^i f \partial_i f' - \partial_i f \partial^i f' \qquad f, f' \in C^\infty(W'').$$

Therefore, one can regard W'' as the momentum phase space of the time-dependent CIS in question around the invariant manifold N.

It is readily observed that the Hamiltonian vector field γ_T of the autonomous Hamiltonian $\mathcal{H}^* = I_0$ is $\gamma_T = \partial_t$, and so is its projection γ_H (6) on W''. Consequently, the Hamilton equation (7) of a time-dependent CIS with respect to the action–angle coordinates take the form $\dot{I}_i = 0, \dot{\phi}^i = 0$. Hence, $(I_i; t, \phi^i)$ are the initial data coordinates. One can introduce such coordinates as follows. Given the fibration ξ (10), let us provide $N_0 \times V \subset V_0^* Q$ in theorem 3 with action–angle coordinates $(\overline{I}_i; \overline{\phi}^i)$ for the CIS $\{i_0^* F_k\}$ on the symplectic leaf $V_0^* Q$. Then, it is readily observed that $(\overline{I}_i; t, \overline{\phi}^i)$ are time-dependent action–angle coordinates on W'' (25) such that the Hamiltonian $\mathcal{H}(\overline{I}_j)$ of a time-dependent CIS relative to these coordinates vanishes, i.e. $\mathcal{H}^* = \overline{I}_0$. Using the canonical transformations (21), one can consider time-dependent action–angle coordinates besides the initial data coordinates. Given a smooth function $\mathcal{H}$ on $\mathbb{R}^m$, one can provide W'' with the action–angle coordinates

$$I_0 = \overline{I}_0 - \mathcal{H}(\overline{I}_j) \qquad I_i = \overline{I}_i \qquad \phi^i = \overline{\phi}^i + t\partial^i \mathcal{H}(\overline{I}_j)$$

such that $\mathcal{H}(I_i)$ is a Hamiltonian of a time-dependent CIS on W''.

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