THE GAUGE TREATMENT OF GRAVITY

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THE GAUGE TREATMENT OF GRAVITY

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Contents:

Abstract:
The gauge gravitation theory, in spite of twenty five years of its history, still remains the single gap in the excellent gauge picture of fundamental interactions. The main disputable point is the gauge status of Einstein's gravitational field, which is a metric or tetrad field, while gauge fields represent connections on fiber bundles. The corner-stones of Einstein's gravitation theory are the Relativity and Equivalence Principles. Having been reformulated in the fiber bundle terms, the gravitation theory turns out to be built from these principles directly as a gauge theory of space-time symmetries, which, however, are spontaneously broken down to the Lorentz symmetries. Metric gravitational fields appear in such a theory as the consequence of this spontaneous symmetry breaking and have the nature of Goldstone type fields. The Lorentz, GL(4, R) and Poincaré gauge theories of gravitation are analyzed from these points of view, and some outlooks of the gauge treatment of gravitation, e.g., as the affine-metric theory, are discussed.

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Why to gauge gravity?

At present Einstein's General Relativity (GR) still remains the most satisfactory theory of classical gravitation for all now observable gravitational fields. GR successfully passed the test of recent experiments on the radiolocation of planets and on the laser-location of the Moon, which have put the end to some other versions of gravitation theory, e.g., the scalar-tensor theory.

At the same time the conventional description of gravity by Einstein's GR obviously faced a number of serious problems [55], and even some corner-stones of gravitation theory still remain disputable up to our day. This is reflected also in the rather curious uninterrupted flow of proposals for new designations of this theory.

Really, it is difficult to find in physics another example of continuous discussions about naming a well-established theory like Einstein's theory of gravitation. The author's own proposal "Allgemeine Relativitätstheorie" as pointing on the generalization of the Special Relativity (SR) is still not admitted by some scientists. A. Friedmann wrote about "small" and "great" principles. V. Fock was a supporter of Fokker's expression "chronogeometry", insisting that GR does not possess any "relativity" (in the sense of SR), and believing with Kretschmann and P. Havas that "general covariance" requirement is a trivial one. H.-J. Treder speaks about "geochronometrical gravity". J.A. Wheeler repeatedly writes about "geometrodynamics", though previously it designated "already unified" theory of gravitation and electromagnetism of Rainich. One would speak now about Newtonian "gravistatics" and Einsteinian "gravidynamics" (after B. de Witt and D. Ivanenko) or "gravitodynamics" (A. Mercier).

It is well known that Einstein repeatedly insisted that the Relativity Principle (RP) is not a priori necessary, i.e., it is not physically trivial. At the same time formulation of this principle is known to be directly connected with establishing the notion of reference frames in gravitation theory, which itself still remains under discussion.

The problem of reference frame definition in GR was not paid sufficient attention up to about the mid-1950s. As was stressed when founding GR, e.g., in the Einstein–Kretschmann–Kottler discussions, revived later by V. Fock, P. Havas and also by H.-J. Treder, the important possibility of a general covariant formulation means that coordinates are only auxiliary quantities losing their immediate physical sense of observable objects, in contrast with, e.g., the coordinates in Minkowski space. Then there arose the necessity to distinguish coordinate systems from reference frames. Einstein, though emphasizing for the first time the role of reference frames, however, did not give himself any formal definition of them. Moreover many contemporary authors mixing coordinates and reference terms simply ignore this problem. But more precise reference system determination is necessary for correct experimentation, for stating the Cauchy problem for gravitational field equations, for describing spinors in GR, and for other problems of gravitation theory.

Most deeply the reference frame problem is examined in the tetrad version of GR in combination with the technique of $(3+1)$ decomposition [76, 91, 47], where tetrads, thought to define local reference frames, are erected in all space-time points. But the dilemma to make up these reference tetrads by a certain choice of physical devices is as yet far from a final solution. All the more in the general case of a curved space-time there may not exist any continuous tetrad distribution, but only up to admitting SO(3)-transformations of tetrads.

The Equivalence Principle (EP) being another corner-stone of GR is also open to question, e.g., one separates "weak", "middle-strong", and "strong" equivalence principles [112, 108].

In GR the Equivalence Principle supplements RP and must establish the existence of a certain reference frame, where all physical laws would take the known special relativistic form; and it seems naturally for some authors to establish the disappearance of a gravitational field as the criterion of transition from GR to SR in some reference frame.

All existent formulations of EP are based upon the empirical equality of inertial mass, gravitational active and passive charges. In the case of a uniform gravitational field it provides the existence of a reference frame searched for, in which the motion law of probe particles is viewed in the same way as free motion in SR. In the general case EP is formulated as a sui generis localization of this equivalence for uniform gravitational fields, i.e., a local inertial frame must exist, where a metric field becomes the Minkowski one, and its Christoffel connection disappears in a given space-time point. But gravitation curvature does generally not vanish in such a reference frame. Does this mean that the Equivalence Principle in gravitation holds only in the "weak" variant, i.e., only for laws, e.g., of probe particle motion, which do not contain more than first-order derivatives of a gravitational field?

Then to what degree is it correct to speak about such a special relativistic attribute as energy-momentum of a gravitational field itself? Maybe it is the cause of the known problem of gravitation energy, which led to vivid disputes, starting with Einstein–Grossman pre-GR works up to recent days, and presenting a broad display of opinions, as, e.g., in the case of gravitational waves: positive energy of waves, or no energy at all!

Note also the widely discussed singularity problem in GR, which shows that either we are unable to gain insight into the nature of singularities as yet, or that GR (at least in its classical version) is incapable to describe extremal gravitational fields.

These and some other difficulties of the GR picture of gravity motivate one's attempts to reformulate gravitation theory from non-conventional standpoints extending the framework of Einstein's GR.

But why gauge gravity? Can the gauge treatment of gravity really solve the above-mentioned problems? Beforehand nobody knows. But today many of these problems seem to be put in the shadow of the urgent goal of the gravity unification with the elementary particle world. Just this goal stimulated by the grand unification program in contemporary particle physics puts the gauge version in the forefront of modern gravitation research.

Today, gauge theory provides the theoretical base of all modern unification attempts in particle physics. It has become clear that weak and electromagnetic interactions can be successfully unified by the Weinberg–Salam gauge model, and there is growing evidence that strong interaction is also mediated by gauge particles or gluons within the framework of chromodynamics. In field theory gauge potentials become a standard tool for describing interactions with very different symmetries. And apparently the single gap in the modern gauge picture still remains gauging the external or space-time symmetries of fields and particles, that includes the gauge gravity also.

Moreover, gauge theory using the mathematical formalism of fiber bundles realizes in fact the known program of the 1920s to build the geometric unified picture of various interactions. And it is strange enough that just the gravitation theory, being the first example of field geometrization, has still not any recognized gauge version. Although the first gauge treatment of gravity was suggested immediately after the gauge theory birth itself [109, 8, 62].

The main dilemma which during 25 years has been confronting the establishment of the gauge gravitation theory, is that gauge potentials represent connections on fiber bundles, while gravitational fields in GR are only metric or tetrad (vierbein) fields.

Connections as fundamental quantities appeared together with the metric in Weyl's and Eddington's generalizations of GR on gravity with nonmetricity and torsion, and in this quality were again recognized by Einstein in his last scientific paper [25]. But even in the gauge gravitation theory connections cannot at all substitute the metric, because there are no groups, whose gauging would lead to the purely gravitational part of space-time connections (Christoffel symbols or Fock–Ivanenko spinorial coefficients). To separate such gravitational components from gauge fields, e.g., of the Lorentz group, metric or tetrad fields have to be introduced.

At the present time the gravitation theory is viewed actually as the affine-metric theory possessing two independent potentials, namely, metric and connection, and just this constitutes the peculiarity of the gauge approach to the gravitation theory in comparison with the gauge models of the Yang–Mills type for internal symmetries.

This article aims to match the gravitation theory and gauge theory within the framework of gauge theory of external symmetries. Because both gravitation and gauge theories have the geometric formulations in terms of the fiber bundle formalism, we shall use the fiber bundle language (for necessary mathematics see [100, 63, 102, 103]).

I. The geometric treatment of gauge theory

This section is not intended to give the complete geometric picture of gauge theory. Here we pay attention only to those aspects of fiber bundle formalization of gauge theory, which are necessary to gain an insight into the gauge nature of gravity, e.g., we shall neglect for a time the topological numbers of bundles and gauge fields, referring the reader interested in details of this subject, to the review articles [75, 18, 24]. Gauge theories of only internal symmetries, i.e., whose transformations do not act on operators of partial derivatives, are discussed in this section.

1. The conventional scheme of gauge theory

One may treat the general gauge theory as a generalization of classical and quantum electrodynamics on non-Abelian symmetries.

The starting point for gauge theory was the known invariance of electrodynamics under gauge transformations of matter fields $\{\varphi\}$ and electromagnetic potentials A_μ. They read

$$\varphi(x) \rightarrow \exp(\mathrm{i}\alpha(x))\,\varphi(x)\,, \qquad A_\mu \rightarrow A_\mu + \partial_\mu\alpha(x) \tag{1.1}$$

and represent local phase transformations, whose parameters $\alpha(x)$ are functions of space-time. The invariance under these transformations (1.1) is provided by the so-called minimal type of electromagnetic interaction, when potentials A_μ appear in a matter field Lagrangian only inside generalized derivatives

$$\mathrm{D}_\mu = \partial_\mu - \mathrm{i}A_\mu\,. \tag{1.2}$$

Hence, to be invariant under local phase transformations, a matter Lagrangian must include electromagnetic interaction. For the first time Weyl pointed out this phenomenon and laid it in the foundation of his unified version of gravity and electromagnetism of 1918 [113], and later he and others repeatedly returned to the idea of introduction of electromagnetic potentials from the condition of the local phase invariance.

The crucial step was taken by Yang and Mills in 1954 [118], who laid down the concept of a non-Abelian gauge theory as a generalization of Weyl's picture of electromagnetism.

In conventional form the basic principle of gauge theory consists in the conservation of the invariance of a field Lagrangian under passing to local symmetry transformations, whose parameters become functions of space-time and vary from point to point.

Then to compensate the invariance-violating terms arising in a Lagrangian under local transformations due to the non-zero derivatives of their parameters, one has to introduce supplementary gauge or compensating fields $A_\mu = A^m_\mu I_m$, possessing the gauge law of local symmetry transformations

$$g(x):\ A_\mu \to gAg^{-1} + \mathrm{d}g \cdot g^{-1} \tag{1.3}$$

and must replace partial derivatives ∂_μ in a matter Lagrangian with generalized derivatives called gauge-covariant or compensative:

$$\mathrm{D}_\mu = \partial_\mu - A^m_\mu I_m \tag{1.4}$$

where I_m are generators of the symmetry group. The transformation law (1.3) of gauge fields and the derivative (1.4) generalize the gauge transformation (1.1) and the derivative (1.2) corresponding to the electromagnetic case of the symmetry group U(1) with the single generator $I = \mathrm{i}$. Interaction with any non-Abelian symmetries is mediated by gauge fields just as the electromagnetic interaction is mediated by vector-potentials A_μ.

Thus the simple requirement of the local symmetry invariance armed physicists with the universal method of the interaction description [64, 1].

On the other hand the gauge derivative (1.3) proves to be regarded as a covariant derivative on a certain fiber bundle over space-time, and gauge potentials represent connections on them. Thereby gauge theory generalizes the gravitation Christoffel connection, which realizes parallel transport in the tangent bundle over space-time, and it takes the bundles into consideration, whose fibers are spaces of internal attributes of the particles, e.g., isospin, "colour", "flavour", etc. Gauge fields define the affine geometry on these bundles.

In particular, it clarifies also the cause of the failure of unified theories of the 1920s (Weyl, Kaluza, Klein, Eddington, Einstein). These theories attempted to describe electromagnetism and other fields just as gravitation in terms of space-time geometry, while even the electromagnetic gauge derivative (1.2) cannot play a covariant derivative role in any space-time geometry.

The known Fock–Ivanenko coefficients of 1929 describing parallel transport of fermions in GR [26, 114] were actually the first non-tangent bundle connections, which were applied in field theory. That the Yang–Mills gauge theory and the affine geometry of fiber bundles were one and the same thing was pointed out by various authors from the mid-1960s [69, 39, 64, 116, 86, 67, 11], and to-day the fiber bundle formulation of classical gauge theory becomes already commonly admitted and gives the adequate image of this theory [75, 18, 24].

Moreover, our opinion is that fiber bundles provide the most relevant mathematical language of the whole field theory, when any physical fields are formalized by cross sections of corresponding fiber bundles, whose geometric and topological structures are thought to characterize entirely the interaction of these fields. In particular, this permits various mathematical ideas, methods, and results of differential geometry and topology to come into play in field theory, and, e.g., global topological attributes of bundles have been widely used as sui generis topological quantum numbers of fields. Moreover, because nothing hinders in principle to take advantage of any type of bundles, and to generalize field notion on their cross sections, one may receive a powerful and universal tool for research of very different physical systems including those, which as yet remain out of the possibilities of other existing methods.

2. Fiber bundles

For convenience of the readers who are not familiar enough with fiber bundles, we have gathered below some of the basic notions adapted from standard textbooks [100, 63, 102, 103] (see also review papers [75, 18, 24]).

A pair of topological spaces M and X possessing the continuous projection π: $M \to X$ is called a fiber bundle $\lambda = (M, \pi, x)$, where M is a total or bundle space, X is a base space, but the inverse image of a base point $x \in X$ with respect to projection π, $M_x = \pi^{-1}(x)$, is called a fiber of a bundle λ at a point x. A topological space V is called a typical fiber of a fiber bundle λ, if all fibers M_x are isomorphic with V.

A cross section or local section of a fiber bundle λ is such a continuous injection s: $U \to M$ of an open subset U of X into M, that $\pi s = \mathrm{id}_U$ is the identity on U. It means that a section s is a mapping, which assigns a preferred point $s(x)$ on each fiber to each point $x \in U$

$$s(x) \in M_x = \pi^{-1}(x) .$$

A section is named global, if it is defined on the whole base X. There exist fiber bundles which have no global sections.

One says a fiber bundle λ is trivial, if a bundle space M is homeomorphic with the direct product $M = V \times X$, where V is a typical fiber of λ.

A fiber bundle λ is called locally trivial, if there is such a family $\{U_i\}$ of open subsets covering X, that restriction of λ onto every U_i is trivialization, i.e., there is a family of homeomorphisms $\{\psi_i\}$ so that for each U_i

$$\psi_i \pi^{-1}(U_i) = V \times U_i .$$

A pair (U_i, ψ_i) is called a chart, and the transition functions $\psi_{ij} = \psi_i \psi_j^{-1}$ must be defined on the overlap of the patches U_i and U_j; they satisfy the cocycle conditions: $\psi_{ii} = \mathrm{Id}_{U_i}$

$$\psi_{ij}\psi_{jk} = \psi_{ik} \qquad \text{for } x \in U_i \cap U_k . \tag{2.1}$$

For each fixed $x \in U_i \cap U_j$ the transition function ψ_{ij} represents the map from V onto V. If all such maps belong to a certain topological group G of transformations of the typical fiber V, i.e., if all transition functions ψ_{ij} represent elements of the group G(X) of all continuous functions on open subsets of X, which take values in G, the group G is called a structure group of a fiber bundle λ.

A set of elements of the group $G(x)$ represents the transition functions of the bundle λ, if and only if the cocycle condition (2.1) holds. Transition functions define a consistent procedure for gluing together the trivial pieces of a locally trivial bundle, and determine it completely. The transition functions of a trivial bundle can be taken to be the identity.

A collection of trivialization charts $\Psi_\lambda = \{U_i, \psi_i\}$ is called an atlas of a fiber bundle λ. Atlases $\Psi_\lambda = \{U_i, \psi_i\}$ and $\Psi'_\lambda = \{U'_i, \psi'_i\}$ are considered to be equivalent, if their combination is again an atlas, i.e., if transition functions between any chart (U_i, ψ_i) of Ψ_λ and any chart (U'_i, ψ'_i) of Ψ'_λ may be determined. Transformation between equivalent atlases is defined by elements of the group G(x), and reads

$$\psi_i'(x) = g_i(x)\,\psi_i(x)\,, \qquad x \in U_i$$

$$\psi_{ij}'(x) = g_i(x)\,\psi_{ij}(x)\,g_j^{-1}(x)\,, \qquad x \in U_i \cap U_j\,. \tag{2.2}$$

Fiber bundles λ and λ' with the same base and structure group are named associated, if their atlases Ψ_λ and $\Psi_{\lambda'}$ within the accuracy of their equivalence transformations (2.2) have the same families of transition functions $\{\psi_{ij}\}$, which as elements of the group G(X) can be defined without regard to the concrete typical fiber of bundles with the same structure group. Associated bundles with the same typical fibers are called isomorphic.

Every fiber bundle λ possessing a structure group G has an associated principal bundle, whose typical fiber is the group G itself, acting by left translations. Continuous sections of a principal bundle compose the group G(X), but its global section exists only if this bundle is trivial.

The class of associated bundles is a topological characteristic of fiber bundles belonging to it. These classes can be described in terms of Chern, Pontrjagin, and some other characteristic classes representing certain elements of cohomology groups of the base space X, e.g., any fiber bundle over a contractible base space which has zero cohomology groups is trivial.

One says that contraction of a structure group G of a bundle λ to a certain subgroup H takes place, if there is an atlas of λ, whose transition functions reduce to elements of the subgroup H(X) of G(X). The necessary and sufficient condition for occurrence of this contraction is the existence of a global section of an associated bundle, whose typical fiber is a quotient space G/H. Such a contraction always takes place, when the base X of λ is a paracompact space, G is a Lie group, and H is its maximal compact subgroup.

Note that the quotient bundle in question may have many global sections, from which one singles out the center section $\sigma(x) = \sigma_0 = \text{const.}$, where σ_0 is the H-fixed center of the quotient space G/H. Other global sections differ from it only on patches of trivialization of the quotient bundle, and all of them can be led to the center section in some atlas corresponding to another variant of the structure group contraction.

For our purpose we shall confine in the following our consideration to differentiable vector bundles.

A differentiable vector bundle

$$\lambda = (X, V, G, \Psi_\lambda) \tag{2.3}$$

is given by the following objects:

1) A base X is a finite-dimensional connected manifold supplied with a coordinate atlas Ψ_X.

2) A typical fiber V is a finite-dimensional topological vector space.

3) A structure group G is a Lie subgroup of the group GL(V) of all isomorphisms of the vector space V, which contracts to G as a structure group of a vector bundle λ.

4) A bundle atlas $\Psi_\lambda = \{U_i, \psi_i\}$ defines some kind of reference frame on a fiber bundle λ, where all bundle attributes are described as their images onto trivial bundles $\{U_i \times V\}$ with respect to trivialization mappings $\{\psi_i\}$. In particular, one can erect a basis of any fiber M_x as the inverse image under ψ_i of a basis of the typical fiber V. Also any section s of a bundle λ is represented in such a reference frame by the collection

of vector-valued functions

$$s_{(i)}(x) = \psi_i(x)\, s(x)\,, \qquad x \in U_i$$

relative to the trivialization charts (U_i, ψ_i). Changes of the atlases Ψ_λ and Ψ_X generate changes of the reference frames (2.2) and the coordinate frame, respectively. It is often convenient to imply that atlases Ψ_λ and Ψ_X have the same family $\{U_i\}$ covering X.

5) All manipulations with λ are assumed to be differentiable a sufficient number of times.

Any vector bundle has global sections, although there may be no global sections which are everywhere non-zero.

We let a tangent bundle T(X) and a cotangent bundle T*(X) be real vector bundles whose fibers over a point $x \in X$ are given by a tangent space T_x and a cotangent space T_x^*, respectively. We observe that if $\Psi_X = \{U_i, \varphi_i\}$ is a coordinate atlas of the base manifold X, then one can choose a holonomic atlas Ψ of a tangent bundle T(X) and a cotangent bundle T*(X), such that $\Psi = \{U_i, \psi_i = \partial\varphi_i\}$, where $\partial\varphi_i(x)\colon T_x \to R^n$ is a differential of the mapping $\varphi\colon U_i \to R^n$ in a point $x \in U_i$, and where basis elements of the fibers T_x and T_x^* consist of differential operators $\{\partial_\mu\}$ and $\{dx^\mu\}$, respectively.

One can build the following constructions on the vector bundles λ and λ' with typical fibers V and V′, respectively:

a dual vector bundle λ^*, whose typical fiber is a dual space V^*.

a Whitney sum bundle $\lambda \oplus \lambda$, whose typical fiber is a direct sum $V \oplus V$.

a tensor product bundle $\lambda \otimes \lambda'$, whose typical fiber is a tensor product $V \otimes V'$.

Sections of a vector bundle (2.3) make up a vector functional space and their differentiation can be defined. But before doing this, one must remember that to compare vectors belonging to fibers over different base points these vectors need to be transported into the same fiber. Therefore differentiation of vector bundle sections may be generally determined only as a covariant differentiation.

Let a bundle of p-linear alternating-sign maps from the tangent bundle T(X) into a vector bundle λ be given. Sections of this bundle represent exterior or skew differential p-forms on X with values in the total space of λ. In particular, 0-forms are sections of λ. Denote Ω_λ^p as the sheaf of such p-forms. Then their covariant differentiation is determined by the operator of the exterior covariant derivative D satisfying the following conditions:

$$\mathrm{D}\colon \Omega_\lambda^p \to \Omega_\lambda^{p+1}$$

$$\mathrm{D}(\omega + \omega') = \mathrm{D}\omega + \mathrm{D}\omega'$$

$$\mathrm{D}(\omega \wedge \omega') = \mathrm{D}\omega \wedge \omega' + (-1)^p \omega \wedge \mathrm{D}\omega'$$

where $\wedge$ is the sign of the exterior product.

In the frame of the atlas Ψ_λ the exterior covariant derivative D on trivialization charts is expressed in the form

$$\psi_i \mathrm{D} \psi_i^{-1} = \mathrm{d} - A \tag{2.4}$$

where d is the ordinary exterior derivative with zero square $\mathrm{dd} \equiv 0$, and A represents a connection 1-form on X with values in the Lie algebra gl(V) of the group GL(V).

In contrast to $\mathrm{dd} \equiv 0$, the square of covariant derivatives is generally non-zero and determines the curvature 2-form

$$\mathrm{DD} = F \tag{2.5}$$

for which the Bianchi identity $\mathrm{D}F = 0$ always holds.

In frames of bundle and coordinate atlases, when the basis $\{\mathrm{d}x^\mu\}$ of the cotangent spaces T^*_x and the basis $\{I_m\}$ of the Lie algebra gl(V) are chosen, the connection and curvature forms are evaluated from the expressions

$$\begin{aligned} &A = A^m_\mu I_m \,\mathrm{d}x^\mu, \qquad F = F^m_{\mu\nu} I_m \,\mathrm{d}x^\mu \wedge \mathrm{d}x^\mu \\ &F^m_{\mu\nu} = \partial_\nu A_\mu - \partial_\mu A_\nu - c^m_{nk} A^n_\mu A^k_\nu \end{aligned} \tag{2.6}$$

where c^m_{nk} are the structure constants of gl(V).

Let us notice that the covariant derivative (just like the curvature form) on a differentiable bundle is determined without regard to any atlas. But its expression (2.4) through a connection form A is true only in some atlas. From this one finds the transformation law of a connection form A under the atlas transformations (2.2):

$$\begin{aligned} &\mathrm{G(X)} \ni g_i(x): \quad \psi_i(x) \to g_i(x)\, \psi_i(x), \qquad x \in \mathrm{U}_i, \\ &g_i(x): \quad A(x) \to g_i(x)\, A(x)\, g_i^{-1}(x) + \mathrm{d}g_i(x)\, g_i^{-1}(x). \end{aligned} \tag{2.7}$$

The connection form A can be defined isomorphically on any fiber bundle associated with λ. In particular, one studies conveniently the properties of the connection, when it is observed on a principal bundle.

A connection form A represents an infinitesimal operator of parallel transports of fibers of a bundle along paths in a base space.

In a general form the connection on a fiber bundle may be introduced by lifting the paths from a base space into a bundle space. In particular, such a lift of loops passed through the same base point $x \in \mathrm{X}$ induces a group of isomorphisms of the fiber over x. This group is called a holonomy group H_x of a connection over a point $x \in \mathrm{X}$.

In the case of a connection given on a differentiable vector bundle (2.3), the holonomy groups H_x over all points $x \in \mathrm{X}$ are isomorphic to each other, and are isomorphic with a certain Lie subgroup H of the group GL(V). The Lie algebra of H_x proves to coincide with the Lie algebra of all values of the curvature form F. Moreover, the group H turning out to be the holonomy group of some connection on a fiber bundle (2.3) is sufficient for the structure group of this bundle to be contracted to H.

The given definition of bundles is close to the physicist's way of thinking in terms of local coordinates, gauge transformations, covariant differentiation, etc. At the same time the readers may find another formulation of the fiber bundle theory in most of the mathematical and some physical literature. Therefore we shall sketch it briefly to clarify some constructions in the bundle and gauge theories.

One starts with the notion of a principal G-bundle. Its total space may be determined as a smooth manifold P realizing a free action of a Lie group G on P to the right, such that the equality $ug = u$ for some $u \in \mathrm{P}$ and $g \in \mathrm{G}$ leads to $g = e$, where e marks the unit element of G. A quotient space P/G of orbits of G on P represents a base X of a principal bundle, but a canonical mapping $\pi: \mathrm{P} \to \mathrm{P/G}$ is a bundle projection. A principal bundle $(P, \pi, X = P/G)$ is considered to be locally trivial with regard to a certain family of trivializations ψ_i: $\pi^{-1}(U_i) \to U_i \times G$. G acts inside fibers of a principal bundle, and the mapping ψ_i satisfies the condition $\psi_i(ug) = \psi_i(u)g$.

Now let $G \times V \to V$ denote an action of G into a manifold V to the left: $G \times V \ni (g, v) \to gv \in V$. If P is a total space of a principal G-bundle, then one defines an action of G in $P \times V$:

$$g: (u, v) \to (ug, g^{-1}v).$$

Let

$$\rho: P \times V \to M = (P \times V)/G \tag{2.8}$$

be the canonical map on the quotient of $P \times V$ by G. The set M has the natural structure of a total space of an associated bundle $\lambda = (M, V, G, X)$ with a typical fiber V and a projection π_M: $M \to X$ defined by $\pi_M \cdot \rho(u, v) = \pi(u)$. Every element $u \in P$ induces a mapping

$$u: V \to \rho(u, V) = V_{x=\pi(u)} = \pi_M^{-1}(\pi(u)) \tag{2.9}$$

such that $u\, g(v) = u(gv)$.

Let φ: $P \to V$ be a mapping equivariant with respect to the actions of G into P and V, e.g., such that for any $g \in G$ and $u \in P$

$$\varphi(ug) = g^{-1} \varphi(u).$$

Then one can construct the section

$$s(\pi(u)) = \rho(u, \varphi(u))$$

of an associated bundle λ, and there is a one-to-one correspondence between equivariant mappings from P to V and sections of λ.

One may also consider another definition of connection on fiber bundles. This definition starts with the notion of a connection on a principal bundle by fixing a so-called "horizontal" subspace $T_u^h(P)$ of a tangent space $T_u(P)$ at every point $u \in P$, such that

$$T_u(P) = T_u^G(P) \oplus T_u^h(P), \qquad T_{ug}^h(P) = (\partial g)\, T_u^h(P).$$

Here (∂g) denotes the differential of the mapping g: $P \to P_g$, and $T_u^G(P)$ is a tangent space over a fiber G at $\pi(u)$. The space $T_u^h(P)$ is isomorphic with the tangent space $T_{\pi(u)}(X)$ over a base manifold X and determines the directions of infinitesimal parallel translations of the point u along the paths through $\pi(u)$ in X.

$T_u^h(X)$ depends smoothly on u and can be described by a connection 1-form ω on P from the condition

$$\omega(t^h) = 0, \qquad t^h \in T_u^h(X).$$

This form ω takes on values in the Lie algebra L_G of the group G and reads $\omega_i = (\theta + A)$ with respect to some atlas $\Psi = \{\psi_i, U_i\}$ of a principal bundle (P, G, X), where θ is a canonical form on G, i.e., $\theta(l) = l$, but A is a connection 1-form on X, which was introduced above. One finds that $A = (\partial\sigma_i)\omega$, where $\sigma_i(x) = \psi_i^{-1}(x, l)$ is the section of (P, G, X) on U_i. Just the component A expresses the individuality of a connection form ω, while θ is a standard part of ω. The form A being transferred to any associated fiber bundle defines a connection on it.

A fiber bundle LX of linear frames of an n-dimensional manifold X exemplifies a principal bundle important for our following researches. A total space of LX represents a manifold of all linear frames in tangent spaces $T_x(X)$ over X and realizes a free action of the group GL(n, R). LX is associated with a tangent bundle T(X), and every linear frame in a point $x \in X$ may be described as a non-singular linear mapping (2.9) of the typical fiber R^n of T(X) on T_x. In turn, this mapping $u(R)$ is represented as the linear isomorphism $u(R)\psi_i$ of R^n with respect to some atlas $\Psi = (\psi_i, U_i)$ of the bundle T(X).

We also define the R^n-valued soldering form θ on LX:

$$\theta(T) = u^{-1}(\partial\pi(T)), \qquad T \in T_u(LX) \tag{2.10}$$

where $(\partial\pi)$: $T(LX) \to T(X)$ is the differential of the projection π: $LX \to X$. θ is the equivariant form with respect to the actions of G in T(LX) and R^n, and represents the identity mapping of T(X) on itself, e.g., $\theta(x) = \tau_i(x)\,\sigma^i(x)$, where $\tau_i(x)$ and $\sigma^i(x)$ are dual bases of $T_x(X)$ and $T^*_x(X)$, such that $\sigma^i(\tau_j) = \delta^i_j$. The soldering form represents a global 1-form $\theta \in \Omega^1_{T(X)}$ on a manifold X. If D is a covariant differential of some connection on T(X), the 2-form of torsion $D\theta = \Omega$ is defined, such that the first Bianchi identity $D\Omega = F \wedge \theta$ holds.

We shall conclude our summary of the necessary fiber bundle mathematics with some words about pseudo-Riemannian spaces.

A manifold X^n is called to be provided with the pseudo-Riemannian structure of the rank $k < n$, if a global section g of the fiber bundle Λ of pseudo-Euclidean bilinear forms in tangent spaces over X^n is defined. Λ is associated with the tangent bundle T(X) possessing the structure group GL(n, R), and it is isomorphic with the fiber bundle in quotient spaces GL(n, R)/O($n - k$, k), whose global section (and consequently a global section of Λ) exists only when the structure group GL(n, R) of the bundle $T(X)^n$ can be contracted to its subgroup O($n - k$, k).

It means that there is an atlas Ψ_η of $T(X^n)$, of which all transition functions are realized by elements of the group $O(n-k, n)(X^n)$, and where g takes the canonical diagonical form η: $\eta_{ii} = 1, i \le k$; $\eta_{ii} = -1$, $i > k$. Of course, if the bundle $T(X^n)$ is nontrivial, the atlas Ψ_η is nonholonomic.

Other global sections of Λ differ from a given g on patches of trivialization of $T(X^n)$ by some elements of GL(n, R)(X), and corresponding atlases diagonalizing them exist.

To define a pseudo-Riemannian structure on a manifold X^n is not always possible. Only a Riemannian structure ($k = 0$) exists on every manifold X^n, because a structure group GL(n, R) can always be contracted to its maximal compact subgroup O(n).

Every connection form Γ on a pseudo-Riemannian space (X, g) can be expanded in a sum of three components: $\Gamma = \{\ \} + K + Q$, where $\{\ \}$ are Christoffel symbols, K is contortion, and Q is nonmetricity. They are calculated from the conditions:

$$(\mathrm{d} - \{\ \})g \equiv 0 \qquad (\mathrm{d} - \{\ \})\theta = 0$$

$$(\mathrm{d} - \Gamma)g = -2Q \qquad (\mathrm{d} - \Gamma)\theta = S$$

and their coefficients read with respect to some holonomic atlas of T(X):

$$\begin{aligned}
&\Gamma_{\epsilon\sigma\mu} = \{\epsilon\sigma\mu\} + K_{\epsilon\sigma\mu} + Q_{\epsilon\sigma\mu}\,,\\
&\{\epsilon\sigma\mu\} = \{\epsilon\mu\sigma\}\,, \qquad K_{\epsilon\sigma\mu} = -K_{\sigma\epsilon\mu}\,, \qquad Q_{\epsilon\sigma\mu} = Q_{\sigma\epsilon\mu}\\
&\{\epsilon\sigma\mu\} = \tfrac{1}{2}(\partial_\mu g_{\epsilon\sigma} + \partial_\sigma g_{\epsilon\mu} - \partial_\epsilon g_{\sigma\mu})\\
&K_{\epsilon\sigma\mu} = S_{\epsilon\sigma\mu} + S_{\sigma\mu\epsilon} - S_{\mu\epsilon\sigma} + Q_{\mu\epsilon\sigma} - Q_{\mu\sigma\epsilon}\\
&S_{\epsilon\sigma\mu} = \tfrac{1}{2}(\Gamma_{\epsilon\sigma\mu} - \Gamma_{\epsilon\mu\sigma})\,.
\end{aligned} \tag{2.11}$$

One says that a connection Γ on a pseudo-Riemannian space (X, g) satisfied the metricity condition, if

$$(\mathrm{d} - \Gamma)g \equiv 0\,. \tag{2.12}$$

The metricity condition (2.12) holds only when a connection Γ is reduced to the group $\mathrm{O}(n-k, n)$, i.e., when its holonomy group belongs to $\mathrm{O}(n-k, n)$, and there is an atlas of T(X), where Γ is represented by $\mathrm{O}(n-k, n)$-valued forms in all charts of this atlas.

A pseudo-Riemannian space depending on a kind of a connection Γ on it is called an Einstein–Cartan space U^n, if $Q \equiv 0$; an Einstein GR space V^n, if $Q \equiv 0$ and $K \equiv 0$; and a space of teleparallelism T^n, if a curvature form R of Γ equals zero.

3. *Fiber bundle representation of gauge theory*

As we have said, the formalization of field systems by fiber bundles is based on the field representation as cross sections of corresponding bundles.

Let $\{\varphi(x)\}$ be a certain multiplet of classical so-called matter fields on an orientable space-time manifold X^4 with values in a vector space V possessing a representation of a Lie group G of internal symmetries. The existence of a nonsingular G-invariant bilinear form on V is also admitted.

In fiber bundle terms the matter fields $\{\varphi(x)\}$ are formalized by global sections of a differentiable vector bundle $\lambda = (\mathrm{V}, \mathrm{X}^4, \mathrm{G}, \Psi_\lambda)$ which possesses the base space X^4, the typical fiber V, and the structure group G. The bundle atlas Ψ_λ and the coordinate atlas Ψ_X of the manifold X^4 fix a reference frame and a coordinate frame, respectively. In regard to these frames fields $\{\varphi\}$ look like V-valued functions on trivialization patches of λ, and changes of atlases Ψ_λ and Ψ_X induce the gauge G(X) and coordinate transformations of these field functions.

Because replacing the atlas represents an equivalence transformation of λ, the invariance of the field system described by λ under such a transformation seems to be quite natural for it to be required. Thus the fiber bundle picture of field systems leads directly to their gauge description, where a gauge principle is manifested as some kind of relativity principle.

Gauge potentials appear naturally also in the fiber bundle description of matter fields. They are identified with coefficients A^m_μ of a certain connection form on the bundle λ. Their gauge transformation law (2.7) represents the non-Abelian generalization of electromagnetic gauge transformations (1.1), and its familiar infinitesimal form reads

$$I_n\,\delta\omega^n(x)\colon\ \delta A^m_\mu(x) = c^m_{nk} A^k_\mu\,\delta\omega^n(x) + \partial_\mu \omega^m(x)\,.$$

Matter fields $\{\varphi\}$ and gauge potentials A compose the dynamic variables of the system in question, and one uses the action principle for describing their evolution. Here we prefer to apply the form of this principle familiar for physicists, although it does not correspond to elegancy of the fiber bundle of the kinematics of gauge theory, but its fiber bundle reformulation [19] is not as yet widespread.

The total action function S of the system of fields $\{\varphi\}$ may be written on any compact-limited range U of X and takes the form

$$S = \int_U [L_\varphi(\varphi, D\varphi) + L_A] . \tag{3.1}$$

Here a matter Lagrangian L_φ is constructed from a free field Lagrangian by replacing the ordinary derivatives ∂ by the covariant ones D. As a consequence L_φ looks, on the one hand, like the Lagrangian of free fields $\{\varphi\}$ in the affine geometry of the bundle λ, and, on the other hand, like the Lagrangian of the fields $\{\varphi\}$ which interact due to the gauge field mediating between them.

A gauge field Lagrangian L_A is always chosen by analogy with the Abelian case of the electromagnetic field Lagrangian in the form

$$L_A = -\frac{1}{4g^2}\langle F \wedge {}^*F\rangle \tag{3.2}$$

where F is the curvature or strength form (2.6) of gauge fields, and $\langle\, ,\rangle$ means a nonsingular G-invariant bilinear form in the Lie algebra of the group G. If G is semisimple, $\langle\, ,\rangle$ is the Killing form, and

$$L_A = \frac{1}{8g^2}\,\mathrm{tr}(F \wedge {}^*F) .$$

Note also that after special rescaling of the gauge fields the coupling constant g and L_A can be inserted into the covariant derivative $D = d - gA$.

The action functional (3.1) is written without regard to any reference frame, and is evidently gauge invariant. It leads to the following Noether's identities:

$$\partial_\mu(\mathcal{J}^\mu_\varphi + \mathcal{J}^\mu_A) = 0 \tag{3.3a}$$

$$\mathcal{J}^\mu_\varphi + \partial L_\varphi/\partial A_\mu \equiv 0 , \qquad \mathcal{J}^\mu_A + \partial L_A/\partial A_\mu \equiv 0 \tag{3.3b}$$

$$\partial L_A/\partial A_{\mu,\nu} + \partial L_A/\partial A_{\nu,\mu} \equiv 0 \tag{3.3c}$$

where

$$\mathcal{J}^\nu_\varphi = \frac{\partial L_\varphi}{\partial \varphi_{,\nu}}\, I(\varphi) , \qquad \mathcal{J}^\nu_A = \frac{\partial L_A}{\partial A_{,\nu}}\, I(A) \tag{3.4}$$

are currents according to symmetry transformations with a generator I.

The identities (3.3b,c) are strong, while (3.3a) is so-called weak, which takes place only for extremal fields satisfying the Euler–Lagrange equations:

$$\frac{\delta L}{\delta \varphi} = \mathrm{D}_\mu \frac{\partial L_\varphi}{\partial \mathrm{D}_\mu \varphi} - \frac{\partial L_\varphi}{\partial \varphi}\bigg|_{\mathrm{D}_\mu\varphi = \mathrm{const}} = 0$$
$$\delta L/\delta A = \mathrm{D}_\mu F^{\mu\nu} - 4g^2 \mathcal{J}^{\nu}_{\varphi} = 0\,. \tag{3.5}$$

These equations describe the interaction of matter fields $\{\varphi\}$ via gauge fields, whose sources are currents of matter fields. Remark that in the general case different boundary sources may also appear on the right hand sides of the conservation law (3.3a) and the equations (3.5).

If we now consider matter fields $\{\varphi\}$ interacting with external gauge fields, the Noether identities become the modified identity (3.3a)

$$\mathrm{D}_\mu \mathcal{J}^{\mu}_{\varphi} = 0\,. \tag{3.6}$$

The identity (3.6) does not express preserving of any integrable physical quantity because it contains the covariant divergence, and to obtain the conservation law (3.3a), one must take the sum of the matter field current and the gauge field current, although the latter proves to be non-covariant under local G(X)-transformations. This situation is analogous to the known difficulties of the formulation of energy-momentum conservation in gravitation theory.

One uses the fiber bundle formalization to describe the field models with space-times which possess very different geometric and topological structures. This is very important, because real space-time is a non-Minkowski space curved by gravitation, and other causes may change space-time topology also, e.g., inside elementary particles, under cosmological conditions. Moreover, some authors believe that many fundamental attributes of particles have the topological nature. The familiar Aharonov–Bohm effects in the case of electromagnetic fields in spaces with non-zero homology groups exemplify the phenomena, which may be caused by a nontrivial global topological structure of space-time.

Following some papers, e.g. [116], we finish this section with the translation table between gauge and bundle terminologies:

Gauge field terminology	Fiber bundle terminology
classical fields	sections of a vector bundle
space-time	base space
symmetry group	structure group
gauge transformation	atlas transformation
gauge principle	relativity principle
gauge potential	connection 1-form on a bundle
field strength	curvature of a connection

4. *Goldstone and Higgs fields in fiber bundle terms*

Not only gauge fields can appear in gauge theory; Goldstone and Higgs fields are well known to arise also in it, when the symmetry is spontaneously broken [4].

One observes various situations looked upon as the spontaneous symmetry breaking in gauge models, and some of them have obtained the fiber bundle versions [71, 107, 108, 42, 43, 6]. Although these versions are applied to describe rather different physical circumstances, the common feature for all of them is bundles with contracted structure groups.

We concentrate our attention here on the situation of spontaneous symmetry breaking, which issues from the contraction of the structure group of the fiber bundle λ describing a certain gauge model from the previous section. We shall see that such a situation, on the one hand, corresponds to the Higgs mechanism of spontaneous symmetry breaking in gauge models of internal symmetries and, on the other hand, is the basis for the definition of gravitational fields in the gauge theory of external symmetries.

Denote by $\{\chi\}$ the multiplet of scalar fields described by global sections of a certain vector bundle Λ. Let also the structure group G of Λ be contracted to its subgroup H, which is required to be a stability group of some non-zero points of the typical fiber W. Denote by W_H the subspace of W composed by these points.

Then there exist non-zero global sections of the bundle Λ, which take on values inside the H-invariant subspace W_H in the reference frame of a certain bundle atlas, whose transition functions belong to the group H(X). Let χ_0 denote one of these global sections, which minimizes the energy functional of the fields $\{\chi\}$. Then χ_0 may be interpreted as sui generis physical vacuum or as a ground state, whose symmetry is broken, and small perturbations over χ_0 may be considered.

In particular, in the most familiar case, when the coupling potential of fields $\chi(x)$ reads

$$L_{\text{int}} = -\mu^2\chi^2 + \lambda^2\chi^4 \tag{4.1}$$

χ_0 is found in the form $\chi_0(x) = w_0 = \text{const}$, where w_0 belongs to the subspace W_H, and $w_0^2 = \mu^2/\lambda^2$.

To describe the perturbations over a state with broken symmetry, note that every point w in a small neighbourhood of the non-zero H-fixed point $w_0 \in W$ can be written as

$$w = g(w_0 + w_H), \qquad g \in G, \qquad w_H \in W_H.$$

For small w, w_H and g being near the unity of G, this decomposition is rewritten in the form

$$w = w_0 + w_H + \alpha \tag{4.2}$$

where $(w_0 + \sigma)$ belongs to the orbit (Gw_0) of the group G in W, and consequently can be identified with a point of the quotient space G/H.

Then the small perturbations $\{\chi\}$ over the ground field χ_0 are found in the known form

$$\chi(x) = \chi_0 + w_H(x) + \sigma(x) \tag{4.3}$$

where the fields $w_H(x)$ are global sections of Λ, which take values in the subspace W_H, but the fields $(\chi_0 + \sigma(x))$ represent global sections of the associated bundle in the quotient space G/H.

H-invariant components $(\chi_0 + w_H(x))$ of fields $\chi(x)$ with spontaneously broken symmetry are usually called Higgs fields, and after being quantized, the fields $\sigma(x)$ prove to represent the Goldstone fields of the known Goldstone's theorem [4].

Fields with broken symmetry of vacuum become part of many modern gauge models to provide the

Higgs mechanism supplying matter and gauge fields with suitable masses due to interaction of these fields with the Higgs vacuum χ_0.

Nevertheless, separating a field χ with spontaneously broken symmetry in Goldstone and Higgs parts fails to be gauge invariant, because in gauge theory with internal symmetries Goldstone components of χ can always be removed by the appropriate gauge, while a previously pure Higgs field turns out to be supplied with such components in the new gauge. Note also that the Goldstone theorem in gauge theory can only be proved in the gauge where vacuum is invariant under translations. But such a gauge fails to exist in general.

Now we point out the particular kind of fields with spontaneously broken symmetries, which are always present in all gauge models of internal symmetries. These are Hermitian metrics. Really, most internal symmetry groups are unitary subgroups of a certain group $GL(n, C)$. Consequently, a fiber bundle λ formalizing a gauge model with such a symmetry group has the structure group $GL(n, C)$ which, however, is always contracted to the structure group $U(n)$ as the maximal compact subgroup of $GL(n, C)$. Then, as stated above, in this gauge model, apart from matter and gauge fields, there appear supplementary fields representing the global sections of the associated bundle in quotient spaces $GL(n, C)/U(n)$, which are isomorphic with the space of Hermitian metrics in C^n. Hence, Hermitian metrics, being these supplementary fields, may be regarded as fields whose symmetries are spontaneously broken, whose ground state is the unit matrix; Higgs type fields are $U(n)$-invariant diagonal metrics, but their deflections play the role of Goldstone type fields. At the same time Hermitian metrics don't compose dynamic fields in gauge models of internal symmetries because every of them can be brought always to the constant diagonal Hermitian metric in a certain gauge.

The situation is changed radically in the gauge theory of external or space-time symmetries.

II. Relativity Principle and Equivalence Principle in the gauge gravitation theory

The single accepted point of the gauge treatment of gravity is the standpoint that this treatment must issue from the gauge theory of some external symmetries. However, it seems that almost every author puts forward his own version of such a theory.

The multitude of the proposed gauge gravitation theories shows that the gauge principle alone is insufficient to describe the gravity.

The key difficulty lies in the determination of the gauge status of the metric or tetrad gravitational field.

At the same time this field gains the excellent description in Einstein's GR based on the Relativity and Equivalence Principles.

This leads one to admit that a gauge model aiming to describe gravitation as close as possible to GR must agree with the basic Relativity and Equivalence Principles of Einstein's theory. Moreover, one can admit that the gauge gravitation theory just like GR must also be based on RP and EP reformulated in gauge theory terms using the fiber bundle formalism [56].

5. Relativity Principle as the gauge type one

In the fiber bundle language Einstein's gravitational field on an orientable space-time manifold X^4 is defined as a global section of the fiber bundle Λ of pseudo-Euclidean bilinear forms in tangent spaces over X^4. Λ is associated with the tangent bundle $T(X^4)$ possessing the structure group $GL^+(4, R)$. Λ is isomorphic with the fiber bundle in quotient spaces $GL^+(4, R)/L$, where $L = SO(3, 1)$ is the Lorentz group. We call this bundle Σ. Its global section $h(x)$, isomorphic with g, describes a gravitational field in tetradic form. h used to be written as a section of the associated principal $GL(4, R)$-bundle up to multiplication of h on the right by elements of the group $L(X)$, i.e., $h \overset{\text{def}}{=} h\ L(X)$.

Then with respect to some atlas of $T(X^4)$ the section h can be realized by a family of matrix functions $h_i(x)$ in R^4, whose gluing is done modulo Lorentz transformations, and the isomorphism of h and g forms $g_{ij}(x) = h_i^a(x)\, h_j^b(x)\, \eta_{ab}$, where η_a is the constant Minkowski metric field. This relationship appears from the fact that h is realized by a global field taking the value in the center of the quotient space $GL^+(4, R)/L$ and consequently by a family of unit matrix functions in R^4 with respect to the atlas Ψ_η, where $g = \eta$.

In every point $x \in X$ a tetrad field h defines a tetrad $\{t(x)\} = \psi_i^{-1}(x) h_i(x)\{t\}$ in T_x, where $\{t\}$ is a basis of R^4, such that a metric $g(x)$ becomes diagonal with respect to $\{\ \}$. $\{\ \}$ is defined up to admitting Lorentz transformations and forms a global section of quotients of the linear frame bundle LX by L.

A gravitational field g or h on a manifold X^4 and its Christoffel connection $\{\ \}$ in $T(X^4)$ define a certain geometry of an Einstein space V^4 on X^4, which is conventionaly understood as a geometry of space-time.

RP in GR, as discussed above, is usually formulated as a requirement for matter field and test particle equations to conserve their form under all changes of reference frames.

Following the general fiber bundle formulation of field theory, a reference frame in the gravitation theory may be defined in fiber bundle terms as the choice of a certain atlas $\Psi = \{U_i, \psi_i\}$ of the tangent bundle $T(X^4)$, and the group of all reference frame changes is the gauge group $GL^+(4, R)(X^4)$.

This definition is close to that used in the tetrad formulation of GR. If an atlas $\Psi = (U_i, \psi_i)$ of $T(X^4)$ is fixed, a tetrad $\{t(x)\} = \psi_i^{-1}(x)\, h_i(x)\{t\}$, where $\{t\}$ is a basis of the typical fiber R^4 of $T(X^4)$, may be erected in every point of a space-time manifold X^4. A family of these tetrads is uniquely defined by an atlas Ψ, and transformations of these tetrads accompany changes of a reference frame. Inversely, if a family of local sections $t_i(x)$ of the linear frame bundle LX^4 on some covering $\{U_i\}$ of X is fixed, it defines a choice of an atlas $\Psi = \{U_i, \psi_i, \rho_{ij}\}$ of $T(X^4)$, such that $\{t_i(x)\} = \psi_i^{-1}(x)\{t\}$ and $\psi_i(x)\{t_j(x)\} = \rho_{ij}(x)\{t\}$.

The conventional (generally covariant) form of GR corresponds to the special case of purely holonomic transformations of reference frames, when a choice of the atlas of the bundle $T(X^4)$ correlates

$$\Psi = \{U_i, \psi_i = \partial\varphi_i\} \tag{5.1}$$

with the coordinate atlas $\Psi_X = \{U_i, \varphi_i\}$ of the manifold X^4, and such a correlation is strictly retained under all reference and coordinate frame changes.

Thus in the fiber bundle formalism of the gravitation theory RP may be formulated as the requirement of the covariance of field equations under the gauge group $GL^+(4, R)(X)$. In this way RP proves to be identical with the gauge principle of a gauge theory of the external symmetry group $GL^+(4, R)$, and the gravitation theory can consequently be built from RP directly as the gauge theory of the group $GL^+(4, R)$.

However the $GL^+(4, R)$-gauge theory turns out to be broader than the general conception of the gravitation theory. For example, it does not distinguish the Minkowski metric form from other possible metrics in tangent spaces. Therefore the Equivalence Principle in the gravitation theory also must be taken into account.

6. Equivalence Principle and broken space-time symmetries

Many authors emphasize the same nature of the basic gauge principle and RP of Einstein's GR.

In contrast with RP, EP in the conventional gauge theory of internal symmetries was not explicitly formulated. The representation of gauge fields by a connection 1-form on fiber bundles leads already itself to the fact that these fields can be zero in a given space-time point in a certain gauge, and in this case matter field equations take the free field form in this point.

In the case of external symmetries gauge fields can be eliminated in a given space-time point also by a certain reference frame choice, but, if these gauge fields are not purely Christoffel symbols, the metric or tetrad gravitational field functions are not flat in a point with respect to such a reference frame. Moreover, if the metric becomes flat simultaneously with connection being zero in the local inertial frame, one must postulate in addition for this flat metric to be just the Minkowski metric and no one with another signature.

Thus, as distinguished from the gauge theory of internal symmetries, the EP formulation turns out to be non-trivial in the gauge gravitation theory.

EP in GR supplements RP and guarantees the transition to Special Relativity in a certain reference frame, as discussed above.

In geometric terms SR may be characterized as the geometry of Lorentz invariants (in the spirit of Klein's Erlanger program). Then EP may be formulated in the gauge gravitation theory to require the existence of such a reference frame, with respect to which Lorentz invariants can be laid down everywhere on a space-time manifold X^4, and they would be conserved under any parallel transport.

This requirement holds when a connection of the tangent bundle $T(X^4)$ over a space-time manifold X^4 can be reduced to the Lorentz connection, i.e., when the holonomy group of this connection belongs to the Lorentz group, or in other words, there are atlases of $T(X^4)$, such that a connection form takes on values in the Lie algebra of the Lorentz group. It leads, in turn, to the contraction of the structure group $GL^+(4, R)$ of the tangent bundle $T(X^4)$ to the Lorentz group.

In other words the Equivalence Principle in the gauge gravitation theory makes gauge fields of external symmetries to be reduced to the Lorentz gauge fields in some reference frames.

The direct consequence of such an EP is the existence of global sections of the quotient bundle Σ and the metric bundle Λ, i.e., the existence of the metric or tetrad gravitational field everywhere on a space-time manifold X^4, that follows from the contraction of a structure group of tangent bundles (see section 2).

In the presence of a gravitational field the usual postulates of EP in GR hold. Indeed, there is a holonomic reference frame, where the gravitational metric field becomes just of the Minkowski type, and its Christoffel symbols vanish in a given space-time point. But in the general case of the Lorentz gauge fields containing also torsion components, the whole connection will not be equal to zero in this frame. Nevertheless, there exists also a reference frame, where the whole connection vanishes in a point, but the gravitational tetrad field remains.

The contraction of the structure group $GL^+(4, R)$ of $T(X^4)$ to the Lorentz group results also in its contraction to the maximal compact subgroup $SO(3)$ of the Lorentz group. It means that the existence of atlases of $T(X^4)$, where all transition functions consist of only spatial rotations and fail to break $SO(3)$-invariants, which therefore can be laid down everywhere on X^4. Non-zero time-directed vectors erected in all space-time points exemplify such $SO(3)$-invariants. In particular, these vectors may be defined as the global section of the associated bundle in quotient spaces $SO(3, 1)/SO(3)$, and thereby they represent some 4-velocities. The latter means that local reference frames in the sense of SR can be set up to admitting spatial rotations in all space-time points.

In particular, this proves the important theorem that in the gravitation theory the well-known (3 + 1) decomposition procedure can be carried out in a relevant reference frame in all points of a space-time manifold. We emphasize this fact because (3 + 1) decomposition has become a part of many modern branches of gravitation theory [47] (see also the works of A. Zelmanov and C. Cattaneo).

EP in gauge gravitation theory defines in fact sui generis Klein–Chern geometry of invariants on the total spaces, sheafs of sections of the tangent, and the associated bundles over a space-time X^4. This enables one to interpret the geometrical aspects of GR in the spirit of Klein's Erlanger program, in contrast to the repeated opinion of Fock, Bondi, Havas and some other authors denying the presence of any symmetries in the gravitation theory.

7. *Gravity as the Goldstone type field*

For the aim of this article it is especially important that the Equivalence Principle formulated in the gravitation gauge theory suggests that the gravitation field looks like a field of the Goldstone type.

In the gauge gravitation theory the EP leads, as stated above, to contraction of the structure group $GL^+(4, R)$ of the tangent bundle $T(X^4)$ to the Lorentz group SO(3, 1) imitating the situation, which is analogous to the spontaneous symmetry breaking. It leads to the existence of a global section of the quotient bundle with the typical fiber $GL^+(4, R)/L$, whose single L-fixed point is the Minkowski metric. Then, in analogy to the case of the spontaneous breaking of internal symmetries, one may look upon the Minkowski metric field as being the vacuum Higgs field χ_0, and small perturbations may seem to play the role of Goldstone fields $\sigma(x)$. These metric perturbations can admittedly be identified with the presence of a gravitation field, which therefore displays itself as a field of the Goldstone type. But the specificity of the external symmetry gauge comes into play.

First of all, as distinguished from Goldstone fields of internal symmetries, the gravitational field fails to be removed by any gauge. The reason lies in the fact that gauge transformations of external symmetries act also on operators of partial derivatives, which are vectors ∂_μ of tangent spaces. But these vectors play the role of derivatives only in holonomic frames. In non-holonomic frames, where the metric gravitational field g becomes the Minkowski one, $g = \eta$, the vector $\partial_i = h_i^\mu(x)\, \partial_\mu$ contains tetrad fields. Consequently tetrad fields cannot, in the general case, be hidden completely in the regauged connection under any gauge transformations, in contrast to the occurrence with the Goldstone fields in the internal symmetry case. And it shows that a non-flat gravitational field remains in any reference frame.

It means also that a gravitational field, as the field breaking space-time symmetries down to the Lorentz group, turns out to be deprived of the purely Higgs vacuum state on space-time manifolds which possess non-trivial tangent bundles. But then its similarity to the splitting as in eq. (4.3), into Goldstone and Higgs parts becomes also impossible. At the same time this splitting can always take place on flat patches of space-time, e.g., locally, although even in this case it used to come into play only in the weak-field limit. But even in this limit the way to assume a Goldstone nature of gravitons must be carefully specified.

We are far from asserting that the standard Goldstone theorem may be exactly preserved in the gauge gravitation theory, because, in particular, one of the essential points of this theorem is that the vacuum must be invariant under translations, what is, however, generally violated in the case of external symmetries.

At the same time one observes the analogy of Goldstone and Higgs fields in the conventional gauge models with a spontaneous breakdown of internal symmetries, and a metric or tetrad gravitational field in the external symmetry gauge case. Just this motivates the treatment of Einstein's gravitational field as a field of the Goldstone or Higgs–Goldstone type. Moreover, gravity seems to present a good chance to be the single macroscopic field of such a type.

For the first time the ideas, that violating Lorentz symmetry due to curvature of space-time leads to the conception that the graviton may be a Goldstone particle, were expressed in the mid 1960s in connection with the comparison of cosmological and vacuum asymmetries (Heisenberg, Ivanenko, supported by H.-J. Treder) [54].

Then these ideas on breaking space-time symmetries were revived when the formalism of non-linear realizations as a kind of induced representations was suggested [17, 58] as the appropriate mathematical tool for discussion of situations of spontaneous symmetry breaking.

Non-linear realizations of a group G are built in the product space $W_H \times G/H$ by combining a linear representation of a Cartan subgroup H of G in some space W_H with the representation of G by left translations in the quotient space G/H. Confining us, as usual, to a small neighbourhood of the unity of G, whose elements can be written in the exponential form $g = (\exp \sigma I)h$, where $h \in H$, but I are non-H generators of G, a non-linear representation of G can be found in the form

$$G \ni g: \qquad W_H \times G/H \ni w = w_0 + \sigma + w_H \to w_0 + \sigma' + w'_H$$

where $(w_0 + \sigma') \in G/H$, and $w'_H \in W_H$ is evaluated from the expression

$$(\exp \sigma' I)h = g(\exp \sigma I), \qquad w'_H = h w_H .$$

In the context of breakdown of G to H-symmetries the non-H parameters σ of G are regarded as Goldstone particles possessing generally the inhomogeneous G-transformation law [17, 58, 111]. The decomposition (4.2) of vectors w near the H-fixed, but non-G-fixed point w_0 exemplifies a construction of such a non-linear realization of a symmetry group G broken down spontaneously to its subgroup H.

There have been built non-linear realizations of different space-time symmetry groups, e.g., the Poincaré group [41, 44], the conformal group [92, 5] and others. Space-time coordinates and the Weyl connection coefficients exemplify a geometric kind of Goldstone fields appearing in these models of space-time symmetry breaking.

The first reference to the idea that a pseudo-Euclidean metric tensor can be reproduced within the framework of the non-linear realization formalism in the context of symmetry breakdown of GL(4, R) to the Lorentz group can be found in ref. [48], but earlier this idea had been partly realized in [82], although only in the weak-field limit and without the symmetry breaking interpretation. This question was further investigated by J. Ne'eman et al. [77, 78]. However in the non-linear realization formalism the Goldstone treatment of metric gravity, based only on the isomorphism of the space of pseudo-Euclidean bilinear forms in R^4 with the quotient space $GL^+(4, R)/SU(3, 1)$, ignores the geometric aspects of gravity.

In the fiber bundle language it has been pointed out by us [94, 95] and by A. Trautman [107, 108] that a pseudo-Riemannian metric field on X^4 can be thought of as field breaking $GL^4(4, R)$-symmetries.

Trautman introduces the notion of a metric gravitational field as related to mapping of the principal GL(4, R)-bundle onto a GL(4, R)-orbit in the space of bilinear forms on R^4, which passes through the Minkowski metric, which gives rise to reduction of this principal bundle in the image of the linear frame bundle LX^4 to its subbundle of orthonormal frames. In contrast with others, Trautman calls a gravitational field, a Higgs field. We see some reasons for this name also [94], but in comparison with conventional Higgs fields a non-flat gravitational field does not possess any stability group.

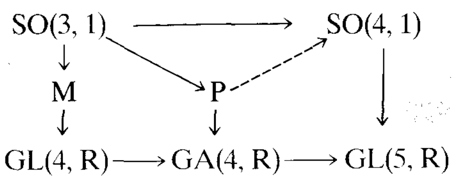
In our gauge approach just the Equivalence Principle secures an external symmetry breakdown to the

Lorentz group, and using the same mathematics we prefer, however, to describe such a breakdown in terms of a structure group contracted on the tangent bundle, which carries all geometric and topological information about space-time. This proves to be related with the existence of a gravitational field, called into being as a field of the Goldstone or Higgs–Goldstone type in the gauge gravitation theory.

Now after clarifying the gauge status of Einstein's metric or tetrad gravitational fields we are ready to review present day's space-time symmetry gauge models, whose part and parcel are the Einstein gravitation theory and its generalizations.

III. Gauge gravitation models

We give the following table of space-time symmetry groups, whose gauging pretends to describe the gravity:



where M and P mark, respectively, the product group of Lorentz rotations and dilatations, and the Poincaré group.

Gauge models of the linear groups SO(3, 1), M, GL(4, R) are based on realizing them as structure and holonomy groups of the tangent bundle $T(X^4)$ over an orientable space-time manifold X^4. All these models have the same pseudo-Riemannian metric structure induced by contraction of the general structure group $GL^+(4, R)$ of $T(X^4)$ to the Lorentz group, but they differ from each other in the structure of connection.

Gauging the affine groups P and GA(4, R) has the essential peculiarity connected with the necessity to consider affine bundles. A version of the affine group gauge is also the insertion of affine groups in the linear groups SO(4, 1), GL(5, R), whose gauge theories are built in the standard way, but are supplied with some conditions of reduction to the Poincaré or Lorentz gauge.

8. Lorentz gauge gravity

The first gauge treatment of gravity suggested by R. Utiyama [109], and D. Ivanenko et al. [8] immediately after the famous work of Yang and Mills, was based on gauging just the Lorentz group. There were considered in [8], in contrast with the standard gauge scheme, non-infinitesimal Lorentz local transformations, and gauge potentials were taken in the special case of the Cartan connection form, $\Gamma = g(x)\, dg^{-1}(x)$, which have zero curvature.

The Lorentz gauge geometry proves to coincide with the Einstein–Cartan geometry of gravity with torsion.

This geometry is defined to supply a space-time with a pseudo-Riemannian metric g and an affine connection Γ satisfying the metricity condition

$$(d - \Gamma)g = 0 \qquad (8.1)$$

which is necessary and sufficient for a connection Γ to be a Lorentz connection.

Inversely, the Lorentz gauge theory turns just out to be the case, where the reduction of the connection on tangent and associated bundles to the Lorentz gauge potentials, that, in our opinion, may be motivated from EP, entails the existence of a gravitational metric or tetrad field on a space-time manifold X^4, and the metricity condition (8.1) holds.

Note that in Utiyama's original work tetrad fields were a priori introduced into the Lorentz gauge model as operators of reference frame changes, that lay outside the conventional gauge scheme and called also to mind various attempts to generalize Utiyama's model for describing tetrad fields as gauge potentials of some space-time symmetry.

Let us point out that a Lorentz connection Γ looks generally like $GL^+(4, R)$-valued gauge potentials, e.g., in a holonomic reference frame, but forms a Lorentz gauge field only with respect to a special atlas Ψ_η of $T(X^4)$, where the metric field g (such that a pair (Γ, g) satisfies the metricity condition) forms the constant Minkowski metric field $g = \eta$. In general this atlas is non-holonomic. Nevertheless one needs it for bundles in the Lorentz gauge theory, whose typical fibers, e.g., a spinor fiber, require only Lorentz transformation. Such bundles, being associated with the tangent bundle, nevertheless admit atlases only with the Lorentz transition functions and Lorentz connections.

Fock, Ivanenko and Weyl were the first, who in reality considered spinor bundles in GR, and Lorentz gauge fields (without torsion) nowadays reproduce the well-known Fock–Ivanenko coefficients of 1929 [26, 114], which described the parallel transport of spinors in GR.

In a holonomic atlas a Lorentz connection is expressed by the formulas (2.11) for the zero non-metricity term Q. In the non-holonomic atlas Ψ_η the Lorentz connection takes values in the Lie algebra of the Lorentz group

$$\Gamma_\mu = \Gamma_\mu^{ab}\, I_{ab} = \tfrac{1}{2}(h^a_\sigma h^b_\nu \Gamma^{\sigma\nu}_\mu - h^a_\nu h^{\nu b}_{,\mu})\, I_{ab} \tag{8.2}$$

where I_{ab} are generators of the Lorentz group in some representation. In the connection expressed by (8.2) one can also separate purely gravitation and torsion parts:

$$\begin{aligned}
\Gamma^{ab}_\mu &= \begin{Bmatrix} ab \\ \mu \end{Bmatrix} I_{ab} + K^{ab}_\mu I_{ab} \\
\begin{Bmatrix} ab \\ \mu \end{Bmatrix} &= \tfrac{1}{2} h^{a\sigma} h^{b\nu} (h_{\nu i} h^i_{[\mu,\sigma]} + h_{\sigma i} h^i_{[\nu,\mu]} + h_{\mu i} h^i_{[\nu,\sigma]}) \\
K^{ab}_\mu &= (S^{ab}_\mu + h^{a\nu} h_{\mu c} S^{bc}_\nu - h^{b\nu} h_{\mu c} S^{ca}_\nu) \\
S^{ab}_\nu &= \Gamma^{ab}_\mu - \Gamma^{ac}_\nu h_{c\mu} h^{b\nu} + (h^a_{\mu,\nu} - h^a_{\nu,\mu}) h^{\nu b}\,.
\end{aligned} \tag{8.3}$$

Note that separating the torsion term in a connection is the specificity of gauge theories of external symmetries, whose generators act also on tangent and cotangent vectors, e.g., the operators ∂_μ and dx^μ. It results that a torsion form $S^\mu = D\,dx^\mu$ and the following specific curvature tensors can be defined

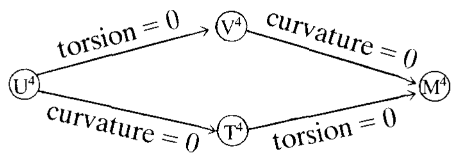
$$R_{\nu\sigma} = R^{ab}_{\mu\nu}\, I^\mu_{ab\sigma}\,, \qquad R = R^{ab}_{\mu\nu}\, I^{\mu\nu}_{ab}\,. \tag{8.4}$$

The expressions (8.4) represent the well known Ricci curvature term and scalar curvature. Similar curvature constructions are impossible in gauge models of internal symmetries.

The quantities $R_{\mu\nu}$ and R from (8.4) behave as tensor and invariant, respectively, under holonomic gauge transformations, and they provide the additional freedom of Lagrangian choice in gravitation gauge theory, in comparison with the Yang–Mills's type of Lagrangians in the internal symmetry gauge scheme.

Note also that a zero curvature alone does not trivialize an Einstein–Cartan space, because for a singly connected manifold (which possesses a certain linear connection) to be isomorphic with the Minkowski space, a zero torsion of this connection is also necessary and sufficient. This is in contradistinction to the case of internal symmetries in gauge theories, and the reason lies in the fact that the flat space-time is a Minkowski affine space, whose affine curvature is the sum of the linear curvature and torsion forms, and both of them must be equal to zero. For example, one can consider spaces with the Minkowski metric, but with non-zero torsion [90, 33].

The relations of the Einstein–Cartan space-time U^4 with its limiting cases are illustrated by the figure [34]



(8.5)

where T^4 = teleparallelism space, V^4 = Einstein GR space, M^4 = Minkowski space.

One sees that in the Lorentz gauge procedure nothing forces the torsion part of Lorentz gauge fields to be zero. Nevertheless the fact that the Einstein gravitation never arose in gauge theories alone did not satisfy some authors as is clear from Utiyama's article of 1956, where (in contradistinction to Ivanenko et al. [8]) the skew part of a Lorentz connection was neglected. Later Utiyama and some others attempted to build the Einstein gravitation by gauging translations alone, but all such attempts included necessary symmetrization of a connection.

Let us now consider the invariance condition in the Lorentz gauge. As distinguished from an internal symmetry case, there are in the gauge models of external symmetries always two kinds of gauge transformations, namely, the familiar atlas transformations of the matter field bundle λ, and also the atlas changes of the tangent bundle $T(X^4)$. The atlases of both these bundles are equivalent, but not always the same, as noticed above. Hence their gauge transformations fail to correlate generally and result in different conservation laws.

Utiyama had remarked this fact and inserted tetrad fields in his gauge model just to secure the Lagrangian invariance under tangent gauge transformations. The invariance with respect to field gauge transformations was in the meantime provided with inserting Lorentz gauge potentials just as in the internal symmetry gauge. Nevertheless, Utiyma's work left the question open about the gauge status of tetrad fields, and also nothing fixed the Minkowski signature of metric in his model. We have seen that a gravitational field has the Goldstone field nature, and its existence results from the Equivalence Principle.

In the Lorentz gauge theory the gauge transformations of matter field lead to the familiar type of gauge conservation laws (3.3), where Noether's current $\mathcal{J}_\varphi$ represents a spin current of matter fields, and this current, as in the case of internal symmetry currents in eq. (3.5), plays the role of a source but only of contortion components of the Lorentz gauge fields. While the invariance of a matter field Lagrangian under $T(X^4)$ gauge transformations leads to the known identity $D_\mu T^{\mu\nu} \equiv 0$, which is weak with respect to matter fields and strong with respect to a gravitational field, and gives the zero value of covariant divergence of the energy-momentum tensor of matter fields

$$\tfrac{1}{2}\sqrt{-g}\,T^{\mu\nu} = \partial L_\varphi/\partial g^{\mu\nu} - \partial_\epsilon\, \partial L_\varphi/\partial g^{\mu\nu}_{,\epsilon}\ .$$

Thus one may consider a matter energy-momentum tensor as a "current" corresponding to $GL^+(4, R)$-gauge transformations of tetrad or metric fields, whose matter source is also this tensor.

Hence, the Lorentz gauge model issued from RP and EP as the result of space-time symmetry breaking turns apparently out to be the minimal gauge picture containing Einstein's gravity. This model represents the adequate gauge picture of the Einstein–Cartan geometry supplied with both an affine Lorentz connection and a pseudo-Riemannian metric as the sui generis Goldstone fields [95, 56, 57].

The Lorentz group gauge, of course, does not pretend to be the single gauge picture of space-time geometry, gravitation and external attributes of elementary particles, whose symmetries are not restricted only by the Lorentz group. At the same time the Lorentz gauge seems to be the minimal gauge model containing Einstein's gravitation, and thereby it must be the kernel of any gauge model, which tries to extend Einstein's gravity. As a rule, such generalizations conserve the metric structure but modify the affine structure of the Lorentz gauge gravitation by using the fact that, for the existence of a gravitational field reduction of a bundle connection to Lorentz gauge potentials is sufficient but not necessary, and hence gauge fields of wider space-time symmetry groups may come into play, but only if they are accompanied by spontaneously breaking of these symmetries down to the Lorentz ones.

9. *GL*(4, *R*)-*symmetry gauge*

The symmetry group GL(4, R) is one of the most natural candidates to generalize the Lorentz gauge gravitation because its gauge fields represent the most general kind of linear connection on tangent bundles. Nevertheless, now one prefers, as a rule [117, 73, 37, 78], to gauge GL(4, R) symmetries in the framework of the affine GA(4, R) gauge.

Sometimes GL(4, R) gauge transformations are by mistake identified with coordinate transformations, what motivates somebody to look upon GL(4, R) as so-called passive symmetries, whose localization has nothing to do with conventional gauging. This question was discussed by us, F. Hehl et al., and J. Cho et al. [16], and the fiber bundle analysis clarifies this point.

GL(4, R) is a structure group of the tangent bundle over a space-time manifold X^4, and in the gauge gravitation theory GL(4, R) (or $GL^+(4, R)$ because of orientability of X^4) gauge transformations have the conventional gauge status as changes of atlases of tangent and associated bundles, while generally coordinate transformations vary the coordinate atlas of a manifold X^4. Thus coordinate and GL(4, R) gauge transformations fail to correlate with each other. Such a correlation must be secured by hand, e.g., operating with holonomic reference frames. In this case coordinate and tangent bundle atlases are taken just as in eq. (5.1), and they both change in such a way that coordinate transformations $x^\mu \to x'^\mu(x^\nu)$ are regarded as generating GL(4, R) gauge transformations of tangent reference frames $\partial_\mu \to \partial'_\mu = (\partial x^\nu/\partial x_\mu)\partial_\nu$. These transformations form a holonomic subgroup of the gauge group $GL(4, R)(X^4)$.

We go into these details because one meets the same mixing of coordinate and gauge transformations in the original versions of the Poincaré gauge theory (see next section).

The GL(4, R) group can be split into a one-parameter group of dilatations D and the SL(4, R) group of volume preserving transformations in the Minkowski space-time. The latter has the Lorentz subgroup SO(3, 1) of angular momentum and spin operators L_{ab} ($a, b = 0, 1, 2, 3$), where $L_{ab} = -L_{ba}$, but the remaining nine generators form the symmetric shear operators I_{ab} ($a, b = 0, 1, 2, 3$), i.e. $I_{ab} = I_{ba}$ and tr $I_{ab} = 0$. The commutation relations of the Sl(4, R) algebra are given by known ones of the Lorentz algebra and by the following expressions:

$$\begin{aligned} [L_{ab}, I_{cd}] &= (\eta_{ad} I_{bc} + \eta_{bc} I_{ad} - \eta_{ac} I_{bd} - \eta_{bd} I_{ac}) \\ [I_{ab}, I_{cd}] &= (\eta_{ac} L_{bd} + \eta_{ad} L_{bc} + \eta_{bc} L_{ad} + \eta_{bd} L_{ac}) . \end{aligned} \tag{9.1}$$

The relevance of GL(4, R)-gauge for gauge gravitation is based on the fact that all world-tensors are classified by finite linear representations of this group. Difficulties arise, however, with physical interpretation of spinorial linear representations of GL(4, R), which is reduced to the infinite sums of O(3)-spinors [30, 83, 97, 78].

At the same time the physical importance of spinorial representations of the Lorentz subgroup of GL(4, R) is common knowledge. It motivates one to find physically relevant spinorial states with GL(4, R) symmetries in a class of non-linear realizations of this group induced by spinorial representations of the Lorentz group [78], which is the Cartan subgroup of GL(4, R) as one can see from commutation relations (9.1). Note that representations of GL(4, R) coincide (because the dilatation operator D commutes with all other generators) with representations of SL(4, R) on which the dilatation law can be defined at will.

Non-linear realizations of SL(4, R) are built in the infinitesimal limit as follows. Denoting the parameters of the Lorentz and shear generators of the algebra Sl(4, R) as $\{u^{ab}\}$ and $\{\sigma^{ab}\}$, one may write a given element g in the neighbourhood of the unity of SL(4, R) as

$$g = (\exp \sigma I)(\exp uL) . \tag{9.2}$$

We find a non-linear representation of SL(4, R) on the product space V = $\Psi \times$ SL(4, R)/L, where Ψ is a space of some linear, e.g., spinorial representation of the Lorentz group. Small elements of V are represented by pairs (σ, ψ), where σ marks a certain left coset of SL(4, R) modulo L, but the group element $(\exp \sigma I)$ in (9.2) is a representative of this coset σ.

The left translation action of SL(4, R) on the coset space elements σ can be regarded as acting on the representatives of cosets

$$g\sigma = \sigma' , \qquad g \exp(\sigma I) = \exp(\sigma' I) \exp(u' L) \tag{9.3}$$

where the L-valued remainder $\exp(u'L)$, being superfluous for the transformation law of cosets σ, may be utilized for action on a Lorentz representation space Ψ. Thus a total realization of SL(4, R) on the space V can be defined

$$g \colon (\sigma, \psi) \to (\sigma', \psi' = \exp(u'L)\psi) \tag{9.4}$$

where σ' and u' are solved from eq. (9.3).

Because the quotient space SL(4, R)/L is isomorphic with the space of pseudo-Euclidean bilinear forms g_{ab} in R^4 (with det $g = -1$) the representation space V looks like a metric-spinor or gravitation-spinor complex composed of elements (g_{ab}, ψ), where $g_{ab} = (\exp(\sigma I)\eta)_{ab}$, and the operators $(\exp \sigma I)$ represent the tetrad coefficients h.

For the first time a similar gravitation-spinor complex was constructed in ref. [82], and we have, with reference to [77, 78], discussed such a realization of SL(4, R) in view of spontaneously breaking space-time symmetries and treating gravity as a Goldstone type field.

However, this realization of SL(4, R) in the form of eq. (9.4) is built only for infinitesimal operators of this group, i.e., in fact for the Lie algebra Sl(4, R) and the weak-gravitation limit. Usually the infinitesimal limit in the non-linear realization scheme satisfies everybody, but in the given case it is insufficient because the geometric nature of the gravitational field depends on the global structure of a space-time manifold X^4.

Therefore one must use a global procedure to induce representations for building the gravitation-spinor complex. This procedure states that representations of SL(4, R), induced by a certain representation of the Lorentz subgroup, are found on the space of Ψ-valued functions in the group space of SL(4, R), which satisfies the following condition

$$\varphi(g, l) = l^{-1}\varphi(g), \qquad l \in \mathrm{L}.$$

This condition as a matter of fact, reduces functions $\varphi(g)$ on the group space to the functions $\varphi(\sigma)$ on the coset space SL(4, R)/L or on the set of the representatives $\{\sigma_r\}$ of cosets $\varphi(\sigma) = \varphi(\sigma_r)$. Then the induced representation of SL(4, R) on these functions is defined as

$$(g\varphi)(\sigma_r) = (g^{-1}\sigma)_r^{-1}(g^{-1}\sigma_r)\,\varphi(g^{-1}\sigma_r)\,. \tag{9.5}$$

We see that a concrete form of the representation (9.5) is determined by a particular choice of representatives of the cosets. The expression (9.2) exemplifies such a choice of representatives near the unity of the group. Generally a family of coset representatives $\{\sigma_r\}$ is defined by a certain global section of the L-principal bundle SL(4, R)→SL(4, R)/L. As this bundle is trivial, it has global sections.

In turn, Ψ-valued functions $\{\psi\}$ represent global sections of fiber bundles over a base SL(4, R)/L with a typical fiber Ψ. If such a bundle is trivial too, one may take non-zero constant functions ψ, i.e., represented by pairs (σ, ψ) like in the case of the infinitesimal representation (9.4).

Thus it is proved that the gravitation-spinor representation (g_{ab}, ψ) of SL(4, R) can be spread on all group transformations and all pseudo-Euclidean metrics g_{ab}.

World-tensors can also be rewritten in the fashion of induced representations of SL(4, R). For instance, a world-vector a_μ may be identified with a pair (h^a_μ, a) of a tetrad coefficient h^a_μ and a Lorentz vector $a_a = h^\mu_a a_\mu$.

The action of the dilatation subgroup D of GL(4, R) on representations of SL(4, R) may be defined at will, e.g., on the world-vectors both by length preserving operators

$$\mathrm{D}\colon a^\mu \to \mathrm{d}a^\mu\,, \qquad a_\mu \to \mathrm{d}^{-1}a_\mu$$

or by conformal scale operators

$$\mathrm{D}\colon a^\mu \to \mathrm{d}a^\mu\,, \qquad a_\mu \to \mathrm{d}a_\mu, \qquad a^2 \to \mathrm{d}^2 a^2\,.$$

However, the last case may be reduced to the former one by multiplication of a_μ by a scalar density s:

$$a^\mu \to a^\mu, \qquad a_\mu \to s^{-2}a_\mu$$

which possesses the transformation law D: $s \to \mathrm{d}s$.

As follows from this GL(4, R)-representation excursion just the induced and world-tensor representations of GL(4, R) form now apparently the chief part of physical applications of this group, although nobody would quite reject its polyfield representations too. So, the bundles occurring in the GL(4, R) gauge theory represent tensor products of tangent and cotangent bundles and bundles in product spaces $GL(4, R)/L \times \Psi$ of induced representations of GL(4, R). But in the last case for a global section, namely for existence of the gravitational part of such a bundle, the contraction of a structure group GL(4, R) to the Lorentz group must occur. It reproduces the situation, which arises due to the Equivalence Principle in the Lorentz gauge gravitation. But here it is caused by the requirement that fermion fields must be defined in any reference frame, e.g., in holonomic frames on a space-time.

In a sense, gravitation enlarges the Lorentz symmetry of fermions up to the spontaneously broken GL(4, R)-symmetry.

Now let us consider the affine GL(4, R) geometry. Gauge fields of GL(4, R) contain Lorentz, shear, and dilatation gauge fields corresponding to *L*-, *I*- and *D*-generators, respectively. In comparison with the Lorentz connection which consists of Christoffel symbols and contortion, GL(4, R) connection includes also a non-metricity part Q associated with shear and dilatation fields. On a tangent bundle Q can be evaluated from the expression

$$\mathrm{D}_\mu g_{\epsilon\sigma} = -2Q_{\epsilon\sigma\mu}$$

and the whole GL(4, R) connection takes the form (2.11). This connection represents the most general kind of linear connections on a tangent bundle and describes the Eddington affine geometry on it.

The Noether current associated with the shear and dilatation transformations is a so-called hypermomentum current [36, 37].

H. Weyl was the first [113] who attempted to generalize gravitation theory by taking a non-metricity connection into consideration in the form of the dilatation gauge field $Q_{\epsilon\sigma\mu} = Q_\mu g_{\epsilon\sigma}$. Later many authors (Eddington, Dirac, Utiyama, Ehlers et al.) followed him [13, 60, 110], and now a non-metricity affine geometry is revived again within the framework of gauge gravitation models.

Nevertheless, in spite of an almost 60-years history, the non-metricity generalization of gravity is not as widespread as the Einstein–Cartan theory of gravity with torsion. In our opinion, the reason lies in the fact of the absence as yet of any observable sources, which may be identified with shear and dilatation currents and would generate non-metricity gauge fields like a spin generating torsion. However, in the orbital representation one finds that hypermomentum can be reduced to the set of time derivatives of gravitation quadrupole momentum [16].

At the same time the shear components of the GL(4, R) gauge fields fail to retain Lorentz invariants, and the GL(4, R) gauge theory violates EP in the version discussed above.

10. What are the Poincaré gauge fields?

The other natural extension of the Lorentz group of space-time symmetries is the Poincaré group. But the Poincaré gauge did not appear as a generalization of the Lorentz gauge gravitation model, but as its competitor.

The Poincaré gauge approach to gravity was brought into play immediately after Utiyama's work in order to correct his gauge gravitation model, whose drawback was seen in the unnatural gauge status of a tetrad gravitational field from the conventional gauge point of view [62, 96, 28]. The Poincaré approach aimed to represent tetrad fields as gauge fields of translations due to the coincidence of tensor ranks of tetrad fields h^a_μ and hypothetic translation gauge fields A^a_μ, and this idea has been most widespread in gauge gravitation researches for almost 20 years.

Why so? The space-time of Special Relativity is the affine Minkowski space, and the Poincaré group, being the motion group of this space, represents the fundamental dynamic group of SR, and its unitary representations are identified with free particle states in SR. Up to that, for a gauge theory of elementary particles to be complete, the Poincaré gauge is believed to supplement the internal gauging and the intrinsic spin symmetries of particles.

However, one faces here the specificity of gauging the Poincaré group as a dynamic group. Without aiming to give the exhaustive definition, one can characterize dynamic symmetries as describing a space distribution and time evolution of a physical system, and these are realized by differential operators acting in a functional space. Wave functions of free particles in SR exemplify the realization of the Poincaré group as a dynamic group with generators expressed via differential operators:

$$P_\mu = \partial_\mu, \qquad L_{\mu\nu} = L^{\mathrm{or}}_{\mu\nu} + L^{\mathrm{sp}}_{\mu\nu}, \tag{10.1}$$

and just the orbital part L^{or} of the Lorentz group generators $L_{\mu\nu}$ provides the canonical commutation relations of the Poincaré group as a semidirect product of translation and Lorentz groups

$$[L_{\mu\nu}, P_\epsilon] = [L^{\mathrm{or}}_{\mu\nu}, P_\epsilon] = (\eta_{\nu\epsilon}P_\mu - \eta_{\mu\epsilon}P_\nu).$$

In contrast with internal symmetries and Lorentz spin transformations which change field functions in a point, Poincaré transformations with the differential generators (10.1) can be thought of, on the one hand, as coordinate transformations, and, on the other hand, as transitions from point to point. Both these interpretations are equivalent in a flat space, but differ under gauging.

Authors of the first works on Poincaré gauge [62, 96, 28] adhered to the coordinate interpretation of the Poincaré generators (10.1). They combined gauging Lorentz spin transformations and coordinate translations $x^\mu \to x^\mu + a^\mu$. Localization of these translations $x^\mu \to x^\mu + a^\mu(x)$ reproduced the group of general coordinate transformations, which induced, in turn, the holonomic subgroup of the gauge group of tangent bundle transformations GL(4, R)(X), as we have discussed in the previous section. Generators of such correlated coordinate and gauge holonomic transformations are Lie derivatives [64], and just the invariance condition of a matter Lagrangian under these transformations, having nothing to do obviously with gauge translations, called to mind the tetrad or metric field in this gauge approach. But the same invariance condition had motivated Utiyama's insertion of tetrad fields in his Lorentz gauge gravitation model, and hence the gauge status of tetrad fields in this Poincaré gauge model has nothing to do with gauge potentials of the translation group.

The procedure of gauging the Poincaré transformations (10.1) interpreted as point to point transitions was proposed by F. Hehl, P. von der Heyde et al. [40, 35, 38]. This procedure does not reduce to localization of group parameters, as usual, but modifies also the generators of the Poincaré group by replacing ordinary derivatives in (10.1) by the covariant ones:

$$\partial_\mu \to \mathrm{D}_\mu = \partial_\mu - \Gamma_\mu \tag{10.2}$$

where $\Gamma_\mu = \Gamma^{\alpha\beta}_\mu L_{\alpha\beta}$ is a certain Lorentz connection. Then the localization of Poincaré transformations

$$P = \exp\{\sigma^\mu\partial_\mu + \omega^{\alpha\beta}(L'^{\mathrm{orb}}_{\alpha\beta} + L^{\mathrm{sp}}_{\alpha\beta})\}$$

takes the non-conventional form

$$P(x) = \exp\{\sigma^{\mu}(x)\,\mathrm{D}_{\mu} + \omega^{\alpha\beta}(x)\,(L'^{\mathrm{orb}}_{\alpha\beta} + {}^{\mathrm{sp}}_{\alpha\beta})\} \tag{10.3}$$

where L'^{orb} results from L^{orb} by substitution of (10.2).

The replacement (10.2) seems quite natural as generalization of translations in a flat space to parallel translations in a curved space. But at the same time it violates the familiar commutation relations of the Poincaré group, e.g., translation generators become non-commutative

$$[\mathrm{D}_{\mu}, \mathrm{D}_{\nu}] = R^{\alpha\beta}_{\mu\nu} L_{\alpha\beta}$$

and transformations (10.3) do not compose the gauge Poincaré group P(X) in the conventional sense.

Moreover, the invariance of a matter field Lagrangian under the gauge Poincaré transformations (10.3) reduces on extremal fields to the ordinary invariance conditions under gauge Lorentz spin transformations and holonomic gauge GL(4, R)-transformations, which are the same as we have seen in the previous gauge gravitation models and they lead to the same Lorentz gauge and tetrad fields.

Thus one observes that both of the discussed Poincaré gauge versions being outside the conventional gauge scheme, fail to provide a gravitational field with the status of the gauge potential of the Poincaré translations, in spite of the previous intentions. At the same time these Poincaré gauge attempts were stimulating for the development of the affine-metric theory of gravitation, e.g., the Einstein–Cartan theory of gravity with torsion.

The conventional gauge techniques can be applied for gauging the Poincaré group, if one degresses for a time from its physical role as a special relativistic dynamic group and regards it as an abstract structure and holonomy group of some fiber bundles [14, 15, 73, 84, 21, 31, 81].

It is a well-known fact that most of the representations of the Poincaré group P with physical meaning are built as induced representations, realizing the translation subgroup T as translations in its own subgroup space T = P/L, which is isomorphic with the affine Minkowski space. Thereby, this space must be a part of any construction of a fiber space of bundles in the standard Poincaré gauge theory. Moreover it seems reasonable to require that the bundles in T are associated with an affine tangent bundle A T(X) over a space-time manifold X^4. Therefore we confine our attention to Poincaré connections on A T(X) and the associated principal bundle AX in affine frame spaces.

The bundles A T(X) and AX differ from the linear bundles T(X) and LX in an affine typical fiber V × T, where V denotes a vector typical fiber of T(X) or LX. The Poincaré group action on V × T reads

$$\mathrm{P} \ni g = (g_{\mathrm{L}} \in \mathrm{L}, g_{\mathrm{T}} \in \mathrm{T}): \quad (v, t) \to (g_{\mathrm{L}} v, g_{\mathrm{L}} t + g_{\mathrm{T}}).$$

AT(X) and AX are associated with T(X) and LX, and the structure affine group GA(4, R) of AT(X) and AX contracts to the linear group GL(4, R).

A Poincaré connection form A splits on a P-bundle in two components $A = A_{\mathrm{L}} + A_{\mathrm{T}}$, where A_{L} denotes a Lorentz connection, and $A_{\mathrm{T}} = A^{a}_{\mu} T_{a}\, \mathrm{d}x^{\mu}$ represents an R^4-valued translation connection form, whose coefficients play the role of translation gauge fields.

We are especially interested in the situation, where the Poincaré structure group of a bundle contracts to its Lorentz subgroup. In this case a global section σ of the associated bundle exists in the quotient spaces P/L. Then one can expand a translation connection form A_{T} in two parts:

$$A_{\mathrm{T}} = A_{\sigma} + \hat{\theta} \tag{10.4}$$

where A_σ is calculated from the following condition

$$(\mathrm{D}-A_\sigma)\sigma \equiv 0\,, \qquad \mathrm{D}=\mathrm{d}-A$$

and reads $(A_\sigma)^a = (\mathrm{D}\sigma)^a$, such that (10.4) takes the form

$$A_\mathrm{T} = (\mathrm{D}\sigma)^a T_a + \hat{\theta}^a T_a\,. \tag{10.5}$$

One easily sees that just the component A_σ is responsible for the inhomogeneous transformation law of the connection A_T under gauge translations, while $\hat{\theta}^a$ remains invariant under these transformations and satisfies the linear law of gauge Lorentz transformations. Moreover, there is always a certain translation gauge, where the inhomogeneous part A_σ of the translation connection A_T equals zero, and A_T coincides the part $\hat{\theta}$. For instance, it is possible to get such a reduction $A_\mathrm{T} = \hat{\theta}$ to occur in all bundle atlases which have only linear group transition functions, if one chooses σ to coincide with the zero function in these atlases.

Let us fix such a translation gauge. Then one returns to consideration of the bundles AX and TA(X), and can make use of the known theorems [63] establishing the one-to-one correspondence between general affine connections A on AX, pairs $(A_l, \hat{\theta})$ of the line connections A_l on LX, and the R^4-valued 1-forms $\hat{\theta}$ on X. This correspondence reads

$$A = \begin{pmatrix} A_l & \hat{\theta} \\ 0 & 0 \end{pmatrix}, \qquad F = \begin{pmatrix} R_l & \mathrm{D}\hat{\theta} \\ 0 & 0 \end{pmatrix} \tag{10.6}$$

where the general affine connection form A and its curvature form F are represented by (5×5)-matrices acting on columns $\binom{r}{1}$, $r \in \mathrm{R}^4$, and D and R_l denote a covariant differential and a curvature form of the linear connection A_l.

In the framework of the discussed Poincaré gauge A_l is a Lorentz connection, whose coefficients represent Lorentz gauge fields, but coefficients of the form $\hat{\theta}$ are homogeneous components of translation gauge potentials.

One sees at once the agreement of the tensor ranks of the translation gauge potentials $\hat{\theta}^a_\mu$ and the tetrad fields h^a_μ. For a long time this superficial agreement stimulated repeated attempts to describe the tetrad gravitational fields in the framework of gauge gravitation theory as gauge potentials of the translation group.

Is there indeed any correlation between gauge fields $\hat{\theta}^a_\mu$ and tetrad fields h^a_μ describing any gravitational field on X^4? Let us look into this question.

Remind that a tetrad gravitational field h is defined as a global section of the fiber bundle Σ in quotient spaces $\mathrm{GL}^+(4,\mathrm{R})/\mathrm{L}$. However, h used to be represented by a family of local sections $\{h_i\}$ of the principal $\mathrm{GL}^+(4,\mathrm{R})$-bundle, which are considered up to Lorentz gauge transformations acting on $\{h_i\}$ to the right. $\{h_i\}$ can be described as matrix fields $\{h^a_\mu(x)\}$ acting in R^4 and corresponding to the gauge transformation between an atlas Ψ of the tangent bundle and the atlas Ψ_η, where the metric gravitational field g (isomorphic with h) looks like the constant Minkowski metric η, i.e., $\psi_{i\eta} = h_i\psi$, where ψ and ψ_η are trivialization mappings of atlases Ψ and Ψ_η, respectively. Changes of a bundle atlas Ψ lead to gauge transformations of a tetrad field

$$h \to gh\,. \tag{10.7}$$

A translation connection form $\hat{\theta}$ determines linear transformations $\hat{\theta}(x)$: $T_x(x) \to T_x(x)$ of tangent spaces at every point $x \in X$. $\hat{\theta}(x)$ can be singular. With respect to some atlas of T(X) a translation form $\hat{\theta}$ is represented by a family of matrix fields $\{\hat{\theta}_i = \psi_i \hat{\theta} \psi_i^{-1}\}$, also acting in R^4. Thus one can locally match tetrad fields h and gauge translation fields $\hat{\theta}$. But the gauge transformation law of a connection form $\hat{\theta}$:

$$\hat{\theta}_i \to g_i \hat{\theta}_i g_i^{-1} \tag{10.8}$$

differs from the transformation law (10.7) of a tetrad field h, and just this difference destroys the hypothetical identification of tetrad gravitational fields h and translation gauge potentials $\hat{\theta}$.

Indeed, let us suppose for a moment that a tetrad gravitational field h and a translation connection $\hat{\theta}$, which are realized, with regard to some atlas Ψ, by the matrix fields $\{h_i\}$ and $\{\hat{\theta}_i\}$ in R^4, respectively, are identified. Let Ψ by Ψ_η. Then $\{h_i\}$ represent Lorentz transformations of R^4, and one can always single out a patch U_i and an atlas Ψ_η, such that $h_i(x) = \mathrm{id}_{R^4}$ on U_i in Ψ_η. In turn, the translation gauge field $\hat{\theta}$ identified with h must be reduced to the soldering form θ on U_i. But then, in virtue of the gauge transformation law (10.8), such a connection has to be equal to this soldering form θ on the whole manifold X^4 and in all atlases of the tangent bundle. This is obviously impossible on a manifold X^4 possessing a non-trivial tangent bundle and with respect to other atlases of $T(X^4)$.

Thus we see that the identification of tetrad fields describing a certain gravitational field and homogeneous components $\hat{\theta}$ of gauge translation fields can only take place on some patch of trivialization of a tangent bundle T(X) and with respect to a certain fixed reference frame Ψ. But even within these limits nothing proves that a tetrad field $h = \hat{\theta}$ represents a tetrad field corresponding to some gravitational field because nothing fixes the Minkowski signature of this field.

Note that some authors [87, 14, 31] proposed to identify h with $(\theta + \hat{\theta})$, but it does not change the main conclusion about the non-coincidence of gravitational and gauge translation fields.

This conclusion brings us back to the problem of the physical meaning of translation gauge fields and the Poincaré translations acting inside fibers of bundles.

In the case of the affine tangent and affine frame bundles the Poincaré translations act inside tangent spaces as translations of tangent vectors

$$T_a: \tau_B \to \tau_b + a_b \delta^B_a\,, \qquad \tau_b \in T_X\,.$$

Some authors [20, 84, 85] considered the realization of such translations of functions $\varphi(x, \tau)$ which do not depend only on space-time points x, but also on tangent vectors τ, i.e., $\varphi(x, \tau)$ possess a sui generis "internal affine" symmetry, and, for instance, even the relevance of such functions for describing hadrons was discussed.

In contradistinction to the application of tangent vectors as arguments of field functions another approach (L. Chang [12], V. Ponomariov [85]) uses them as values of field functions, which are considered to take values in a space $V \times T_X$ of some non-linear realization of the Poincaré group, where V is a space of the Lorentz group representation, but tangent spaces T_X play the role of spaces of values of the Goldstone fields corresponding to the translation group T = P/L. So Poincaré translations act only on Goldstone fields $\tau(x)$, which, however, can be removed by a certain translation gauge, but a translation connection $\hat{\theta}$ remains, though both its physical relevance and its geometrical sense (in the framework of linear geometry on a manifold X^4) seems to be not quite clear.

Indeed, a generalized affine connection A on the affine frame bundle AX defines a linear connection

A_l and a supplementary R^4-valued form $\hat{\theta}$ on a linear frame bundle, and an affine curvature of A represents the sum of the linear curvature of A_l and the linear covariant derivative $D_l\hat{\theta}$ of the form $\hat{\theta}$. Only if a translation connection from $\hat{\theta}$ is reduced to the soldering form θ, the covariant derivative $D_l\theta$ represents a familiar geometric object, namely, the torsion form Ω of a linear connection A_l.

This fact leads some authors to restrict their attention only to affine connections, i.e., when a translation connection coincides with the soldering form θ [87, 31, 73]. However such a coincidence can only occur if a linear group principal subbundle of an affine bundle is isomorphic with a linear frame bundle, but in this case the soldering form represents the canonical attribute of all linear frame bundles, and moreover it fails to contain any information about the specificity of each of them.

In spite of the opinion of some authors, the soldering form itself is unable to define any torsion and tetrad gravitational fields, because a connection is constructed without use of the soldering form. A parallel transport of θ does not define a torsion field, but picks out the torsion components of a connection as only these components take part in θ transport. The coefficients $h^i_\mu(x)$ of the soldering form $\theta = \tau_i\sigma^i$, $\sigma^i = h^i_\mu\, dx^\mu$, written with respect to a certain atlas Ψ, make only sense of tetrad coefficients, if a gravitational field has been defined, and Ψ is the atlas, where this field becomes a Minkowski one.

Note that with use of the results of the GL(4, R) gauge (see the previous section) the Poincaré gauge theory is generalized easy on gauging the affine group GA(4, R) both in the framework of non-conventional approaches [3, 35, 68, 34, 38] and in the standard-like gauge scheme [15, 84, 81], but here one faces the same difficulties as in the Poincaré gauge theory.

A version of the Poincaré gauge is also based on the insertion of the Poincaré group in some linear groups, e.g., GL(5, R), the de Sitter group, and the conformal group, whose gauge theories are built in the standard way, but must be supplied with some conditions of reduction to the Poincaré or Lorentz gauges, which, in particular, results in various kinds of Goldstone and Higgs fields appearing in the framework of non-linear realizations of these groups [9, 70, 72, 46, 61, 45, 21]. Nevertheless these theories are blurred by a number of hypothetic fields connected with non-Lorentz symmetries, and whose physical sense remains unclear as yet.

Thus one has the impression that at the present time only the Lorentz gauge theory supplied with the mechanism of spontaneous symmetry breaking can pretend to the quite satisfactory gauge description of gravity supplemented with torsion. Indeed, only in the Lorentz gauge gravitation, in contrast with other gauge generalizations of Einstein's gravitation possesses observable sources for all gauge, Goldstone and other fields.

IV. Some outlooks

In this concluding section the main outcomes of the gauge treatment of gravity for modern gravitation theory are briefly sketched.

11. Gauge gravitation as a metric-affine one

The structure of gravitation theory as likely as any field theory is determined by establishing a family of fundamental dynamic variables and a form of their action functional defining the field equations and conservation laws in classical theory, and generating functionals in quantum theory.

For a long time the majority of gravitationists followed the path of Einstein's GR and believed that the metric (or tetrads, or 2-spin field etc. in various reformulations of GR) is the single gravitational variable possessing the Hilbert–Einstein scalar curvature Lagrangian. At present time the choice of variables and of a Lagrangian of gravitation theory is again widely discussed.

As we have mentioned above, Weyl, Eddington and Cartan were the first who drew their attention to the fundamental role of connections in the geometrization of field theories [113, 23, 10]. Later these ideas were developed in connection with the description of fermions in GR [26, 114, 115]. But the decisive step on this way was made in the framework of the gauge gravitation approach [62, 96, 28, 32], where all versions admit two kinds of fundamental gravitational potentials: metric (or tetrad field) and connection.

Metric and connection represent two independent geometric objects of space-time geometry, but in gravitation theory this independence is not absolute because EP, postulating parallel translations preserving Lorentz invariants, establishes the metricity constraint

$$(\mathrm{d} - \Gamma)g \equiv 0 \tag{11.1}$$

on the pseudo-Riemannian metric g and the Lorentz connection Γ, whose variations can, therefore, not be independent. Consequently this constraint is introduced on physical reasons and does not issue from the geometrical nature of metric and connection. It holds in metric-affine gravitation theories with Lorentz connections, e.g., in GR and the Einstein–Cartan theory.

In these theories one chooses one from two possible versions of the choice of dynamic variables. In the first case they are metric (or tetrad) fields and the connection Γ is constrained by (11.1). In the second one the constraint (11.1) is resolved as a sui generis kinematic condition, which results in splitting of the connection Γ in Christoffel and contortion parts as in (8.2). Then a family of dynamic variables of an affine-metric gravitation theory will consist of only tensor quantities, metric g and contortion K, and even in the limit of zero contortion these different variants of the possible choice prove not to be equivalent.

Note that in absence of the metricity constraint (11.1) one can either prefer independent variables g and Γ, or one can split Γ in the sum (2.11) of the Christoffel symbols, contortion K and non-metricity Q to deal with only tensor dynamic variables g, K (or torsion S) and Q.

Let us discuss the chief points of the Lagrangian problem in the gauge gravitation theory. It is necessary to emphasize that the gauge scheme itself establishes only the kinematics of gauge theory, i.e., a family of field variables, their transformation laws etc., but fails to determine directly a form of Lagrangian. However, in gauge models of internal symmetries the choice of a gauge field Lagrangian in the Yang-Mills form is rather unique, but in the gauge gravitation theory one has more freedom. The reason lies in the specificity of gauge models of space-time symmetries, where Goldstone fields are dynamic, and there is the possibility to pair group and space-time indexes of gauge fields. That enriches the gauge gravitation models with different variants of Lagrangians, which are impossible in the gauge theories of internal symmetries.

First of all let us point out that the conventional scalar curvature Lagrangian

$$L_{\text{H-E}} = \frac{\sqrt{-g}R}{2\kappa}$$

turns out not to be as correct as it seemed.

Indeed, the variation goal based on $L_{\mathrm{H-E}}$ in GR can be stated correctly only for asymptotically flat metrics, because boundary conditions turned out to be incompatible in general with the Hilbert–Einstein Lagrangian $L_{\mathrm{H-E}}$ including the second-order derivatives in metric. To remove this incompatibility one can reduce the rank of gravitation equations either by using the Palatiny variation with respect to metric and connection variables constrained by the metricity condition (11.1), or to convert $L_{\mathrm{H-E}}$ by adding the special divergence term to the Lagrangian without second-order derivatives of the metric. In connection with this we remark that having regard to one or another boundary term in a gravitation, the action turns out to be highly essential both in classical and in quantum gravitation theories.

In metric-affine generalizations of GR the Lagrangian

$$L_{\mathrm{H-E}} = \sqrt{-g}\,\frac{\tilde{R}}{2\kappa}$$

where $\tilde{R}$ denotes a scalar curvature of a general linear connection (while in the following R will mark a curvature only of the Christoffel part of a connection), turns out to be not quite satisfactory too, because it produces only algebraic equations of torsion fields and strikes off free dynamic torsion ("tordions").

In quantum gravitation theory $L_{\mathrm{H-E}}$ faced the problem of renormalization. Quantum "tordions" is also impossible in the theory with only the Hilbert–Einstein Lagrangian.

One tries to resolve these problems by introducing the quadratic curvature Lagrangians, and the discussion on gravitation Lagrangian option centres to-day in the branch of quantum arguments.

The simplest generalization of Einstein's GR consists in adding quadratic curvature terms in a gravitation Lagrangian

$$L = \sqrt{-g}\,(\lambda + a_{\mathrm{R}}R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu}R^{\mu\nu} + \alpha_3 R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}) \tag{11.2}$$

(Eddington, Buchdal, Lanczos, etc.). They were proposed also by Weyl in his version of unified theory, and proved to be quite necessary in quantum gravitation theory as counter terms due to the one-loop contributions. Some constraints are imposed usually on values of the constants α in eq. (11.2).

At the same time gravitational field equations derived from the Lagrangians (11.2) are of the fourth order in metric, which leads to other gravitation solutions in comparison with GR.

We emphasize also the necessity in virtue of a variety of serious arguments [55] to include also the cosmological term λ in Lagrangians of the gravitation theory (these are formal arguments missed in 1915 by Hilbert and Einstein; empirical cosmological data of McVittie and B. Tinsley also lead to retaining λ introduced first by Einstein in 1917; quantum field theory necessarily leads to induced λ, which may prove to be variable).

Now let metric g, contortion K (or torsion S), and non-metricity Q compose the family of independent dynamic variables in the metric-affine generalizations of GR. A metric-affine Lagrangian L used to be constructed from a complete connection Γ, and the fields S and Q turn out to be inserted in L only via the connection Γ in this case. However one may construct a total Lagrangian also as the sum

$$L = L_{\mathrm{g}} + L_{\mathrm{S}}(S, \mathrm{D}S) + L_{\mathrm{Q}}(Q, \mathrm{D}Q) \tag{11.3}$$

of a metric gravitation Lagrangian L_g, e.g., the Hilbert–Einstein Lagrangian of GR, and Lagrangians L_S of torsion S and L_Q of non-metricity Q. This is possible because both S and Q are tensors under gauge transformations. Moreover, a covariant derivative of fields S and Q in L_S and L_Q may be chosen either as a complete covariant derivative $\tilde{\mathrm{D}}$ with contortion and non-metricity terms or reduced to the covariant derivative D only of the Christoffel connection.

Of course, Lagrangians of the form (11.3) seem to be unnatural from the standpoint of the geometric unity of Christoffel and torsion, and non-metricity components of a complete connection Γ. Nevertheless, two reasons can motivate us to neglect for a moment the considerations on the geometric elegancy of Lagrangians constructed in the metric-affine theory. In the first place, the gravitation L_g, torsion L_S and non-metricity L_Q components of a total Lagrangian (11.3) may be chosen independently of each other, e.g., L_g is the Hilbert–Einstein Lagrangian of GR, but L_S and L_Q are Lagrangians of the Yang–Mills type. Secondly, nothing requires that coupling constants of the torsion and non-metricity Lagrangians L_S and L_Q in (11.3) coincide with the gravitation constant in L_g. In particular, torsion and non-metricity coupling constants may be chosen much stronger than the gravitational one, which opens a door to the hypothesis about the possibility of strong torsion inside elementary particles or quarks, whose effect would be comparable with weak or strong interaction effects.

In recent years torsion has attracted great attention as an affine generalization of the metric gravity. The reason that torsion comes to the front lies in the fact that at present we only know two observable space-time characteristics of particles, namely, mass (energy-momentum) and spin. And just energy-momentum and spin of matter turn out to be the sources of metric gravity and torsion, respectively. But because we do not observe any object possessing macrovalues of spin polarization, torsion theory as yet cannot rival with Einstein's gravitation theory.

The possibility of introducing torsion was indicated first by Cartan in 1922 [10], but for a long time it remained in the shadow. For instance, Eddington who developed some affine generalizations of the metric structure of GR, explicitly rejected the torsion [23]. Essentially such scepticism was due to the lack of success of the attempt of Einstein and others (1928) to use the torsion generalization of Riemannian geometry for building the unified theory of gravity and electromagnetism.

In field theory torsion was revived again in 1950 in the works of Weyl [115], and also by us [52, 53], but just the gauge approach showed torsion as the indispensable attribute of space-time geometry [62, 96, 28, 33, 37] since it was proved that one has no symmetry group, whose gauge would describe only metric gravitation.

A. Trautman [105, 106] made an important contribution to the formulation of the gravitation theory of the Einstein–Cartan type, and was among the first to use the treatment of the underlying structure of this theory in terms of fiber bundles.

Referring the readers for details of the theory and effects of classical torsion to the review of Hehl et al. [35], we want to draw their attention again to the fundamental phenomenon of torsion, inducing nonlinearity in the spinor Dirac equation.

Let us consider a system of Dirac massless fermions ψ in the Einstein–Cartan space U^4 and supply this system with the Lagrangian

$$L = \frac{1}{2\kappa}\sqrt{-g}\,\tilde{R} + L_\psi = \frac{1}{2\kappa}\sqrt{-g}\,R + \frac{\sqrt{-g}}{2\kappa}K^2 + L_\psi\,,$$

where L_ψ is the Dirac Lagrangian in the space U^4.

The variation of L over the matter fields ψ yields the Dirac (Weyl) equation in the space U^4

$$\gamma^{\mu}\tilde{D}_{\mu}\psi = 0\,. \tag{11.4}$$

The variation of L over metric fields yields the Einstein-type equation

$$\tilde{R}_{\mu\nu} - \tfrac{1}{2}\tilde{R}g_{\mu\nu} = \kappa\tilde{T}_{\mu\nu}(\psi)\,. \tag{11.5}$$

At last the variation over contortion yields the specific equation connecting contortion K with the spin current of fermion fields $\mathcal{J}_{\Psi}$

$$K^{\alpha\beta\gamma}\epsilon_{\alpha\beta\gamma\epsilon} = K_{\epsilon} = \kappa\bar{\psi}\gamma_{\epsilon}\gamma_5\psi = \kappa S_{\epsilon}\,. \tag{11.6}$$

Substituting K evaluated from (11.6) in the equations (11.4, 5), one finds the field equations (11.4–6) in the form:

$$R_{\mu\nu} - \tfrac{1}{2}R\,g_{\mu\nu} = \kappa T_{\mu\nu}(\psi) \tag{11.7a}$$

$$\gamma^{\mu}D_{\mu}\psi - \tfrac{3}{8}\kappa\gamma^{\mu}\gamma_5(\bar{\psi}\gamma_{\mu}\gamma_5\psi)\psi = 0 \tag{11.7b}$$

$$K_{\mu} = \kappa\bar{\psi}\gamma_{\mu}\gamma_5\psi\,. \tag{11.7c}$$

Here (11.7a) is the familiar Einstein gravitation equation, but with the modified right hand side corresponding to the energy-momentum tensor of nonlinear fermions described by eq. (11.7b) representing the non-linear generalization of the Dirac equation.

The appearance of the non-linearity in the Dirac equations due to torsion was noted for the first time by V. Rodichev [90] in the special case of the teleparallelism space T_4, and was investigated later in the general space endowed with arbitrary curvature and torsion by many authors (Hehl, Datta, Peres, Krechet, Ponomariov [65], etc.).

Non-linearities due to torsion arise in other fields of non-vanishing spin, e.g., in electromagnetic and Proca fields (V. Ponomariov, E. Smetanin [99], V. de Sabbata). At the same time the question of torsion interaction with gauge fields is not quite clear as yet because such an interaction breaks the corresponding gauge invariance.

Not discussing other interesting consequences of torsion (e.g., specific spin precession, and left–right neutrino oscillations of de Sabbata) we point here at the important possibility of preventing cosmological collapse by account of torsion, which can violate the energy dominance condition of the Hawking–Penrose theorems. We shall return below to some effects of the quantum torsion.

The account of non-metricity also leads to analogous important effects: for Dirac fermionic matter it contributes to the collapse, not preventing it in contradistinction with the torsion case (V. Krechet); like torsion non-metricity induces non-linearity, but of vector, not of pseudo-vector type in spinor equations. Such non-linear equations are analogous to sine-Gordon equations possessing important solitonic solutions.

The spinor case is especially important as the non-linear Dirac equation just of this cubic type has been established (Ivanenko, 1938) and had been proposed as the basis of the unified spinor theory (Heisenberg, Ivanenko, the 1950th) long before its geometrical reproduction. Referring for all details and references to our work [55], we point here out that this version of unification led to many promising results; there were evaluated the masses of hadrons (up to the Ω^- particle) and the coupling constants; in an impressive way the fine structure constant was obtained (1/115–1/120 instead of 1/137). At present we observe the revival of the modernized spinorial non-linearity for describing sub-quarks (preons) [104] as in the standard quark models of Grand unification one is obliged to introduce too many arbitrary parameters (Higgs fields etc.).

Nevertheless, one can object that the constant $l^2 = \frac{3}{8}\kappa$ in front of the non-linear term in (11.7b) is too weak. However, as we have pointed out, the torsion coupling is not obliged to coincide with the gravitation constant. In particular, if the torsion coupling constant is due to the Salam "strong gravity" constant introduced from the condition of equality of sizes of a proton and a "black hole" possessing the proton mass, the self-interaction constant arising in (11.7b) turns out to be of the order of the non-linearity constant of Heisenberg and our version of unified theory [55, 98]. Further analogies of particles and microuniverses also arise (Ivanenko–Krechet, P. Roman, E. Recami [89]).

12. Gauge version of quantum gravity

There are some different approaches to dealing with quantum gravity. One of them, worked out in the framework of concepts of "supergeometry", is based on the Wheeler–De Witt equation [7]

$$\left\{\frac{\delta}{\delta g^{ik}} G^{iklm} \frac{\delta}{\delta g^{em}} + \sqrt{{}^{(3)}g}\,{}^{(3)}R\right\}|{}^{(3)}g\rangle = 0\,, \qquad H^k|{}^{(3)}g\rangle = 0\,.$$

In this approach only a number of quantum cosmological models, reduced to one-parametric functional spaces, were quantized.

Another version of gravity quantization is the covariant quantization. Its actions used to be taken in the general form

$$S = \int_X \mathrm{d}^4x\, \sqrt{-g}\,\{\alpha_1 \Lambda + \alpha_2 R + \alpha_3 R^2 + \alpha_4 R_{\mu\nu} R^{\mu\nu} + \alpha_5\, \epsilon^{\mu\nu\alpha\beta}\, \epsilon_{\mu\nu\alpha\beta}\, R^{\lambda\rho}_{\mu\nu}\, R^{\gamma\delta}_{\alpha\beta}\} + \text{boundary terms.} \tag{12.1}$$

In some versions of the action (12.1) the renormalizable quantum gravitation can be built [49, 101], although it is non-unitary.

The difficulties in quantizing gravity stimulated some authors to consider gravity as a gauge field for using the well-tested quantization and the renormalization procedure of gauge theory. But one faces many peculiarities in quantizing gauge gravity in comparison with quantizing both Yang–Mills and metric gravity.

These peculiarities lie in the affine-metric nature of gauge gravity possessing both connection and metric variables. We shall point at some typical situations which arise in quantizing the affine-metric gravity without entering in details.

1. A space-time connection Γ is treated as a standard gauge field of a certain group, and its gauge

theory differs from the ordinary Yang–Mills formulation only in additional freedom in the choice of the Lagrangian. A metric tensor in such a theory is then regarded as an auxiliary quantity to satisfy the general covariance [74], and is considered as a certain classical background without dynamical role.

2. For another type of metric-affine quantization we refer, e.g., to refs. [101, 79], where both tetrad fields h^a_μ and Lorentz connection $\Gamma_{\mu\alpha\beta}$ are quantized independently. In this approach a Lagrangian, constructed as combination of different pairing curvature and torsion tensors, depends on nine coupling parameters, whose special selection may satisfy some requirements, e.g., the unitarity of a propagator, the decrease of a number of "ghost" fields, etc. This can motivate a definite Lagrangian choice in the metric-affine theory from the quantum standpoint. Nevertheless, this quantization version fails to be fit for, e.g., the Einstein–Cartan metric-affine theory, where the metricity constraint relating connection and metric fields must be taken into account in the procedure of quantization.

3. The metricity constraint in the Einstein–Cartan theory can be resolved before one sets to quantization, which then is modified for the quantization of a contortion component $K_{\mu\nu\sigma}$ of the Lorentz connection instead of quantizing the total connection. Contortion is a tensor under gauge transformations, which causes a certain peculiarity in quantizing it in comparison with gauge fields. In the gauges, where a metric field is reduced to the Minkowski metric, a contortion tensor reduces to a pseudovector field $K_\mu = \epsilon_\mu^{\nu\lambda\sigma} K_{\nu\lambda\sigma}$, but, in contrast to the general gauge fields, the longitudinal component of K_μ cannot be removed by any Lorentz gauge transformation, and quantizing a torsion field is consequently non-renormalizable in general; however, in some particular cases renormalization via, e.g., special choice of Lagrangian may occur, e.g., in the case of the Yang–Mills type Lagrangian for a contortion field. The symmetry properties of the Lagrangian admit additional gauge-like transformations of a torsion field, whose gauge theory then coincides with the standard formulation. Interaction of a contortion with spinor matter fields is renormalizable too.

The quantum torsion in the space of teleparallelism was also investigated in our group by P. Poznanin [88], who considered the model with the contortion Lagrangian

$$L = -\frac{1}{12\kappa} K^\alpha K_\alpha - \frac{\lambda^2}{9} K^{[\sigma,\alpha]} K_{[\sigma,\alpha]} + \frac{\lambda_1}{18} K^{\sigma,\alpha} K_{\sigma,\alpha} + \frac{\lambda_1}{36} (K^\alpha_{,\alpha})^2$$

and the processes of scattering of "tordions" and interactions between particles realized by the exchange of "tordions". The interaction potential of two fermions due to exchange of a single "tordion" contains a long-range part, which in the non-relativistic limit has exactly the form of a dipole-type magnetic interaction

$$\left[3\frac{(\sigma_2)}{r^5} - \frac{(\sigma_1\sigma_2)}{r^3}\right].$$

Another interesting phenomenon, to which we want to draw attention, is that vacuum polarization due to quantized spinor matter induces quadratic terms in the Lagrangian of the Einstein–Cartan field (P. Pronin) quite like the well-known case of the Einstein gravity field (in the last case such terms can lead to non-singular de-Sitter type inflationary cosmology (Guth, Starobinski, Gurovich). The calculation of the one-loop corrections leads to the appearance of counterterms in the Lagrangian, which have the form of the quadratic torsion Lagrangian:

$$L_K \sim F_{\mu\nu ab}F^{\mu\nu ab}, \qquad F_{\mu\nu ab} = \partial_{[\mu}K_{\nu]ab}.$$

Such phenomena attract attention as a possible mechanism of the origin of induced gravitation and other gauge fields by interactions of matter fields [2, 55].

13. Topological features of gravitational fields

Investigation of topological characteristics of gauge fields stimulated the research of analogous characteristics of gravitational fields from the general point of view of topological classification of corresponding fiber bundles.

Here we have to refer the readers for the necessary mathematics to, e.g., refs. [100, 59, 24].

Remind that gravitational fields on an orientable manifold X^4 are defined as global sections of the bundle Λ in spaces of pseudo-Euclidean bilinear forms in tangent spaces over X^4 or of the associated bundle in the quotient spaces $GL^+(4, R)/SO(3, 1)$, and the condition of the existence of the gravitational field everywhere on X^4 is the contraction of the structure group $GL(4, R)$ of $T(X^4)$ to the Lorentz group $SO(3, 1)$.

It defines the first step in the topological classification of gravitational fields into the characteristic classes of tangent bundles admitting the Lorentz structure group.

Such bundles are characterized by the Euler class $e \in H^4(X)$ and the first Pontrjagin class $p_1 \in H^4(X)$, expressed via the Chern classes $p_1 = c_1^2 - 2c_2$, $c_i \in H^{2i}(X)$ of $T(X^4)$ as the bundle admitting the following injection chain of the structure groups

$$SO(3) \to SO(3, 1) \to SL(2, C) \to GL^+(4, R) \to GL(4, C). \tag{13.1}$$

$H^*(X)$ denote the simplicial cohomology groups of a manifold X^4. Due to the injection of these groups into the De Rham cohomology groups of real differential forms on X^4 the characteristic classes e, p_1 can be imaged by the cohomology classes of the closed characteristic forms:

$$e = \frac{1}{32\pi^2}\epsilon_{abcd}R^{ab} \wedge R^{cd}, \qquad p_1 = -\frac{1}{8\pi^2}\mathrm{Tr}\, R \wedge R \tag{13.2}$$

where R is a curvature 2-form of some connection on $T(X^4)$. These classes are independent on the chosen connection, but the Euler form e must be computed only by using the $SO(4-k, k)$-valued curvature R. The Gauss–Bonnet theorem relates the Euler characteristic χ of a compact manifold X^4 to the Euler class of its tangent bundle $T(X^4)$: $\chi(X^4) = \int_{X^4} e(T(X^4))$.

The Stiefel–Whitney classes of a tangent bundle over X^4 are defined too. These classes $\omega_i \in H^i(X, Z_2)$ are not given in terms of curvature, but it is necessary that ω_1, ω_2 are equal to zero for defining spinor fields on X^4.

Tangent bundles possessing the structure group which contract to the Lorentz group and consequently to $SO(3)$ have the classes e, c_1, $\omega_{1,2}$ equal to zero. Thus to classify the manifolds, admitting a gravitation structure, we have only the single topological characteristic – the Pontrjagin class p_1, but the zero value Euler class of $T(X^4)$ represents the necessary and sufficient condition for a gravitational field to exist on X^4, that takes place on non-compact manifolds X^4 and compact manifolds X^4 possessing the trivial Euler characteristic.

The second step of the topological classification of gravitational fields is to classify them on the same manifold X^4.

For this purpose it is convenient to use the one-to-one correspondence between pseudo-Riemannian metrics g and non-vanishing vector fields τ on a manifold X^4, which follows from the contraction of the Lorentz structure group of $T(X^4)$ to SO(3) and reads

$$g^{\tau}_{\alpha\beta} = -g^{R}_{\alpha\beta} + 2g^{R}_{\alpha\gamma}g^{R}_{\beta\epsilon}\tau^{\gamma}\tau^{\epsilon}/\tau^2 \tag{13.3}$$

where g^R is some Riemannian metric on X^4. The vector field τ is time-like with respect to the pseudo-Riemannian metric g^τ.

Every field as such determines a one-codimension transversal (e.g., orthogonal) foliation on X^4 [66, 29], which defines a space-time structure on a manifold X^4, corresponding to the gravitational field g^τ evaluated from (13.3). Inversely, any one-codimension foliation on X^4 may be considered as describing a certain space-time structure on X^4, and a gravitational field corresponding to it can be reconstructed.

Hence, the investigation of gravitational fields on a manifold can be reduced to describing one-codimension foliations on X^4. Their characteristic classes correspond to homotopy classes of morphisms of X^4 into a certain universal space, whose entire topological structure remains unknown up to this time. The single characteristic class of foliations which we have in our hands is the Godbillon–Vey class γ. It is defined as a cohomology class of the closed Godbillon–Vey 3-form $\gamma = \theta \wedge d\theta$, where the 1-form θ is evaluated from the expression $d\tau^* = \tau^* \wedge \theta$, which holds for the dual 1-form τ^* corresponding to the vector field τ, defined by a given one-dimensional foliation on X^4.

In particular, if a foliation is a bundle, i.e., its leaves represent fibers of some bundle $\pi: X^4 \to T^1$, the forms θ and γ are zero, but such a foliation can be described by a real function f evaluated from the expression $\tau^* = df$, when the form τ^* turns out to be exact. The latter is a typical situation in gravitation models, and, e.g., gravitation singularities may be described apparently in terms of critical points of such real functions [57, 93].

We have indicated the chief topological attributes of the pseudo-Riemannian gravitation. At the same time as in the Yang–Mills theory the quantization of gravity in the framework of the path integral formalism urges us to work with the Euclidean version of gravitation theory, i.e., with the Riemannian gravitation. In this theory the finite action solutions of the Einstein equations with self-dual curvature are especially appealing because they have interesting mathematical properties and bear the strongest similarity to the self-dual Yang–Mills instantons. Many four-dimensional manifolds admitting metrics with self-dual Riemannian curvatures have been discovered and, in particular, their Euler and Pontrjagin numbers have been computed (for a review, see [24]).

14. Some more radical versions

We have pointed out the general unification goal as the main stimulus for gauge treatment of gravity. We observe that the gauge gravitation theory, although possessing some specificities, is analogous to the theories of electroweak and strong interactions.

The Grand unified theories try to unify all types of interactions and all types of fields and particles (hadrons built from fermionic quarks of 3 colours and leptons which themselves represent lepto-quarks of the fourth colour, where the interactions are mediated by various gauge (compensating) bosonic fields: gluons, photons, intermediate heavy bosons $W^{\pm}$, Z). The minimal GUT is based on SU(5) group unifying hadronic baryons and mesons with weak (atenonic) leptons, predicting, e.g., important proton decay due to mediation via ultra heavy X bosons.

To include gravity one intensively develops now the supergravity theory which is the gauge theory of supersymmetry unifying bosons and fermions in a single multiplet; this necessitates the existence of a fermionic gravitino particle with spin 3/2 as a supersymmetric partner of the $s = 2$ conventional graviton. We may emphasize here that one must take into account the inclusion of not only Einstein's GR, but of some gauge extensions of GR in a supergravity theory (e.g., Einstein–Cartan torsionic versions). Nevertheless, the arguments preventing the supersymmetry theory to be fully successful is that a priori supersymmetries themselves fail to contain internal symmetries.

Our program of unification is proposed on the Coxeter symmetry groups, whose generator elements $\{S\}$ like reflections satisfy the conditions $S^2 = 1$, and the individuality of each group is defined by establishing the conditions $(SS')^m = 1$ for all pairs of generator elements. The well-known Weyl groups of the simple Lie algebras are examples of Coxeter groups, and finite-dimensional representations of these algebras, including ones describing particle multiplets, may be built as representations of corresponding Coxeter groups. At the same time the space-time symmetry groups are Coxeter groups generated by reflections in different hyper-planes of a space-time too. That's why, in our opinion, Coxeter groups are apparently a good pretender for a Grand unified group, and "prespinors" possessing the simplest Coxeter "yes–no" symmetry may play the role of universal preons, whose various composites (when "yes–no" transformations of different prespinors fail to commute) form quarks, leptons, etc. [51].

Moreover, the functor from the category of Coxeter groups into the category of topological space exists, and the Coxeter symmetry groups may be realized as first homotopy groups of some topological spaces playing the role of topological models of particles and subparticles. Such topological models are produced by different gluing of real projective spaces RP^1 corresponding to prespinors themselves. Prespinors may play also the role of sui generis "space-time" preons in the spirit of Wheeler's "pregeometry".

Just the conceptions of the unified "prematter-pregeometry" complex, which seem to be plausible at extremal conditions of superhigh density, temperature, curvature (and torsion), etc. realized presumably inside particles, inside collapsed stars, and in pre-Big Bang conditions, when violent transmutations of gravitons and particles and fluctuations of the metric in the Planck-length region could break the space-time topology (leading to discrete space, worm holes, etc.), may represent the essence of the unification program.

Let us draw our attention on the "Big numbers" $\approx 10^{40}$ which are met in many strange relations between gravitational, cosmological and quantum atomic quantities (e.g., the ratio of Coulomb and Newton forces, the ratio of observed Metagalaxy and nuclear dimensions, etc.). Dirac even tried to develop a new cosmology including "Big numbers" as a fundamental feature [22]. The ratio of Salam's strong gravity and Newton–Einstein gravity constants is also of the order of such "Big numbers". We have with V. Krechet obtained some exact solutions of Einsteinian gravity coupled to the vector Proca field which with a strong gravity constant imitates an elementary particle in the spirit of modernized hierarchical ideas and modernized Machian ideas (Bertotti, Goldoni, etc.). It may be that the cooling of the universe from its "prematter-pregeometry" preonic state by means of a chain of phase transitions passes through some hierarchical stages before reaching the present quark particle state.

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