FOLIATION ANALYSIS OF GRAVITATION SINGULARITIES

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Gravitation singularities are examined as singularities of space-time foliations which represent critical points of real functions on a space-time.

Singularities remain one of the principal problems of contemporary gravitation theory [1]. One faces them in the majority of physically significant solutions of Einstein's equations, but even the notion of gravitation singularities still remains under discussions. (Here we are dealing with Einstein general relativity.)

It seemed natural to identify gravitation singularities with singular values of metric or curvature components and their scalar combinations. However, firstly, such a notion depends on choosing a reference frame and includes fictious singularities which being real for some observers are absent for others. Secondly, even regularity of all metric and curvature quantities fail to garantee one against facing such singular situations as noncomplete geodesics and causality breaking. At present the criterion of gravitation singularities which is based on the notion of so called "bundle-completeness" generalizing the familiar geodesic completeness is that mostly considered [2]. In virtue of this criterion a gravitation singularity is absent if each smooth curve in a space-time can be prolonged up to any finite value of its generalized affine parameters. But this criterion is not devoid of defects too. We see the main of them in that a behaviour of one or two curves tells only a little about the structure of gravitation singularities.

We try to base our approach to the problem of gravitation singularities on the fact that a gravitation singularity leads to a singularity of a space-time structure, and we aim to describe gravitation singularities via their space-time images.

In the gravitation theory a space-time is usually defined to be a 4-dimensional smooth manifold X^4 admitting a pseudo-Riemanian metric g and a g-compatible space-time structure representing the $(3+1)$ decomposition of a tangent space at each point of X^4 into space and time directions. If the $(3+1)$ decomposition is integrable, one finds a space-time X^4 to be foliated in spatial hypersurfaces. A space-time foliation represents a certain topological construction on a space-time [3], and its behaviour around a singularity point may display a topological structure of gravitation singularities.

1. Gravitational fields. A gravitational field on a orientable smooth manifold X^4 is defined to be a global section g of the fiber bundle of pseudo-euclidean bilinear forms in tangent spaces over X^4. This bundle is associated with the tangent bundle $T(X^4)$ possessing the structure group GL(4, R), and is isomorphic with the fiber bundle W in quotient spaces GL(4, R)/SO(3,1), whose global section h describes a gravitational field in the tetrad form.

The necessary and sufficient condition for a gravitational field g to exist on a manifold X^4 is the contraction of the structure group GL(4, R) of the tangent bundle to the Lorentz group SO(3,1), and consequently to SO(3). This means the existence of an atlas $\Psi^g = \{U_\kappa, \psi_\kappa^g\}$ of $T(X^4)$ such that the transition functions gluing charts (U_κ, ψ_κ) of trivializations of $T(X)$ reduce to elements of the gauge groups SO(3,1)(X) or SO(3)(X), but the gravitational field functions $\{g_\kappa = \psi_\kappa^g g\}$ are the constant Minkowski metric $g_\kappa(x) = \eta, x \in U_\kappa$ in all charts of the atlas Ψ^g.

The tetrad gravitational field h written in this

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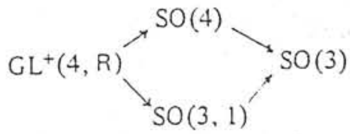
atlas takes on values in the center of the quotient space $GL(4, R)/SO(3, 1)$. Then in any other atlases Ψ the field h can be represented by a family of matrix functions $\{h_\kappa, U_\kappa\}$ which act in the typical fiber R^4 of the tangent bundle and realize a gauge transformation between atlases Ψ^g and $\Psi = \{U_\kappa, \psi_\kappa = h_\kappa \psi_\kappa^g\}$ such that the known relation $g_\kappa = h_\kappa \eta$ holds. The tetrad functions $\{h_\kappa\}$ are determined up to multiplication of them on the right by gauge Lorentz transformations, and this freedom reflects the nonuniqueness of choosing the atlas Ψ^g.

The diagram



of contractions of the structure group of the tangent bundle results in the following theorem.

Theorem 1. Let g be a gravitational field on a manifold X^4. There exists a nonvanishing 1-form ω and a Riemannian metric g^R such that

$$g = g^R - 2\omega \otimes \omega/|\omega|^2 , \qquad (1)$$

where $|\omega|^2 = g^R(\omega, \omega) = -g(\omega, \omega)$. Inversely, let ω be a nonvanishing 1-form on a manifold X^4. For any Riemannian metric g^R on X^4 there exists a pseudo-Riemannian metric g on X^4 such that the collection (g, ω, g^R) satisfies eq. (1). And for any such a collection there is an atlas Ψ^g such that the forms g, g^R, $\omega/|\omega|$ look respectively as the Minkowski metric η, the Euclidean metric η^E, and the constant form $\omega/|\omega| = (1, 0, 0, 0)$ in the frame of Ψ^g.

The last point indicates that the form $\omega/|\omega|$ in theorem 1 coincides with a tetrad form $h^\perp = h_\mu^0 dx^\mu$ of a gravitational field g, and that forms ω and ω' defining the same gravitational field g by eq. (1) differ from each other in some gauge Lorentz transformations and gauge dilatations. Such forms fail to be antipodal in any point $x \in X^4$, and thereby they are homotopic with each other as sections of the bundle in 3-spheres under X^4.

A pair (ω, g^R) defining a gravitational field g by eq. (1) defines also the $(3 + 1)$ decomposition of the tangent bundle $T(X) = T^{\|}(X) \oplus T^\perp(X)$ in a 3-subbundle $T^{\|}(X)$ evaluated from the equation $\omega(T^{\|}(X)) = 0$ and in its orthocomplement $T^\perp(X)$ relative to the Riemannian metric g^R. Metrics g^R and g coincide with each other $\gamma = g\downarrow T^{\|}(X) = g^R \downarrow T^{\|}(X)$ in a subbundle $T^{\|}(X)$.

2. Space-time foliations. Let X be an n-dimensional connected smooth manifold without boundary. One says that a smooth manifold foliation F of codimension $p < n$ is given on X, if X is represented as a union of disjoint sets possessing the following properties. For each point $x \in X$ there is a coordinate chart $(U_\kappa \ni x, \varphi_\kappa)$ such that φ_κ maps linearly connected components of intersections $F_\alpha \cap U_\kappa$ onto $(n - p)$-planes in R^n, which are parallel to the plane $x^1 = ... = x^p = 0$. The sets are named slices of a foliation $F = \{F_\alpha\}$, and X is called a total manifold of F. Slices of a foliation are provided with a topology of $(n - p)$-manifolds.

The tangent bundle $T(X)$ of a foliation total manifold X has a subbundle $T(F)$ of all tangent vectors to foliation slices. The corresponding quotient bundle $N(F)$ is called a normal bundle of a foliation.

A smooth map f: $Y \to X$ of a monifold Y into a foliation total manifold X is called transversal to F, if $T_x(F)$ and $\mathrm{Im}(df)_x$, where df: $T(Y) \to T(X)$, generate the whole tangent space $T_x(X)$ at each point $x \in X$. If f is transversal to F, then preimages of slices of F compose the induced foliation f^*F on a manifold Y.

For foliation mathematics see, e.g., ref. [4,5]. Our application of foliations is based on the following theorem:

Theorem 2. Any orientable foliation F of codimension 1 on a manifold X can be evaluated from the equation $d\omega = 0$, where ω is some nonvanishing 1-form on X, which satisfies the integrability condition

$$\omega \wedge d\omega = 0 . \qquad (2)$$

The form ω is determined up to its multiplication by arbitrary nonvanishing real function on X, and any such a form defines a 1-codimensional foliation on X.

We shall say that an orientable 1-codimensional foliation F on a smooth connected manifold X^4 without boundary is a space-time foliation relative to a gravitational field g, if a generating form ω of F is a tetrad form $h^\perp$ of a given field g.

The following theorem is a corollary of theorems 1,2.

Theorem 3. Any 1-codimensional foliation on a manifold X^4 is a space-time corresponding to a certain gravitational field on X^4. Inversely, a gravitational field g whose tetrad form h satisfies the integrability condition (2) defines a space-time foliation generated by the form $\omega = h^{\perp}$.

A gravitational field g on a manifold X^4 admits a family of space-time foliations whose generating forms differ from each other in gauge Lorentz and dilation transformations. To choose a certain space-time foliation means in a sense to choose a certain reference frame and also to fix a Riemannian metric g^R on X^4 such that observers connected with different foliation frames perceive the same space-time as different Riemannian spaces. The well-known relativistic change of sizes of moving bodies exemplifies this phenomenon.

We shall say that a space-time foliation F is causal if nobody moving along any transversal curve to F intersects any slice of F more than once. It means that slices of a causal foliation are linearly ordered, and for verifying the causality to travel along one or two transversals is sufficient. The effective description of causal foliations can be found as follows:

Theorem 4. A space-time foliation F is causal only if it represents a foliation of level surfaces of some smooth function f possessing the nonvanishing differential $\mathrm{d}f$ on X^4.

A tetrad form $h^{\perp}$ generating a causal foliation takes the form

$$h^{\perp} = N \mathrm{d} f , \qquad (3)$$

where N denotes a nonvanishing real function on X^4.

Remark also that the integrability of a (3 + 1) decomposition may be interpreted as the condition of sui generis local causality which claims that there exist a neighborhood of any point $x \in X$ where the tetrad form $h^{\perp}$ of a given (3 + 1) decomposition can take the form (3). For comparison, Hawking's strong causality [2] is local in space and global time.

There are gravitational fields not admitting space-time foliation or causal foliation. Such fields being regular themselves define singular space-time structures on a manifold X^4.

3. Space-time singularities. Now some more remarks about a space-time manifold X^4. Until now we have failed to discuss singularities destroying the structure of a connected smooth manifold without boundary on X^4. Such a manifold is metrizable, and its diffeomorphisms must be uniformly continuous. Also it seems reasonable to think that X^4 is complete as a metric space.

Thus we shall restrict our discussion to singularities retaining the locally euclidean topological structure of a space-time at singularity points.

We shall say that a gravitational field g on a manifold X^4 possesses a singularity, if there is no g compatible space-time foliation on X^4.

It makes sense to distinguish two types of gravitation singularities. The first type includes gravitational fields admitting regular (3 + 1) decompositions, but no causal space-time foliations. Such singularities, destroying only the causality of a space-time, need not possess singular values of gravitational quantities. The second type of gravitation singularities includes gravitational fields not admitting (3 + 1) decompositions and regular space-time foliations.

Singular foliations are defined as closing the class of foliations under the operation of the foliation induction when a smooth map $f: Y \to X$ fails to be transversal to a foliation F on X. However in this case the induced construction f^*F makes a certain geometric sense too, and may be interpretated as a singular foliation [5].

The following theorem describes singularities in causal space-time foliations:

Theorem 5. Singular foliations closing the class of causal space-time foliations represent foliations of level surfaces of real smooth functions f on a space-time.

Singularities in such foliations are identified with critical points of functions f, i.e. with points where $\mathrm{d}f = 0$. It may look promising for describing correspondent gravitation singularities because such points are well studied and classified.

In a general case, however, a 1-form ω going to zero at some points of x^4 fails to define any singular foliation (Haefliger structure). But such a form may be considered as defining a singular (3 + 1) decomposition on X^4, and a singular gravitational field as follows.

Let g^R be a Riemannian metric on X^4 and let S denote a closed set of points where a given form ω comes to zero. Then the forms g^R and ω being restricted on the manifold (X^4-S) define a gravitational field g on $(X^4 - S)$ in virtue of eq. (1), which in a general case cannot be expanded on the whole manifold X^4. In so far as forms ω defining the same singular gravitational field g on X^4 differ from each other in gauge Lorentz and dilatation transformations the following theorem is true:

Theorem 6. If a gravitational field g admits a singular (3 + 1) decomposition generated by a form ω, any other g-compatible (3 + 1) decomposition or a space-time foliation generated by some form ω' possesses a singularity $\omega' = 0$ at the same point where ω does, and if this point is isolated, this singularity is characterized by the same index of ω as the index of ω'.

However there are situations when a form ω going to zero at a point x, X can define a regular (3 + 1) decomposition or foliation on X^4. It takes place when the form $\omega/|\omega|$ existing on the manifold $(X^4 - x)$ can be expanded regularly on the whole manifold X^4. In this case the (3 + 1) decomposition and a gravitational field defined by on $(X^4 - x)$ can be expanded regularly on X^4 too, and thereby a singularity indicated by the vanishing of such a form ω turns out to be fictitious.

For to discern such a fictional singularity one can use the additional criterium that the divergence of the vector field $h_\perp = h_0^\mu \partial_\mu$ must enlarge infinitely about a true singularity. For instance, it takes place when the field $h_\perp$ possesses a nontrivial index at an isolated singular point.

Let a form ω define a singular foliation F on a manifold X^4. Then in virtue of the known theorems the divergence div $h_\perp$ can be connected with values of the second fundamental form (the exterior curvature) on slices of F as follows:

$$K = -\gamma^{ab}\Gamma_{ab} = -\operatorname{div} h_\perp - \tfrac{1}{2}\,\partial_\perp \ln|g^R|\,,$$

where Γ is a gravitational connection, but $a, b = 1, 2, 3$ denote the basis indices on slices. It shows that the exterior curvature of foliation slices enlarges infinitely about a foliation singularity, and this fact can play the role of the criterium of a foliation singularity too.

The last criterium seems especially preferable because the Einstein equations can be rewritten as the evolution equations for components of the exterior curvature K [6]. For instance, the evolution equation for the scalar exterior curvature reads

$$\partial_\perp K = K_n^m K_m^n + \tfrac{1}{2}T - N^{-1}\nabla^2 N\,, \tag{4}$$

where T is the Euclidean spur of the energy-momentum tensor of matter sources, but N is a time scale function which can be set equal to a constant because of our thesis about regularity of a space-time as a Riemannian space. Let $T \geqslant 0$, which is natural for most matter sources. Then, taking into account the algebraic relation $K_n^m K_m^n \geqslant K^2/3$, one can rewrite the evolution equation (4) in the form:

$$\partial K/\partial s \leqslant \tfrac{1}{3}K^2 \tag{5}$$

where s denotes a parameter along an integral curve of the field $h_\perp$. It is easy to prove that any $K \neq 0$ obeying eq. (5) becomes infinite at some finite value of a parameter s. This means that space-time singularities are the inevitable attribute of the most physically significant solutions of the Einstein equations.

In a sense this result correlates with the known theorems by Hawking and Penrose, because if a gravitational field possesses a singularity from the foliation point of view, it possesses also a singularity expressed by the noncompleteness of some curve, namely, an integral curve of the field $h_\perp$.

References

[1] D. Ivanenko, in: Einstein centenary (Johnson, New York, 1979) p. 295.
[2] S. Hawking and G. Ellis, The large scale structure of a space-time (Cambridge Univ. Press, Cambridge, 1973).
[3] G. Whiston, Gen. Rel. Grav. 5 (1974) 517, 525.
[4] H. Lawson, Bull. Am. Math. Soc. 80 (1974) 369.
[5] D. Fuks, in: Modern problems of mathematics, VINITY, Vol. 10 (1978) p. 179 (in Russian).
[6] J. Isenberg and J. Nester, in: General relativity and gravitation (Plenum Press, New York, 1980) p. 23.