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Subcritical Catalytic Branching Random Walk with Finite or Infinite Variance of Offspring Number

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Abstract—Subcritical catalytic branching random walk on the d-dimensional integer lattice is studied. New theorems concerning the asymptotic behavior of distributions of local particle numbers are established. To prove the results, different approaches are used, including the connection between fractional moments of random variables and fractional derivatives of their Laplace transforms. In the previous papers on this subject only supercritical and critical regimes were investigated under the assumptions of finiteness of the first moment of offspring number and finiteness of the variance of offspring number, respectively. In the present paper, for the offspring number in the subcritical regime, the finiteness of the moment of order $1+\delta$ is required where δ is some positive number.

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1. INTRODUCTION AND MAIN RESULTS

The paper is devoted to the investigation of a *catalytic* branching random walk (CBRW) on the integer lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$. This modification of the branching random walk (BRW) with a single source of branching was proposed by V.A. Vatutin, V.A. Topchii and E.B. Yarovaya in [1], and it covers the *symmetric* BRW studied earlier (see, e.g., [2]).

Recall the description of a CBRW on $\mathbb{Z}^d$. Let at the initial time $t = 0$ there be a single particle located at a point $\mathbf{x} \in \mathbb{Z}^d$ on the lattice. If $\mathbf{x} \neq \mathbf{0}$, then the particle performs the continuous-time random walk until it hits the origin. We assume that the random walk outside the origin is specified by an infinitesimal matrix $A = (a(\mathbf{u},\mathbf{v}))_{\mathbf{u},\mathbf{v}\in\mathbb{Z}^d}$ and is symmetric, homogeneous, with finite variance of jumps and irreducible (i.e., the particle passes from an arbitrary point $\mathbf{u} \in \mathbb{Z}^d$ to any point $\mathbf{v} \in \mathbb{Z}^d$ in finite time with positive probability). This means that

$$a(\mathbf{u},\mathbf{v}) = a(\mathbf{v},\mathbf{u}), \qquad a(\mathbf{u},\mathbf{v}) = a(\mathbf{0},\mathbf{v}-\mathbf{u}) := a(\mathbf{v}-\mathbf{u}), \qquad \mathbf{u},\mathbf{v} \in \mathbb{Z}^d,$$

$$\sum_{\mathbf{v}\in\mathbb{Z}^d} a(\mathbf{v}) = 0, \quad \text{where} \quad a(\mathbf{0}) < 0 \quad \text{and} \quad a(\mathbf{v}) \ge 0 \quad \text{for} \quad \mathbf{v} \neq \mathbf{0}, \qquad \sum_{\mathbf{v}\in\mathbb{Z}^d} \|\mathbf{v}\|^2 a(\mathbf{v}) < \infty$$

(here $\|\cdot\|$ denotes an arbitrary norm in $\mathbb{R}^d$). If $\mathbf{x} = \mathbf{0}$ or the particle has just hit the origin, then it spends there random time distributed according to the exponential law with parameter 1. Afterwards it either dies with probability α, producing a random offspring number ξ just before the death, or leaves the origin with probability $1-\alpha$, so that the intensity of transition from the origin to a point $\mathbf{v} \neq \mathbf{0}$ equals $-(1-\alpha)a(\mathbf{v})/a(\mathbf{0})$. At the origin the branching of the particle is specified by a probability generating function

$$f(s) := \mathsf{E}\, s^{\xi} = \sum_{k=0}^{\infty} f_k s^k, \qquad s \in [0,1].$$

At the birth moment the newborn particles are located at the origin. They evolve in accordance with the scheme described above independently of each other and of the history of their parents.

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The natural objects of study in a CBRW are total and local particle numbers. Denote by $\mu(t)$ the number of particles existing on the lattice $\mathbb{Z}^d$ at time $t \ge 0$. In a similar way we define local numbers $\mu(t;\mathbf{y})$ as quantities of particles located at separate points $\mathbf{y} \in \mathbb{Z}^d$ at time t.

It was established in [3] that exponential growth (as $t \to \infty$) of both total and local particle numbers in a CBRW on $\mathbb{Z}^d$ holds if and only if $\mathsf{E}\,\xi > 1 + h_d\alpha^{-1}(1-\alpha)$. Here h_d is probability of the event that a particle which has left the origin will never come back. Thus, the value $\mathsf{E}\,\xi = 1 + h_d\alpha^{-1}(1-\alpha)$ is critical, and, similarly to many types of branching processes (see, e.g., [4]), a CBRW is classified as *supercritical*, *critical* or *subcritical* if the mean offspring number $\mathsf{E}\,\xi$ is greater than, equal to or less than $1 + h_d\alpha^{-1}(1-\alpha)$, respectively. In view of the random walk properties such as recurrence or transience, $h_1 = h_2 = 0$, whereas $0 < h_d < 1$ for $d \ge 3$. Recall that $\mathsf{E}\,\xi = f'(1)$ and for the classical Galton–Watson branching processes the critical value of $f'(1)$ is equal to 1.

Critical and subcritical CBRWs on $\mathbb{Z}^d$ are of special interest since in these cases there arise diverse kinds of limit (in time) behavior of the total and local particle numbers depending on the dimension d. For instance, for $d = 1$ and $d = 2$ the probability $\mathsf{P}_{\mathbf{x}}(\mu(t) > 0)$ of non-extinction of the population tends to zero as time grows (the index $\mathbf{x} \in \mathbb{Z}^d$ denotes the starting point of a CBRW), whereas for $d \ge 3$ this probability has a positive limit. Such an effect is due to the positive probability of existence, on the lattice $\mathbb{Z}^d$ for $d \ge 3$, of particles which are ever alive and never hit the source of branching. The total particle number in the critical CBRW on the integer line $\mathbb{Z}$, i.e., for $d = 1$, was studied in the fundamental paper [1]. In [5–8] the investigation was continued for critical and subcritical CBRWs on $\mathbb{Z}^d$ for any $d \in \mathbb{N}$.

The analysis of local particle numbers is much more hard. For critical CBRWs on $\mathbb{Z}^d$ the limit distributions of local particle numbers were studied in a series of papers by V.A. Vatutin, V.A. Topchii, Y. Hu, E.B. Yarovaya and the author (see, e.g., [9–12]). It should be noted that in all these papers the finiteness of the second moment $\mathsf{E}\,\xi^2$ of the offspring number was assumed. In the present paper the limit distributions of local particle numbers in *subcritical* CBRWs on $\mathbb{Z}^d$ are studied for the first time, and the corresponding conditions on the moments of the offspring number ξ are less restrictive than those in the critical case. Namely, we find the asymptotic behavior of the mean local numbers $m(t;\mathbf{x},\mathbf{y}) := \mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})$ for fixed $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$ and $t \to \infty$. Under the assumption that $\mathsf{E}\,\xi^{1+\delta} < \infty$ for some $\delta \in (0,1]$, a similar problem is solved for the non-extinction probability $q(t;\mathbf{x},\mathbf{y}) := \mathsf{P}_{\mathbf{x}}(\mu(t;\mathbf{y}) > 0)$ of local particle numbers. Moreover, under the same restriction on the moment $\mathsf{E}\,\xi^{1+\delta}$, conditional limit theorems are proved for $\mu(t;\mathbf{y})$ as $t \to \infty$.

To formulate the main results, let us introduce additional notation. Let

$$q(s,t;\mathbf{x},\mathbf{y}) := 1 - \mathsf{E}_{\mathbf{x}}\,s^{\mu(t;\mathbf{y})} \qquad \text{and} \qquad J(s;\mathbf{y}) := \int_0^\infty \Phi(q(s,t;\mathbf{0},\mathbf{y}))\,dt$$

where $\Phi(s) := \alpha(f(1-s) - 1 + f'(1)s)$ and $s \in [0,1]$, $t \ge 0$, $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$. Consider the *transition probabilities* $p(t;\mathbf{x},\mathbf{y})$, $t \ge 0$, $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, of the random walk generated by the matrix A. According to [9; 13, Theorem 2.1.1], for fixed $\mathbf{x}$ and $\mathbf{y}$, as $t \to \infty$, the following asymptotic relations hold true:

$$p(t;\mathbf{x},\mathbf{y}) \sim \frac{\gamma_d}{t^{d/2}}, \qquad p'(t;\mathbf{0},\mathbf{0}) \sim -\frac{d\gamma_d}{2t^{d/2+1}}, \qquad p(t;\mathbf{0},\mathbf{0}) - p(t;\mathbf{x},\mathbf{y}) \sim \frac{\widetilde{\gamma}_d(\mathbf{y}-\mathbf{x})}{t^{d/2+1}} \tag{1.1}$$

where $\gamma_d := \big((2\pi)^d|\det\phi''_{\theta\theta}(\mathbf{0})|\big)^{-1/2}$, $\phi(\theta) := \sum_{\mathbf{z}\in\mathbb{Z}^d} a(\mathbf{0},\mathbf{z})\cos(\mathbf{z},\theta)$, $\theta \in [-\pi,\pi]^d$,

$$\phi''_{\theta\theta}(\mathbf{0}) := \left(\frac{\partial^2\phi(\theta)}{\partial\theta_i\partial\theta_j}\bigg|_{\theta=\mathbf{0}}\right)_{i,j\in\{1,\dots,d\}}, \qquad \widetilde{\gamma}_d(\mathbf{z}) := \frac{1}{2(2\pi)^d}\int_{\mathbb{R}^d}(\mathbf{v},\mathbf{z})^2 e^{(\phi''_{\theta\theta}(\mathbf{0})\mathbf{v},\mathbf{v})/2}\,d\mathbf{v}, \quad \mathbf{z} \in \mathbb{Z}^d,$$

and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^d$. Set $G_\lambda(\mathbf{x},\mathbf{y}) := \int_0^\infty e^{-\lambda t}p(t;\mathbf{x},\mathbf{y})\,dt$, $\lambda \ge 0$, $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$; i.e., $G_\lambda(\mathbf{x},\mathbf{y})$ is the Laplace transform of the transition probability $p(\cdot;\mathbf{x},\mathbf{y})$. By virtue of (1.1) for $d \geq 3$ the *Green's function* $G_0(\mathbf{x},\mathbf{y})$ is finite for all $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, which means the transience of the random walk. However, for $d = 1$ or $d = 2$, we deal with recurrent random walk since $\lim_{\lambda\to 0+} G_\lambda(\mathbf{x},\mathbf{y}) = \infty$. Due to the same formula (1.1) the function $\lim_{\lambda\to 0+}(G_\lambda(\mathbf{0},\mathbf{0}) - G_\lambda(\mathbf{x},\mathbf{y}))$ is finite for all $d \in \mathbb{N}$ and $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$. So we may introduce the function

$$\rho_d(\mathbf{z}) := \begin{cases} (1-\alpha)a^{-1} - \beta \displaystyle\int_0^\infty \big(p(t;\mathbf{0},\mathbf{0}) - p(t;\mathbf{0},\mathbf{z})\big)\,dt & \text{if } \mathbf{z} \neq \mathbf{0}, \\ 1 & \text{if } \mathbf{z} = \mathbf{0} \end{cases}$$

where for the sake of convenience we set $a := -a(\mathbf{0})$ and $\beta := \alpha(f'(1) - 1)$. As shown in [9], $h_d = (aG_0(\mathbf{0},\mathbf{0}))^{-1}$. This implies that in the subcritical regime $\beta < (1-\alpha)(aG_0(\mathbf{0},\mathbf{0}))^{-1}$ and, therefore, $\rho_d(\cdot)$ is a strictly positive function for all $d \in \mathbb{N}$. Note also that in accordance with [13, (2.1.15)] the inequality $p(t;\mathbf{0},\mathbf{0}) \geq p(t;\mathbf{x},\mathbf{y})$, $t \geq 0$, is valid and the function $p(\cdot;\mathbf{x},\mathbf{y})$ is symmetric and homogeneous in $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$.

For $d = 1$ and $\mathbf{x},\mathbf{y} \in \mathbb{Z}$, we define

$$C_1(\mathbf{0},\mathbf{y}) := \frac{1-\alpha}{2a\gamma_1\pi\beta^2}\rho_1(\mathbf{y}),$$

$$C_1(\mathbf{x},\mathbf{y}) := \frac{1}{2\gamma_1\pi\beta^2}\rho_1(\mathbf{x})\rho_1(\mathbf{y}) + \widetilde{\gamma}_1(\mathbf{x}) + \widetilde{\gamma}_1(\mathbf{y}) - \widetilde{\gamma}_1(\mathbf{y}-\mathbf{x}), \qquad \mathbf{x} \neq \mathbf{0}.$$

In a similar way, for $d = 2$ and $\mathbf{x},\mathbf{y} \in \mathbb{Z}^2$ we set

$$C_2(\mathbf{0},\mathbf{y}) := \frac{1-\alpha}{a\gamma_2\beta^2}\rho_2(\mathbf{y}), \qquad C_2(\mathbf{x},\mathbf{y}) := \frac{1}{\gamma_2\beta^2}\rho_2(\mathbf{x})\rho_2(\mathbf{y}), \quad \mathbf{x} \neq \mathbf{0}.$$

Finally, for $d \geq 3$, we specify functions $C_d(\mathbf{x},\mathbf{y})$, $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, as follows:

$$C_d(\mathbf{0},\mathbf{y}) := \frac{(1-\alpha)a\gamma_d}{(1-\alpha-a\beta G_0(\mathbf{0},\mathbf{0}))^2}\rho_d(\mathbf{y}),$$

$$C_d(\mathbf{x},\mathbf{y}) := \frac{a^2\gamma_d}{(1-\alpha-a\beta G_0(\mathbf{0},\mathbf{0}))^2}\rho_d(\mathbf{x})\rho_d(\mathbf{y}), \qquad \mathbf{x} \neq \mathbf{0}.$$

Theorem 1. *Let* $\mathsf{E}\,\xi < 1 + h_d\alpha^{-1}(1-\alpha)$. *Then, as* $t \to \infty$, *for fixed* $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, *the following relations are true*:

$$m(t;\mathbf{x},\mathbf{y}) \sim \frac{C_1(\mathbf{x},\mathbf{y})}{t^{3/2}}, \qquad d = 1,$$

$$m(t;\mathbf{x},\mathbf{y}) \sim \frac{C_2(\mathbf{x},\mathbf{y})}{t\ln^2 t}, \qquad d = 2,$$

$$m(t;\mathbf{x},\mathbf{y}) \sim \frac{C_d(\mathbf{x},\mathbf{y})}{t^{d/2}}, \qquad d \geq 3,$$

and the functions $C_d(\cdot,\cdot)$, $d \in \mathbb{N}$, *introduced above are strictly positive.*

The statement of Theorem 1 generalizes the corresponding results of [13, Ch. 5] concerning the asymptotic behavior of the first moments of local particle numbers in subcritical *symmetric* BRWs on $\mathbb{Z}^d$. Here, in contrast to [13], we use approaches such as the representation of complex-valued measures in terms of Banach algebras (see [9]) and Tauberian theorems for derivatives of the Laplace transform (see [14, Sect. 7.3]).

Theorem 2. *If* $\mathsf{E}\,\xi < 1 + h_d\alpha^{-1}(1-\alpha)$ *and there exists* $\mathsf{E}\,\xi^{1+\delta}$ *for some* $\delta \in (0,1]$, *then, for fixed* $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$, *as* $t \to \infty$, *one has*

$$q(t;\mathbf{x},\mathbf{y}) \sim \frac{C_1(\mathbf{x},\mathbf{y}) - C_1(\mathbf{x},\mathbf{0})J(0;\mathbf{y})}{t^{3/2}}, \qquad d = 1,$$

$$q(t;\mathbf{x},\mathbf{y}) \sim \frac{C_2(\mathbf{x},\mathbf{0})(\rho_2(\mathbf{y}) - J(0;\mathbf{y}))}{t\ln^2 t}, \qquad d = 2,$$

$$q(t;\mathbf{x},\mathbf{y}) \sim \frac{C_d(\mathbf{x},\mathbf{0})(\rho_d(\mathbf{y}) - J(0;\mathbf{y}))}{t^{d/2}}, \qquad d \ge 3,$$

where $C_1(\mathbf{x},\mathbf{y}) - C_1(\mathbf{x},\mathbf{0})J(0;\mathbf{y}) > 0$ *for all* $\mathbf{x},\mathbf{y} \in \mathbb{Z}$ *and* $J(0;\mathbf{y}) < \rho_d(\mathbf{y})$ *for* $d \ge 2$ *and* $\mathbf{y} \in \mathbb{Z}^d$.

To prove Theorem 2, we apply the Hölder inequality combined with results on connection between fractional moments of random variables and fractional derivatives of their Laplace transforms (see, e.g., [15]). Theorem 3 can be considered as a corollary of Theorem 2.

Theorem 3. *Let* $\mathsf{E}\,\xi < 1 + h_d\alpha^{-1}(1-\alpha)$ *and* $\mathsf{E}\,\xi^{1+\delta}$ *be finite for some* $\delta \in (0,1]$. *Then, for fixed* $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$ *and each* $s \in [0,1]$, *the following equalities are valid*:

$$\lim_{t\to\infty} \mathsf{E}_{\mathbf{x}}\big(s^{\mu(t;\mathbf{y})} \,\big|\, \mu(t;\mathbf{y}) > 0\big) = \frac{sC_1(\mathbf{x},\mathbf{y}) - C_1(\mathbf{x},\mathbf{0})(J(0;\mathbf{y}) - J(s;\mathbf{y}))}{C_1(\mathbf{x},\mathbf{y}) - C_1(\mathbf{x},\mathbf{0})J(0;\mathbf{y})}, \qquad d = 1,$$

$$\lim_{t\to\infty} \mathsf{E}_{\mathbf{x}}\big(s^{\mu(t;\mathbf{y})} \,\big|\, \mu(t;\mathbf{y}) > 0\big) = \frac{s\rho_d(\mathbf{y}) - (J(0;\mathbf{y}) - J(s;\mathbf{y}))}{\rho_d(\mathbf{y}) - J(0;\mathbf{y})}, \qquad d \ge 2.$$

Remark. Comparing the formulations of Theorems 1–3 and the results on local particle numbers in *critical* CBRWs on $\mathbb{Z}^2$ (see, e.g., [16, 17]) suggests the following conclusion. The local particle numbers in critical CBRWs exhibit “subcritical” behavior in the case of random walk on the integer plane. Moreover, the scheme of proofs of Theorems 1–3 can easily be carried over to the case of critical CBRWs on $\mathbb{Z}^2$. Consequently, we may state that the results of [16, 17] concerning the local numbers in critical CBRWs on $\mathbb{Z}^2$ are valid under less restrictive conditions on the moments of the offspring number. Namely, it is sufficient to require the finiteness of $\mathsf{E}\,\xi^{1+\delta}$ for some $\delta \in (0,1]$ instead of the condition $\mathsf{E}\,\xi^2 < \infty$.

Concluding the first part of the paper, let us note the close relation between the CBRW and superprocesses, namely, the catalytic super-Brownian motion with a single point of catalysis (see, e.g., [18] and references therein). It is of interest that in view of [1] the CBRW may be considered as a queueing system with a random number of independent servers. This opens up wide opportunities for applications of the established results.

2. PROOF OF THEOREM 1

Similarly to the proof of Theorem 1 in [17], we use backward and forward integral equations for the family of functions $\{m(\cdot;\mathbf{x},\mathbf{y})\}_{\mathbf{x},\mathbf{y}\in\mathbb{Z}^d}$. These equations coincide with equations (8) and (9) in [17], which are derived for the mean local particle numbers in the *critical* CBRW on $\mathbb{Z}^d$, upon the replacement of the critical value $\beta_{\mathrm{c}} = (1-\alpha)a^{-1}G_0^{-1}(\mathbf{0},\mathbf{0})$ by β; that is,

$$m(t;\mathbf{x},\mathbf{y}) = p(t;\mathbf{x},\mathbf{y}) + \left(1 - \frac{a}{1-\alpha}\right)\int_0^t p(t-u;\mathbf{x},\mathbf{0})m'(u;\mathbf{0},\mathbf{y})\,du$$
$$+ \frac{a\beta}{1-\alpha}\int_0^t p(t-u;\mathbf{x},\mathbf{0})m(u;\mathbf{0},\mathbf{y})\,du, \tag{2.1}$$

$$m(t;\mathbf{x},\mathbf{y}) = p(t;\mathbf{x},\mathbf{y}) + \left(\frac{1-\alpha}{a} - 1\right)\int\limits_0^t m(u;\mathbf{x},\mathbf{0})p'(t-u;\mathbf{0},\mathbf{y})\,du$$
$$+\beta\int\limits_0^t m(u;\mathbf{x},\mathbf{0})p(t-u;\mathbf{0},\mathbf{y})\,du. \tag{2.2}$$

Relations (2.1) and (2.2), as well as equations (8) and (9) in [17], are obtained by means of the variation of constants formula applied to the backward and forward differential equations (5) and (6) in [17] established in the Banach space $l_\infty(\mathbb{Z}^d)$.

We also need the following auxiliary statement that can be proved in the same way as Lemma 3.3.5 in [13] and Lemma 1 in [17].

Lemma 1. *For each $\mathbf{y}\in\mathbb{Z}^d$ the function $m(t;\mathbf{y},\mathbf{y})$ does not increase in the variable t.*

Now we turn directly to the proof of Theorem 1. First consider the case $\mathbf{x}=\mathbf{y}=\mathbf{0}$. We apply the Laplace transform to both sides of equality (2.2) and use the obtained relation to express the function $\widehat{m}(\lambda) := \int_0^\infty e^{-\lambda t}m(t;\mathbf{0},\mathbf{0})\,dt$, $\lambda\ge 0$. Then

$$\widehat{m}(\lambda) = \frac{G_\lambda(\mathbf{0},\mathbf{0})}{1-((1-\alpha)a^{-1}-1)\int_0^\infty e^{-\lambda t}p'(t;\mathbf{0},\mathbf{0})\,dt - \beta G_\lambda(\mathbf{0},\mathbf{0})}. \tag{2.3}$$

Differentiating each side of the last relation with respect to λ and taking into account the identity $\int_0^\infty e^{-\lambda t}p'(t;\mathbf{0},\mathbf{0})\,dt = \lambda G_\lambda(\mathbf{0},\mathbf{0}) - 1$, we have

$$\widehat{m}'(\lambda) = \frac{(1-\alpha)a^{-1}G'_\lambda(\mathbf{0},\mathbf{0}) + ((1-\alpha)a^{-1}-1)G^2_\lambda(\mathbf{0},\mathbf{0})}{\big((1-\alpha)a^{-1} - ((1-\alpha)a^{-1}-1)\,\lambda G_\lambda(\mathbf{0},\mathbf{0}) - \beta G_\lambda(\mathbf{0},\mathbf{0})\big)^2}. \tag{2.4}$$

According to Tauberian Theorem 2 in [19, Ch. XIII, § 5] along with [14, Corollary 43], relation (1.1) implies

$$G_\lambda(\mathbf{0},\mathbf{0}) \sim \frac{\gamma_1\sqrt{\pi}}{\sqrt{\lambda}}, \qquad G'_\lambda(\mathbf{0},\mathbf{0}) \sim -\frac{\gamma_1\sqrt{\pi}}{2\lambda^{3/2}}, \qquad d=1,$$
$$G_\lambda(\mathbf{0},\mathbf{0}) \sim \gamma_2\ln\frac{1}{\lambda}, \qquad G'_\lambda(\mathbf{0},\mathbf{0}) \sim -\frac{\gamma_2}{\lambda}, \qquad d=2,$$

as $\lambda\to 0+$. Substituting these asymptotic equalities into (2.4) for $d=1$ and $d=2$, respectively, we find

$$\widehat{m}'(\lambda) \sim -\frac{1-\alpha}{2a\gamma_1\sqrt{\pi}\beta^2\sqrt{\lambda}}, \quad d=1, \qquad \widehat{m}'(\lambda) \sim -\frac{1-\alpha}{a\gamma_2\beta^2\lambda\ln^2\lambda}, \quad d=2,$$

as $\lambda\to 0+$. Applying [14, Corollary 43] to the above relations, we come to the assertion of Theorem 1 when $\mathbf{x}=\mathbf{y}=\mathbf{0}$ and $d=1$ or $d=2$.

For $d\ge 3$, we employ another approach, namely, the representation of complex-valued measures in terms of Banach algebras. In view of (1.1) two cumulative distribution functions with the Laplace transforms $G_\lambda(\mathbf{0},\mathbf{0})$ and $\int_0^\infty e^{-\lambda t}p'(t;\mathbf{0},\mathbf{0})\,dt$, respectively, possess tails equivalent to constants (the second constant being zero) multiplied by the same function $t^{1-d/2}$. Hence, due to [9, Lemma 6] and (2.3), the following asymptotic equality holds:

$$\int\limits_t^\infty m(u;\mathbf{0},\mathbf{0})\,du \sim \frac{2(1-\alpha)a\gamma_d}{(d-2)(1-\alpha-a\beta G_0(\mathbf{0},\mathbf{0}))^2t^{d/2-1}}, \qquad t\to\infty.$$

By Lemma 1 and the classical results on differentiating asymptotic formulae (see, e.g., [20, Ch. 7, Sect. 3]), this implies the assertion of Theorem 1 when $\mathbf{x} = \mathbf{y} = \mathbf{0}$ and $d \geq 3$.

Let us consider the case $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} = \mathbf{0}$. Integration by parts permits us to rewrite the family of equations (2.1) as follows:

$$m(t;\mathbf{x},\mathbf{0}) = \frac{a}{1-\alpha}p(t;\mathbf{x},\mathbf{0}) + \left(1 - \frac{a}{1-\alpha}\right)\int_0^t m(t-u;\mathbf{0},\mathbf{0})p'(u;\mathbf{x},\mathbf{0})\,du + \frac{a\beta}{1-\alpha}\int_0^t p(t-u;\mathbf{x},\mathbf{0})m(u;\mathbf{0},\mathbf{0})\,du, \tag{2.5}$$

$$\frac{a}{1-\alpha}m(t;\mathbf{0},\mathbf{0}) = \frac{a}{1-\alpha}p(t;\mathbf{0},\mathbf{0}) + \left(1 - \frac{a}{1-\alpha}\right)\int_0^t m(t-u;\mathbf{0},\mathbf{0})p'(u;\mathbf{0},\mathbf{0})\,du + \frac{a\beta}{1-\alpha}\int_0^t p(t-u;\mathbf{0},\mathbf{0})m(u;\mathbf{0},\mathbf{0})\,du. \tag{2.6}$$

Subtracting equation (2.5) from (2.6), we come to

$$\frac{a}{1-\alpha}m(t;\mathbf{0},\mathbf{0}) - m(t;\mathbf{x},\mathbf{0}) = \frac{a}{1-\alpha}\big(p(t;\mathbf{0},\mathbf{0}) - p(t;\mathbf{x},\mathbf{0})\big) + \left(1 - \frac{a}{1-\alpha}\right)\int_0^t m(t-u;\mathbf{0},\mathbf{0})\big(p'(u;\mathbf{0},\mathbf{0}) - p'(u;\mathbf{x},\mathbf{0})\big)\,du + \frac{a\beta}{1-\alpha}\int_0^t m(t-u;\mathbf{0},\mathbf{0})\big(p(u;\mathbf{0},\mathbf{0}) - p(u;\mathbf{x},\mathbf{0})\big)\,du. \tag{2.7}$$

Employing the results on differentiating of asymptotic formulae once again (see, e.g., [20, Ch. 7, Sect. 3]), as well as relation (1.1) and the inequality $p''(t;\mathbf{0},\mathbf{0}) \geq p''(t;\mathbf{x},\mathbf{0})$, $t \geq 0$, implied by [13, (2.1.15)], we find that

$$p'(t;\mathbf{0},\mathbf{0}) - p'(t;\mathbf{x},\mathbf{0}) \sim -\frac{(d+2)\widetilde{\gamma}_d(\mathbf{x})}{2t^{d/2+2}}, \qquad t \to \infty. \tag{2.8}$$

Therefore, on account of the "lemma on convolutions" [13, Lemma 5.1.2] along with formula (1.1) and the proved part of Theorem 1, we deduce from (2.7) that

$$m(t;\mathbf{x},\mathbf{0}) \sim m(t;\mathbf{0},\mathbf{0})\left(1 - \frac{a\beta}{1-\alpha}\int_0^\infty \big(p(u;\mathbf{0},\mathbf{0}) - p(u;\mathbf{x},\mathbf{0})\big)\,du\right) - \frac{a\widetilde{\gamma}_d(\mathbf{x})}{(1-\alpha)t^{d/2+1}}\left(1 + \beta\int_0^\infty m(u;\mathbf{0},\mathbf{0})\,du\right), \qquad t \to \infty. \tag{2.9}$$

However, by virtue of the already established part of Theorem 1 we see that $\int_0^\infty m(u;\mathbf{0},\mathbf{0})\,du < \infty$ for all $d \in \mathbb{N}$ and, moreover, according to (2.3) for $\lambda = 0$ one gets

$$\int_0^\infty m(u;\mathbf{0},\mathbf{0})\,du = -\beta^{-1} \qquad \text{if} \quad d = 1 \quad \text{or} \quad d = 2. \tag{2.10}$$

Hence we conclude that only the first term on the right-hand side of (2.9) contributes to the asymptotic behavior of $m(t;\mathbf{x},\mathbf{0})$. So, the assertion of Theorem 1 for $\mathbf{x}\neq\mathbf{0}$ and $\mathbf{y}=\mathbf{0}$ is entailed by relation (2.9) and the proved part of this theorem when $\mathbf{x}=\mathbf{y}=\mathbf{0}$.

Let now $\mathbf{x}\in\mathbb{Z}^d$ and $\mathbf{y}\neq\mathbf{0}$. In view of (2.2) one has

$$m(t;\mathbf{x},\mathbf{0})-m(t;\mathbf{x},\mathbf{y})=p(t;\mathbf{x},\mathbf{0})-p(t;\mathbf{x},\mathbf{y})\\ +\left(\frac{1-\alpha}{a}-1\right)\int_0^t m(t-u;\mathbf{x},\mathbf{0})\big(p'(u;\mathbf{0},\mathbf{0})-p'(u;\mathbf{0},\mathbf{y})\big)\,du\\ +\beta\int_0^t m(t-u;\mathbf{x},\mathbf{0})\big(p(u;\mathbf{0},\mathbf{0})-p(u;\mathbf{0},\mathbf{y})\big)\,du.$$

Then, taking into account formulae (1.1) and (2.8) along with [13, Lemma 5.1.2] and the proved part of Theorem 1, we find that

$$m(t;\mathbf{x},\mathbf{y})\sim m(t;\mathbf{x},\mathbf{0})\left(\frac{1-\alpha}{a}-\beta\int_0^\infty\big(p(u;\mathbf{0},\mathbf{0})-p(u;\mathbf{0},\mathbf{y})\big)\,du\right)\\ +\frac{\widetilde{\gamma}_d(\mathbf{x})-\widetilde{\gamma}_d(\mathbf{y}-\mathbf{x})}{t^{d/2+1}}-\frac{\beta\widetilde{\gamma}_d(\mathbf{y})}{t^{d/2+1}}\int_0^\infty m(u;\mathbf{x},\mathbf{0})\,du,\qquad t\to\infty. \tag{2.11}$$

Applying once again the proved part of Theorem 1, we come to the inequality $\int_0^\infty m(u;\mathbf{x},\mathbf{0})\,du<\infty$. Moreover, similarly to the verification of equality (2.10) we check that $\int_0^\infty m(u;\mathbf{x},\mathbf{0})\,du=-\beta^{-1}$ for $d=1$ or $d=2$. Thus, the statement of Theorem 1 for $\mathbf{x}\in\mathbb{Z}^d$ and $\mathbf{y}\neq\mathbf{0}$ is implied by relation (2.11) and the proved part of Theorem 1 for $\mathbf{x}\in\mathbb{Z}^d$ and $\mathbf{y}=\mathbf{0}$.

To complete the proof of Theorem 1, one only has to make sure that the functions $C_d(\cdot,\cdot)$, $d\in\mathbb{N}$, are strictly positive. It is easy except for the case $d=1$ when $\mathbf{x}\neq\mathbf{0}$ and $\mathbf{y}\neq\mathbf{0}$. Let us show that in this case $C_1(\mathbf{x},\mathbf{y})>0$ as well. For this purpose we turn to a function $H_{\mathbf{x},\mathbf{0}}(t)$, $t\ge 0$, which is the cumulative distribution function of time from starting at the point $\mathbf{x}$ to hitting the point $\mathbf{0}$ for the first time in the framework of the random walk generated by the matrix A. Clearly,

$$p(t;\mathbf{x},\mathbf{y})-\int_0^t p(t-u;\mathbf{0},\mathbf{y})\,dH_{\mathbf{x},\mathbf{0}}(u)\ge 0,\qquad t\ge 0. \tag{2.12}$$

Then by virtue of the evident identity $p(t;\mathbf{x},\mathbf{0})=\int_0^t p(t-u;\mathbf{0},\mathbf{0})\,dH_{\mathbf{x},\mathbf{0}}(u)$ one has

$$p(t;\mathbf{x},\mathbf{y})-\int_0^t p(t-u;\mathbf{0},\mathbf{y})\,dH_{\mathbf{x},\mathbf{0}}(u)=p(t;\mathbf{x},\mathbf{y})-p(t;\mathbf{x},\mathbf{0})\\ +\int_0^t\big(p(t-u;\mathbf{0},\mathbf{0})-p(t-u;\mathbf{0},\mathbf{y})\big)\,dH_{\mathbf{x},\mathbf{0}}(u).$$

Since with the help of relation (1.1), [13, Lemma 5.1.2] and [11, Lemma 3] one can find the asymptotic behavior of the right-hand side of the latter equality, we establish that, as $t\to\infty$,

$$p(t;\mathbf{x},\mathbf{y})-\int_0^t p(t-u;\mathbf{0},\mathbf{y})\,dH_{\mathbf{x},\mathbf{0}}(u)\sim\frac{\widetilde{\gamma}_1(\mathbf{x})+\widetilde{\gamma}_1(\mathbf{y})-\widetilde{\gamma}_1(\mathbf{y}-\mathbf{x})}{t^{3/2}}+\frac{(1-\alpha-a\rho_1(\mathbf{x}))(1-\alpha-a\rho_1(\mathbf{y}))}{2a^2\gamma_1\pi\beta^2t^{3/2}}.$$

Hence, it follows from (2.12) that

$$\frac{\rho_1(\mathbf{x})\rho_1(\mathbf{y})}{2\gamma_1\pi\beta^2}+\widetilde{\gamma}_1(\mathbf{x})+\widetilde{\gamma}_1(\mathbf{y})-\widetilde{\gamma}_1(\mathbf{y}-\mathbf{x})\geq\frac{1-\alpha}{2a\gamma_1\pi\beta^2}\left(\rho_1(\mathbf{x})+\rho_1(\mathbf{y})-\frac{1-\alpha}{a}\right).$$

However, the left-hand side of this inequality appears to be $C_1(\mathbf{x},\mathbf{y})$, whereas the right-hand side is strictly positive since $\rho_1(\mathbf{z})>(1-\alpha)a^{-1}$ for $\mathbf{z}\neq\mathbf{0}$. In turn, the last inequality is satisfied due to the definition of the function $\rho_1(\cdot)$ and negativeness of β in the subcritical regime for $d=1$.

Therefore, Theorem 1 is completely proved. □

3. PROOF OF THEOREMS 2 AND 3

It is not difficult to verify (following the scheme in [17]) that in subcritical CBRWs on $\mathbb{Z}^d$, as well as in critical CBRWs, for all $\mathbf{x},\mathbf{y}\in\mathbb{Z}^d$, $s\in[0,1]$ and $t\geq 0$, the nonlinear integral equations

$$q(s,t;\mathbf{x},\mathbf{y})=(1-s)m(t;\mathbf{x},\mathbf{y})-\int\limits_0^t m(t-u;\mathbf{x},\mathbf{0})\Phi(q(s,u;\mathbf{0},\mathbf{y}))\,du \tag{3.1}$$

hold true. So, the following upper estimate for $q(s,t;\mathbf{x},\mathbf{y})$ ensues since the functions $m(\cdot;\mathbf{x},\mathbf{0})$ and $\Phi(\cdot)$ are nonnegative:

$$q(s,t;\mathbf{x},\mathbf{y})\leq(1-s)m(t;\mathbf{x},\mathbf{y}). \tag{3.2}$$

Lemma 2. *If* $\mathsf{E}\,\xi<1+h_d\alpha^{-1}(1-\alpha)$ *and* $\mathsf{E}\,\xi^{1+\delta}<\infty$ *for* $\delta\in(0,1]$, *then for some positive constants* K_1 *and* K_2 *the inequalities*

$$\Phi(s)\leq K_1 s^{1+\delta},\qquad s\in[0,1], \tag{3.3}$$

$$\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})^{1+\delta}\leq K_2 m(t;\mathbf{x},\mathbf{y}),\qquad t\geq t_0(\mathbf{x},\mathbf{y}),\quad \mathbf{x},\mathbf{y}\in\mathbb{Z}^d, \tag{3.4}$$

are valid with a certain nonnegative function $t_0(\cdot,\cdot)$.

Proof. First we consider the case $0<\delta<1$. Let us take advantage of the connection between fractional moments of random variables and fractional derivatives of their Laplace transforms. For the first time such results were obtained in [15], where the traditional notion of Riemann–Liouville fractional derivative was used. However, it is more convenient to involve the up-to-date counterpart of these results, namely, Lemma 2.1 in [21], which gives

$$\mathsf{E}\,\xi^{1+\delta}=\frac{\delta(1+\delta)}{\Gamma(1-\delta)}\int\limits_0^\infty\frac{\Phi(1-e^{-v})+\alpha f'(1)(e^{-v}-1+v)}{\alpha v^{2+\delta}}\,dv. \tag{3.5}$$

Since $\mathsf{E}\,\xi^{1+\delta}$ is finite, the latter equality entails $\int_0^1 v^{-2-\delta}\Phi(v)\,dv<\infty$. In view of the inequality $\Phi'(s)\geq 0$, $s\in[0,1]$, integration by parts implies relation (3.3) for $\delta\in(0,1)$.

Now we proceed to verify (3.4) when $0<\delta<1$. Applying [21, Lemma 2.1] once again, we come to the following equality:

$$\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})^{1+\delta}=\frac{\delta(1+\delta)}{\Gamma(1-\delta)}\int\limits_0^\infty\frac{vm(t;\mathbf{x},\mathbf{y})-q(e^{-v},t;\mathbf{x},\mathbf{y})}{v^{2+\delta}}\,dv.$$

Substituting formula (3.1) into the latter relation, we get

$$\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})^{1+\delta} = m(t;\mathbf{x},\mathbf{y})\frac{\delta(1+\delta)}{\Gamma(1-\delta)}\int_0^\infty \frac{e^{-v}-1+v}{v^{2+\delta}}\,dv + \frac{\delta(1+\delta)}{\Gamma(1-\delta)}\int_0^\infty \frac{1}{v^{2+\delta}}\int_0^t m(t-u;\mathbf{x},\mathbf{0})\Phi(q(e^{-v},u;\mathbf{0},\mathbf{y}))\,du\,dv. \tag{3.6}$$

Obviously, the first integral in (3.6) converges. Let us estimate the double integral. According to (3.2) and the inequality $\Phi(\kappa s) \le \kappa\Phi(s)$, $\kappa, s \in [0,1]$, guaranteed by the convexity property of the function $\Phi(\cdot)$, one has

$$\int_0^\infty \frac{1}{v^{2+\delta}}\int_0^t m(t-u;\mathbf{x},\mathbf{0})\Phi(q(e^{-v},u;\mathbf{0},\mathbf{y}))\,du\,dv \le \int_0^t m(t-u;\mathbf{x},\mathbf{0})m(u;\mathbf{0},\mathbf{y})\,du \int_0^\infty \frac{\Phi(1-e^{-v})}{v^{2+\delta}}\,dv.$$

On the right-hand side of the above inequality, the integral over u is equivalent (up to a constant factor) to the function $m(t;\mathbf{x},\mathbf{y})$, as $t\to\infty$, on account of Theorem 1 and [13, Lemma 5.1.2]. Furthermore, in the same inequality the integral over v converges by virtue of formula (3.5) combined with the finiteness of $\mathsf{E}\,\xi^{1+\delta}$. Therefore, this argument, together with relation (3.6), leads to (3.4) with $0<\delta<1$.

Now we consider the case $\delta = 1$. The identity $f''(1) = \mathsf{E}\,\xi(\xi-1)$ and the conditions of the lemma imply the existence of $f''(1)$ and, consequently, the existence of $\Phi''(0)$. Moreover, since $\Phi(0)=0$ and $\Phi'(0)=0$, inequality (3.3) for $\delta=1$ is proved. Let us take the second left derivatives at $s=1$ on each side of equality (3.1). Using the relations $m(t;\mathbf{x},\mathbf{y}) = -\partial_s q(s,t;\mathbf{x},\mathbf{y})|_{s=1}$ and $\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})(\mu(t;\mathbf{y})-1) = -\partial^2_{ss}q(s,t;\mathbf{x},\mathbf{y})|_{s=1}$, we obtain

$$\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})(\mu(t;\mathbf{y})-1) = \alpha f''(1)\int_0^t m(t-u;\mathbf{x},\mathbf{0})(m(u;\mathbf{0},\mathbf{y}))^2\,du.$$

In accordance with Theorem 1 and [13, Lemma 5.1.2], the integral in the latter equality behaves as $m(t;\mathbf{x},\mathbf{0})$ up to a constant factor, as $t\to\infty$. Hence, by virtue of Theorem 1 this entails the desired inequality (3.4) when $\delta=1$. Lemma 2 is completely proved. $\square$

Let us proceed to the proof of Theorem 2. It is easily verified that due to formulae (3.2) and (3.3) along with Theorem 1 and the proof scheme of Lemma 4 in [16], one gets

$$\int_0^t m(t-u;\mathbf{x},\mathbf{0})\Phi(q(s,u;\mathbf{0},\mathbf{y}))\,du \sim m(t;\mathbf{x},\mathbf{0})J(s;\mathbf{y}), \qquad t\to\infty. \tag{3.7}$$

Observe that the function $J(\cdot;\cdot)$ involved in the formulations of Theorems 2 and 3 is well defined in view of the upper estimate (3.2) for $q(s,t;\mathbf{x},\mathbf{y})$. Let us find a lower estimate for $q(t;\mathbf{x},\mathbf{y})$. By the Hölder inequality one has

$$\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y}) = \mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})\mathbb{I}(\mu(t;\mathbf{y})>0) \le \big(\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})^{1+\delta}\big)^{1/(1+\delta)}\big(\mathsf{E}_{\mathbf{x}}\,\mathbb{I}(\mu(t;\mathbf{y})>0)^{(1+\delta)/\delta}\big)^{\delta/(1+\delta)}$$

where $\mathbb{I}(B)$ denotes the indicator of a set B. We rewrite the last inequality as follows:

$$q(t;\mathbf{x},\mathbf{y}) \ge \frac{(m(t;\mathbf{x},\mathbf{y}))^{(1+\delta)/\delta}}{(\mathsf{E}_{\mathbf{x}}\,\mu(t;\mathbf{y})^{1+\delta})^{1/\delta}}, \qquad t>0.$$

Now, employing (3.4), we come to the desired lower estimate for $q(t;\mathbf{x},\mathbf{y})$:

$$q(t;\mathbf{x},\mathbf{y}) \geq K_2^{-1/\delta} m(t;\mathbf{x},\mathbf{y}), \qquad t \geq t_0(\mathbf{x},\mathbf{y}), \quad \mathbf{x},\mathbf{y} \in \mathbb{Z}^d. \tag{3.8}$$

Combining formulae (3.1) and (3.7), when $s = 0$, with estimate (3.8), we conclude that, as $t \to \infty$,

$$q(t;\mathbf{x},\mathbf{y}) \sim m(t;\mathbf{x},\mathbf{y}) - m(t;\mathbf{x},\mathbf{0})J(0;\mathbf{y}) \qquad \text{and} \qquad J(0;\mathbf{y}) < \lim_{t\to\infty} \frac{m(t;\mathbf{x},\mathbf{y})}{m(t;\mathbf{x},\mathbf{0})}$$

for fixed $\mathbf{x},\mathbf{y} \in \mathbb{Z}^d$. These relations amount to the validity of Theorem 2. $\square$

Let us prove Theorem 3. Applying formulae (3.1) and (3.7) once again, we have

$$q(s,t;\mathbf{x},\mathbf{y}) \sim (1-s)m(t;\mathbf{x},\mathbf{y}) - m(t;\mathbf{x},\mathbf{0})J(s;\mathbf{y}), \qquad t \to \infty, \quad \mathbf{x},\mathbf{y} \in \mathbb{Z}^d.$$

Then with the help of the identity

$$\lim_{t\to\infty} \mathsf{E}_{\mathbf{x}}\big(s^{\mu(t;\mathbf{y})} \,\big|\, \mu(t;\mathbf{y}) > 0\big) = 1 - \lim_{t\to\infty} \frac{q(s,t;\mathbf{x},\mathbf{y})}{q(t;\mathbf{x},\mathbf{y})}$$

and Theorem 2 we obtain the assertion of Theorem 3. $\square$

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