Einstein derivations and narrow Lie algebras

Dmitry Millionshchikov

Lomonosov Moscow State University

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Ricci operator

Consider a metric nilpotent Lie algebra $(\mathfrak{g}, (\,,\,))$, where $(\,,\,)$ is an inner product on $\mathfrak{g}$. Fix an orthonormal basis $e_1, \dots, e_n$ of $\mathfrak{g}$.

Definition

Define the Ricci operator $R : \mathfrak{g} \to \mathfrak{g}$

$$R = \frac{1}{4} \sum_{i=1}^{n} ade_i (ade_i)^* - \frac{1}{2} \sum_{i=1}^{n} (ade_i)^* ade_i.$$

where $adx(y) = [x, y]$ and $(adx)^*$ is the adjoint operator to adx.

The definition of R does not depend on the choice of the basis $e_1, \dots, e_n$.

Ricci tensor and Ricci operator

Let G be a nilpotent Lie group with a left invariant Riemannian metric g and tangent Lie algebra $\mathfrak{g}$. Then for the corresponding Ricci tensor $Ric_{\mathfrak{g}}(,)$ (symmetric bilinear form) we have

$$Ric_{\mathfrak{g}}(X,Y) = \frac{1}{4}\sum_{i,j=1}^{n}(ade_i(e_j),X)(ade_i(e_j),Y) - \frac{1}{2}\sum_{i=1}^{n}(ade_i(X))(ade_i(Y))$$

on left invariant vector fields $X, Y \in \mathfrak{g}$.

Finally we have the defining relation of the Ricci operator R

$$(R(X),Y) = Ric_{\mathfrak{g}}(X,Y).$$

Einstein derivation

Definition

The derivation D of nilpotent Lie algebra $\mathfrak{g}$ satisfying

$$R = cId + D, \; c \in \mathbb{R}, \tag{1}$$

is called the **Einstein derivation** of $\mathfrak{g}$.

Theorem (J. Heber, 1998)

The eigenvalues λ_i of the Einstein derivation D are, up to scaling, natural numbers.

Heber also proved that if the Einstein derivation D exists, then it is **unique** up to multiplication by a positive constant.

From Einstein derivation to Einstein solvemanifold

Definition

Einstein manifold is an Riemannian manifold (M^n, g) satisfying

$$Ric_g = cg,$$

Ric_g denotes the Ricci tensor of g.

Let D be the Einstein derivation of a metric nilpotent Lie algebra $\mathfrak{g}$. Define the metric solvable Lie algebra $\mathfrak{s} = \langle H \rangle \oplus \mathfrak{g}$ by

$$[H, x] = D(x), x \in \mathfrak{g}, \quad H \perp \mathfrak{g}, \; \|H\|^2 = TrD.$$

Then the corresponding simply connected solvable Lie group S with a left invariant Riemannian metric defind by $(,)$ is Einstein solvemanifold. Its scalar curvature is $c(\dim \mathfrak{g} + 1)$.

Example: Heisenberg metric Lie algebra $\mathfrak{h}_3$

Defined by the orthonormal basis e_1, e_2, e_3 and $[e_1, e_2] = e_3$.

$$ade_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, ade_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, ade_3 = 0.$$

$$R = \frac{1}{4}\left(ade_1(ade_1)^* + ade_2(ade_2)^*\right) - \frac{1}{2}\left((ade_1)^* ade_1 + (ade_2)^* ade_2\right),$$

$$R = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

With $c = -\frac{3}{2}$ and the diagonal derivation D is $d(1, 1, 2)$.

Does a nilpotent Lie algebra admit the Einstein derivation?

Recall that both operators R and Id are self-adjoint. Hence **the Einstein derivation D is diagonalizable**.

$D \neq adx$ since adx is nilpotent operator for any $x \in \mathfrak{g}$.

This imposes restrictions on our nilpotent Lie algebra $\mathfrak{g}$, in particular,

$\mathfrak{g}$ **cannot be characteristically nilpotent**.

Lauret proved in 2007 that **any Einstein solvmanifold is** *standard* which means that in the corresponding metric solvable Lie algebra $\mathfrak{s}$ **the orthogonal complement** $\mathfrak{a}$ to the derived algebra $\mathfrak{n} = [\mathfrak{s}, \mathfrak{s}]$ of $\mathfrak{s}$ is an **abelian** subalgebra.

Earlier Heber has shown that in this situation there exists an one-dimensional subspace $\mathfrak{a}_1 \subset \mathfrak{a}$ such that $\mathfrak{s}_1 = \mathfrak{a}_1 \oplus \mathfrak{n}$ is again Einstein metric solvable Lie algebra

All Einstein metric solvable Lie algebras with the nilradical $\mathfrak{g}$ can be obtained from $\mathfrak{g}$ **by adjoining appropriate derivations**. In particular, the **geometry and the algebra** of an Einstein metric solvable Lie algebra $\mathfrak{s}$ is completely determined by its nilradical.

A nilpotent Lie algebra which can be a nilradical of an Einstein metric solvable Lie algebra is called an **Einstein nilradical**.

From Einstein derivation to Ricci nilsolitons

Let M^n is a Riemannian manifold with a metric g_0 satisfying the **Ricci soliton equation**

$$Ric_{g_0} = cg_0 - \frac{1}{2} L_X g_0 \tag{2}$$

where c is a real scalar and L_X is the Lie derivative with respect to some complete vector field X.
Then the one-parametric family of metrics $g(t)$ on M^n

$$g(t) = (-2ct + 1)\phi_t^*(g_0) \tag{3}$$

is a special solution of the Ricci flow

$$\frac{\partial g}{\partial t} = -2Ric_g,\ g(0) = g_0, \tag{4}$$

where $\{\phi_t : M^n \to M^n\}$ is a one-parametric family of diffeomorphisms that corresponds to the vector field X

Derivations and gradings

The eigenspaces $\mathfrak{g}_{\lambda_1}, \dots, \mathfrak{g}_{\lambda_s}$ of D define an $\mathbb{N}$-grading of $\mathfrak{g}$.

$$\mathfrak{g} = \mathfrak{g}_{\lambda_1} \oplus \mathfrak{g}_{\lambda_2} \oplus \dots \oplus \mathfrak{g}_{\lambda_s}.$$

If $[\mathfrak{g}_{\lambda_i}, \mathfrak{g}_{\lambda_j}] \neq 0$, then $\lambda_i + \lambda_j = \lambda_k \in S = \{\lambda_1, \dots, \lambda_s\} \subset \mathbb{N}$.

$$D[x,y] = [Dx,y] + [x,Dy] = [\lambda_i x, y] + [x, \lambda_j y] = \\ = (\lambda_i + \lambda_j)[x,y], x \in \mathfrak{g}_{\lambda_i}, y \in \mathfrak{g}_{\lambda_j}.$$

We denote a diagonal derivation D with eigenvalues $\lambda_1, \dots, \lambda_n$

$$D = d(\lambda_1, \lambda_2, \dots, \lambda_n)$$

We will call the subset $S = \{\lambda_1, \dots, \lambda_s\} \subset \mathbb{N}$ the **support** of the $\mathbb{N}$-grading, defined by the derivation D.

Equivalent $\mathbb{N}$-gradings

Definition

If $\Gamma' : \mathfrak{g} = \oplus_{s' \in S'} \mathfrak{g}'_{s'}$ is another $\mathbb{N}$-grading of $\mathfrak{g}$ with support $S' \subset \mathbb{N}$, we say that Γ' is equivalent to Γ if there is a bijection $\sigma : S \to S'$ and a automorphism $\varphi \in Aut(\mathfrak{g})$ such that $\varphi(\mathfrak{g}_s) = \mathfrak{g}'_{\sigma(s)}$. It follows that $\sigma(s_1 + s_2) = \sigma(s_1) + \sigma(s_2), s_1, s_2 \in S$.

Example (filiform 5-dimensional Lie algebra $\mathfrak{m}_0(4)$)

$$[e_1, e_2] = e_3, [e_1, e_3] = e_4, [e_1, e_4] = e_5.$$

$c = -6$. The Gram matrix G and derivation D are

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 12 & 0 \\ 0 & 0 & 0 & 0 & 36 \end{pmatrix}, \; D = \frac{1}{2} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 11 & 0 & 0 \\ 0 & 0 & 0 & 13 & 0 \\ 0 & 0 & 0 & 0 & 15 \end{pmatrix},$$

This grading Γ' with the support $S' = \{2, 9, 11, 13, 15\}$ is obviously equivalent to the standard $\mathbb{N}$-grading of $\mathfrak{m}_0(4)$ with the support $S = \{1, 2, 3, 4, 5, 6\}$.

Example (5-dimensional quotient of the Witt algebra)

$$[e_1, e_2] = \sqrt{3}e_3, [e_1, e_3] = \sqrt{3}e_4, [e_1, e_4] = \sqrt{2}e_5, [e_2, e_3] = \sqrt{2}e_5.$$

With $c = -11$ and the diagonal $D = \frac{3}{2}d(1, 2, 3, 4, 5)$.

The data for the Einstein derivation D are $(1 < 2 < 3 < 4 < 5; 1, 1, 1, 1, 1)$.

Gradings and derivations

Consider a $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{k=1}^{+\infty} \mathfrak{g}_\alpha,\ \alpha \in \mathbb{N}$. Define operator $\tau : \mathfrak{g} \to \mathfrak{g}$ as scalar operator αId on each homogeneous subspace $\mathfrak{g}_\alpha$:

$$\tau(x) = \alpha x, x \in \mathfrak{g}_\alpha,\ \alpha \in \mathbb{N}.$$

Obviously τ is a positive derivation of $\mathfrak{g}$

$$\begin{gathered}(\alpha+\beta)[x,y] = \tau([x,y]) = \\ = [\tau(x),y]+[x,\tau(y)] = [\alpha x,y]+[x,\beta y], \\ x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta.\end{gathered}$$

Classification in small dimensions

n-dimensional nilsolitons or rank-one Einstein solvmanifolds of dimension $n+1$

- $n \leq 5$ – Jorge Lauret (2003);
- $n = 6$ – Cyntia Will (2003) (based on Morosov's classification);
- $n = 7$ – Fernandez-Culma (2014) (based on Gong's classification of 1998);

The direct sum of nilpotent Lie algebras is nilsoliton if and only if each summand is nilsoliton (Nikolaevsky and independently Jablonsky, 2011),

List of some nilpotent Lie algebras $\mathfrak{g}$ that are Einstein nilradicals.

- $\mathfrak{g}$ is abelian;
- all nilpotent Lie algebras dimensions ≤ 6;
- $\mathfrak{g}$ has an abelian ideal of codimension one (Lauret, 2003);
- Heisenberg Lie algebras;
- Lie algebra of upper strictly lower triangular matrices (T. Payne);

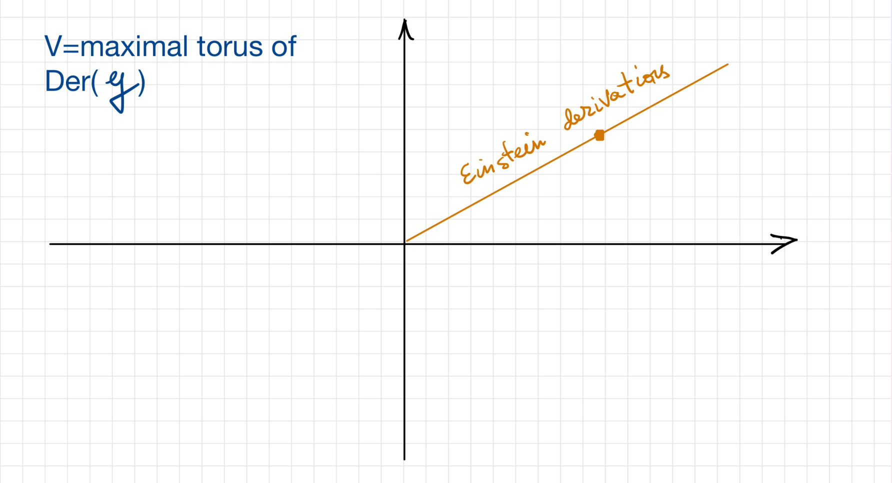
Maximal torus of $Der(\mathfrak{g})$

Let $\mathfrak{g}$ be a nilpotent Lie algebra. Its automorphism group $Aut(\mathfrak{g})$ is a solvable algebraic group ($\mathfrak{g} \neq \mathfrak{h}_3$).

Consider the maximal abelian Lie subalgebra $\mathfrak{t}(\mathfrak{g}) \subset Der(\mathfrak{g})$, consisting of semisimple derivations. It is called the **maximal torus** of $Der(\mathfrak{g})$.

$\dim \mathfrak{t}(\mathfrak{g})$ is called the **rank** of $\mathfrak{g}$.

Einstein derivations inside of the maximal torus



Carnot algebras

A finite dimensional $\mathbb{N}$-graded metric Lie algebra $\mathfrak{g}$ is called **Carnot algebra** in subRiemannian geometry

$$\mathfrak{g} = \oplus_{i=1}^{s} \mathfrak{g}_i, \ [\mathfrak{g}_1, \mathfrak{g}_i] = \mathfrak{g}_{i+1}, 1 \le i \le n{-}1;$$
$$\mathfrak{g}_i \perp \mathfrak{g}_j, \quad i \neq j.$$

It follows that $\mathfrak{g}$ is generated by $\mathfrak{g}_1$.

The corresponding derivation D has the type

$$(1 < 2 < \cdots < s; d_1, \ldots, d_s), \ \ d_i = \dim \mathfrak{g}_i, 1 \le i \le s.$$

A naive question: is it true that the Einstein derivation of a Carnot algebra determines a Carnot grading? As it was in our very first example of Heisenberg Lie algebra $\mathfrak{m}_0(3)$!

However the Einstein derivation D of the simplest filiform Lie algebra $\mathfrak{m}_0(n)$ for $n \geq 4$, has the simple spectrum.

Einstein derivations of filiform Lie algebras

A nilpotent Lie algebra $\mathfrak{g}$ of dimension n is called **filiform** if its lower central series has the maximal possible length s (nil-index), $s = n - 1$.

Consider a maximal abelian Lie subalgebra $\mathfrak{t} \subset Der(\mathfrak{g})$ consisting of semi-simple derivations τ of the Lie algebra $\mathfrak{g}$.

Let $\mathfrak{g}$ be a filiform Lie algebra. It is two-generated and its rank $r(\mathfrak{g})$ does not exceed two

$$r(\mathfrak{g}) = \dim \mathfrak{t} \leq 2.$$

Let $r(\mathfrak{g}) = 2$. Then $\mathfrak{g}$ is isomorphic to either $\mathfrak{m}_0(\mathrm{n})$ or $\mathfrak{m}_1(2k)$.

Nikolaevsky used the classification list of graded filiform Lie algebras of width one (Millionshchikov 2002) and found all **filiform Einstein nilradicals**. In particular, he proved that, starting from some dimension $n \geq n_0$, there are only three families of Einstein filiform nilradicals

$$\mathfrak{m}_0(n), \mathfrak{m}_1(n), W_n^+$$

Graded filiform Lie algebras

algebra	*dimension*	*commutating relations*
$\mathfrak{m}_0(n),\ n \geq 3$	n	$[e_1, e_i] = e_{i+1}, \quad i = 2, \dots, n-1$
$\mathfrak{m}_2(n),\ n \geq 5$	n	$[e_1, e_i] = e_{i+1}, \quad i = 2, \dots, n-1;$ $[e_2, e_i] = e_{i+2}, \quad i = 3, \dots, n-2.$
Алгебра Витта $\mathcal{V}_n,\ n \geq 12$	ЕСТЬ НИЛЬСОЛИТОН!!!! n	$[e_i, e_j] = \begin{cases} (j-i)e_{i+j}, i+j \leq n; \\ 0, i+j > n. \end{cases}$
$\mathfrak{m}_{0,1}(2k+1),$ $k \geq 3$	$2k+1$	$[e_1, e_i] = e_{i+1}, \quad i = 2, \dots, 2k;$ $\cancel{[e_l, e_{2k-l+1}]} = (-1)^{l+1} e_{2k+1}, \quad l = 2, \dots, k.$
$\mathfrak{m}_{0,2}(2k+2),$ $k \geq 3$	$2k+2$	$[e_1, e_i] = e_{i+1},\ i=2, \dots, 2k+1;$ $[e_l, e_{2k-l+1}] = (-1)^{l+1} e_{2k+1},\ l=2, \dots, k;$ $[e_j, e_{2k-j+2}] = (-1)^{j+1}(k-j+1)e_{2k+2},\ j=2, \dots, k.$
$\mathfrak{m}_{0,3}(2k+3),$ $k \geq 3$	$2k+3$	$[e_1, e_i] = e_{i+1}, \quad i=2, \dots, 2k+2;$ $[e_l, e_{2k-l+1}] = (-1)^{l+1} e_{2k+1},\ l=2, \dots, k;$ $[e_j, e_{2k-j+2}] = (-1)^{j+1}(k-j+1)e_{2k+2},\ j=2, \dots, k;$ $[e_m, e_{2k-m+3}] = (-1)^m \left((m-2)k - \frac{(m-2)(m-1)}{2}\right) e_{2k+3},$ $m=3, \dots, k+1.$

Narrow Lie algebras after Zelmanov and Shalev

Definition

A $\mathbb{N}$-graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ is called a Lie algebra of bounded width, if

$$\exists C \geq 0,\ \dim \mathfrak{g}_i \leq C, \forall i \in \mathbb{N}.$$

Definition

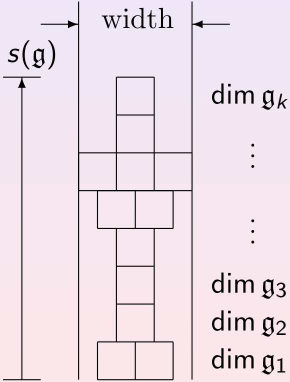
The width $d(\mathfrak{g})$ of a positively graded Lie algebra $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ of bounded width is

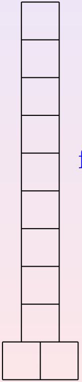
$$d(\mathfrak{g}) = \max_{i \in \mathbb{N}} \dim \mathfrak{g}_i.$$

We consider **narrow** Lie algebras, i.e. with $d(\mathfrak{g}) \leq 2$.

$\mathbb{N}$-graded Lie algebras and lego towers

Let $\mathfrak{g} = \oplus_{i=1}^{k} \mathfrak{g}_i$ be a $\mathbb{N}$-graded Lie algebra. Draw instead of each basis vector a lego brick.





filiform tower

The narrowest Carnot algebra

Theorem (M. Vergne, 1970)

Let $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ be a Carnot algebra such that

$$\dim \mathfrak{g}_1 = 2, \dim \mathfrak{g}_i = 1, \forall i \geq 2.$$

Then $\mathfrak{g}$ is isomorphic to $\mathfrak{m}_0$.

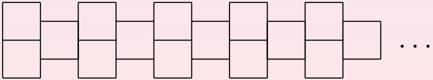
Two loop subalgebras

Two real forms $\mathfrak{so}(3,\mathbb{R})$, $\mathfrak{so}(1,2)=\mathfrak{sl}(2,\mathbb{R})$ of $\mathfrak{sl}(2,\mathbb{C})$ can be defined by the basis u,v,w and commutating relations

$$[u,v]=w,\ [v,w]=\pm u,\ [w,u]=v.$$

Now we consider two subalgebras $\mathfrak{n}_1^{\pm}$ in loop algebras $\mathfrak{so}(3,\mathbb{R})\otimes\mathbb{R}[t]$ and $\mathfrak{so}(1,2)\otimes\mathbb{R}[t]$.

$$\begin{array}{c} u{\otimes}t^1 \\ v{\otimes}t^1 \end{array}\ w{\otimes}t^2\ \begin{array}{c} u{\otimes}t^3 \\ v{\otimes}t^3 \end{array}\ w{\otimes}t^4\ \begin{array}{c} u{\otimes}t^5 \\ v{\otimes}t^5 \end{array}\ w{\otimes}t^6\ldots$$



Carnot algebras of width 3/2

Theorem (Millionshchikov, 2019, Mat Smornik)

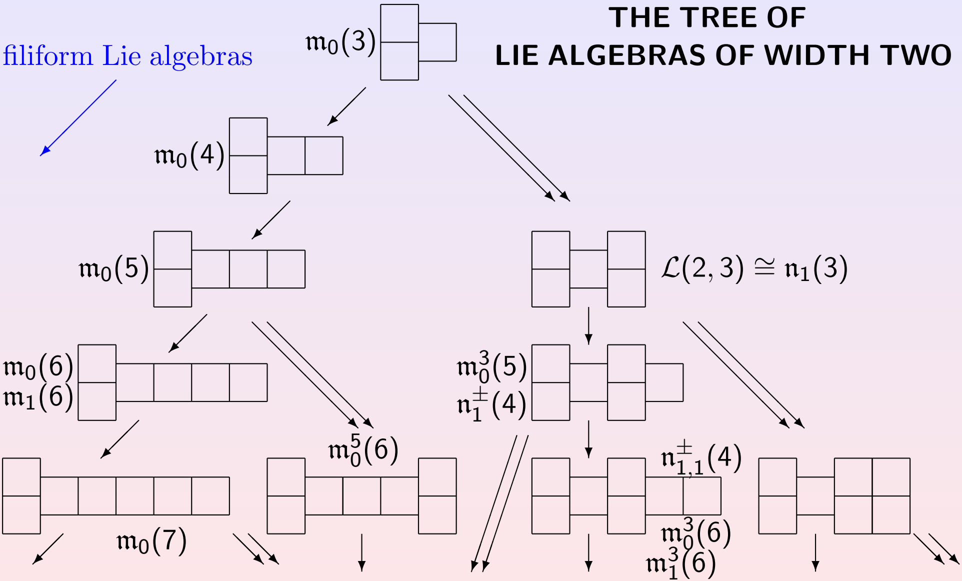
Let $\mathfrak{g} = \oplus_{i=1}^{+infinity} \mathfrak{g}_i$ *be a Carnot algebra such that*

$$\dim \mathfrak{g}_i + \dim \mathfrak{g}_{i+1} \leq 3, i \geq 1.$$

Then $\mathfrak{g} = \oplus_{i=1}^{+\infty} \mathfrak{g}_i$ *is isomorphic to the only one Lie algebra from*

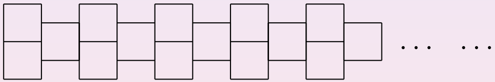
$$\mathfrak{m}_0, \mathfrak{n}_1^{\pm}, \mathfrak{n}_2, \mathfrak{n}_2^3, \left\{ \mathfrak{m}_0^S \right\}.$$

THE TREE OF LIE ALGEBRAS OF WIDTH TWO



Theorem (Millionshchikov, 2022)

$\mathfrak{n}_1^{\pm}(s) = \mathfrak{n}_1^{\pm}/(\mathfrak{n}_1^{\pm})^{s+1}$ *is an Einstein nilradical and the corresponding positive grading coinsides with its Carnot grading for all* s.



Gram matrices for quotients $\mathfrak{n}_1^{\pm}/(\mathfrak{n}_1^{\pm})^{s+1}$

s	dim	Gram matrix =diagonal $d(g_{11}, \dots, g_{nn})$	c
3	5	$d(1, 1, 4, 4{\cdot}3, 4{\cdot}3)$	-6
4	6	$d(1, 1, 4, 4{\cdot}5, 4{\cdot}5, 4{\cdot}5{\cdot}4)$	-10
5	8	$d(1, 1, 6, 6{\cdot}5, 6{\cdot}5, 6{\cdot}5{\cdot}6, 6{\cdot}5{\cdot}6{\cdot}5, 6{\cdot}5{\cdot}6{\cdot}5)$	-15
6	9	$d(1, 1, 6, 6{\cdot}7, 6{\cdot}7, 6{\cdot}7{\cdot}6, 6{\cdot}7{\cdot}6{\cdot}7, 6{\cdot}7{\cdot}6{\cdot}7, 6{\cdot}7{\cdot}6{\cdot}7{\cdot}6)$	-21

Affine variety of Lie algebras

Consider L_n the **affine variety of Lie algebra products** on a fixed n-dimensional vector space V. Also we fix some basis $e_1, \dots, e_n$.

The variety L_n consists of skew-symmetric bilinear maps

$$\mu : V \wedge V \to V, \quad \mu(e_i, e_j) = \sum_{k=1}^{n} c_{ij}^r e_r,$$

satisfying the Jacobi identity

$$\sum_{r=1}^{n} \left(c_{ij}^r c_{kr}^s + c_{jk}^r c_{ir}^s + c_{ki}^r c_{jr}^s \right) = 0$$

One can define also the **affine variety N_n of nilpotent Lie algebras**.

There is the natural **"change of basis" action** of $G = GL_n$ on L_n

$$(g \cdot \mu)(x, y) = g\left(\mu(g^{-1}x, g^{-1}y)\right), \; g \in G, \; x, y \in V.$$

Obviously the **isomorphism class of the given Lie algebra** $\mu \in L_n$ **corresponds to the orbit** $O(\mu)$ of this action.

A Lie algebra μ is called **rigid if its orbit** $O(\mu)$ **is open**.

Affine variety of nilpotent Lie algebras from Riemannian viewpoint (after J.Lauret)

Fix some euclidian inner product $\langle , \rangle$ on V and choose an **orthonormal basis** $e_1, \dots, e_n$ in V.

There is the natural **"change of orthonormal basis" action** of $O(n, \mathbb{R})$ on N_n.

Study of orbit spaces (metric classification of real nilpotent Lie algebras) by E. Wilson (1982), J. Lauret, (dimension 3 and 4, 1997), A. Figula and P.T. Nagy, (dimension 5, 2018), ...

Ricci curvature operator and functional F

Lauret defined a functional F on the variety N_n of nilpotent Lie algebras

$$F(\mu) = tr\, Ric_\mu^2.$$

F written in coordinates

$$F(\mu) = tr Ric_\mu^2 = \sum_{pr} \left(\sum_{ij} \left(-\frac{1}{2} c_{pij} c_{rij} + \frac{1}{4} c_{ijp} c_{ijr} \right) \right)^2, \mu = \{c_{ijk}\} \in N_n.$$

$$gradF|_\mu = -\delta_\mu(Ric_\mu).$$

$\delta : C^1(\mathfrak{g}_\mu, \mathfrak{g}_\mu) \to C^2(\mathfrak{g}_\mu, \mathfrak{g}_\mu)$ is the Chevalley coboundary operator (cochaines with coefficients in the adjoint representation).

Variational method by Lauret

Lemma (J. Lauret, 2003)

Let S denote the sphere $\sum_{ijk} c_{ijk}^2 = 1$ in $\Lambda^2 V^* \otimes V$. Then the following conditions are equivalent

- $Ric_\mu \in \mathbb{R} \oplus Der(\mu)$;
- μ *is a critical point of* $F : S \to \mathbb{R}$;
- μ *is a critical point of* $F : \mathcal{O}(\mu) \cap \mathcal{S} \to \mathbb{R}$.

It follows that if a (nilpotent) $\mu \in N_n$ is a critical point of F then (V, μ) is $\mathbb{N}$-graded Lie algebra.

Recall the following famous open conjecture by Vergne (1970): *there are no rigid Lie algebras in the variety N_n of nilpotent Lie algebras of dimension $n \geq 8$.*

A Lie algebra μ' is called a degeneration of μ, if $\mu' \in \bar{O}(\mu)$, where $\bar{O}(\mu)$ stands for the Zarissky closure of the orbit $O(\mu)$.

There is another open conjecture by Grunewald and O'Halloran: *every nilpotent Lie algebra of dimension two or more is the degeneration of some other Lie algebra.*