UNIFORM OVER THE WHOLE LINE $\mathbb{R}$ ESTIMATE FOR THE RATE OF CONVERGENCE OF THE SPECTRAL EXPANSION RELATED TO THE SCHRODINGER OPERATOR WITH SUMMABLE POTENTIAL

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Estimate

$$|\sigma_\lambda(x, f) - f(x)| = \lambda^{\frac{1}{2}} - \frac{2}{\pi} \sigma(1), \quad \lambda \to +\infty$$

(that is uniform over the whole real line $\mathbb{R} = (-\infty, \infty)$) of the difference between the spectral expansion $\sigma_\lambda(x, f)$ related to the self-adjoint extension $\mathcal{H}$ of the Schrödinger operator $Hu = -u'' + q(x)u, x \in \mathbb{R}$, and the original function $f(x)$ is obtained. Here $f(x)$ is an arbitrary function from the Sobolev–Liouville class $L^2_0(\mathbb{R})$, the potential $q(x)$ belongs to the space $L_p(\mathbb{R}) (1 \leq p < \infty)$ of $p$-summable functions, and the derivation order $\alpha$ is related to the summability degree $p$ by the inequalities: $1/2 < \alpha < 2$ for $1 \leq p < \infty$ and $1/2 < \alpha < 3/2$ for $1 \leq p < 2$.

In this paper we obtain the estimate

$$|\sigma_\lambda(x, f) - f(x)| = \lambda^{\frac{1}{2}} - \frac{2}{\pi} \sigma(1), \quad \lambda \to +\infty,$$  \hfill (1)

(that is uniform over the entire real line $\mathbb{R} = (-\infty, \infty)$) of the difference between the spectral expansion $\sigma_\lambda(x, f)$ related to the self-adjoint extension $\mathcal{H}$ of the Schrödinger operator $Hu = -u'' + q(x)u, x \in \mathbb{R}$, and the original function $f(x)$, provided that $f(x)$ is in the Sobolev–Liouville class $L^2_0(\mathbb{R})$, the potential $q(x)$ belongs to the space $L_p(\mathbb{R}) (1 \leq p < \infty)$ of $p$-summable functions, and the derivation order $\alpha$ is related to the summability degree $p$ by the inequalities: $1/2 < \alpha < 2$ for $1 \leq p < \infty$ and $1/2 < \alpha < 3/2$ for $1 \leq p < 2$.

We proceed with the precise formulation of results and their verification.

Let $\mathcal{H}$ be the self-adjoint extension of the formal differential Schrödinger operator

$$Hu = -u'' + q(x)u$$ \hfill (2)

defined over the whole line $\mathbb{R}$ with potential $q(x) \in L_p(\mathbb{R}), 1 \leq p < \infty$.

It is known that the operator $\mathcal{H}$ is bounded below [2, pp.15,17] and is uniquely defined by the differential expression (2); namely, for $p \geq 2$ the operator $\mathcal{H}$ is essentially self-adjoint on $C^\infty_0(\mathbb{R})$ [2, p.28], and for $1 \leq p < 2$, the set $C^\infty_0(\mathbb{R})$ is an essential domain of the form that generates $\mathcal{H}$ [2, pp.12,18]. Without loss of generality we assume that the operator $\mathcal{H}$ is bounded below by $\lambda_0 \geq 1$.

It follows from the generalization of the Gårding–Browder–Mautner theorem (obtained in [3]) that $L^2(\mathbb{R})$ has an ordered spectral representation with respect to $\mathcal{H}$ that is characterized by a multiplicity $m \leq 2$, multiplicity sets $e_i, i = 1, m$, a spectral measure $\mu(\lambda)$, and fundamental functions (or generalized eigenfunctions) $u_i(x, \lambda), i = 1, m$. These fundamental functions satisfy the equation $\mathcal{H}u_i(x, \lambda) = \lambda u_i(x, \lambda)$ almost everywhere on $\mathbb{R}$, while the generalized Fourier transforms

$$\hat{f}_i(\lambda) = \int_{-\infty}^{\infty} f(y)u_i(y, \lambda) \, dy, \quad i = 1, m,$$ \hfill (3)

of an arbitrary function $f \in L^2(\mathbb{R})$ satisfy the generalized Parseval formula

$$\int_{-\infty}^{\infty} |f(y)|^2 \, dy = \sum_{i=1}^{m} \int_{\lambda_0}^{\infty} |\hat{f}_i(\lambda)|^2 \, d\lambda(\lambda).$$ \hfill (4)
As in [1], the following lemma on the generalized Fourier transforms (3) for functions of the Sobolev–Liouville class is crucial for the proof of this paper’s main theorem.

**Lemma.** Let \( \mathcal{H} \) be the strictly positive selfadjoint extension of Schrödinger operator (2) with potential \( q(x) \) that belongs to \( L^p(\mathbb{R}) \) for some \( p \in [1, \infty) \). Let \( f(x) \) be an arbitrary function from the Sobolev–Liouville class \( L_2^\alpha(\mathbb{R}) \) with a chosen derivation order \( \alpha \) such that \( \alpha \in \left( \frac{1}{p}, \frac{3}{2} \right) \) for \( p \geq 2 \) and \( \alpha \in \left( \frac{1}{p}, \frac{1}{2} \right] \) for \( p \in [1, 2) \). Then the generalized Fourier transforms \( \hat{f}_i(\lambda) \) by the system of fundamental functions of the ordered spectral representation of \( L_2^\alpha(\mathbb{R}) \) with respect to \( \mathcal{H} \) satisfy the inequality

\[
\sum_{i=1}^{m} \int_{\lambda_0}^{\lambda} |\hat{f}_i(t)|^2 \ t^\alpha \ dp(t) \leq C \| f \|_{L_2^\alpha(\mathbb{R})}^2.
\]  

**(5)**

**Proof.** We denote\(^3\) by \( h(x) \) the function from \( L_2(\mathbb{R}) \) such that

\[
f(x) = \int_{\mathbb{R}} \frac{\tau_{\alpha/2}(|x-y|)}{\sqrt{2\pi}} \ dy = \int_{\mathbb{R}} \frac{\tau_{\alpha/2}(x-y)}{t} \ dy,
\]

where \( \tau_{\alpha/2}(x,y) = \tau_{\alpha/2}(|x-y|) \) is the so-called Bessel–McDonald kernel of order \( \alpha/2 > 0 \) that is defined by equality

\[
\tau_{\alpha/2}(\rho) = 2^{1-\alpha/2} \left[ \sqrt{2\pi} \Gamma \left( \frac{\alpha}{2} \right) \right]^{-1} \rho^{(\alpha-1)/2} K_{(1-\alpha)/2}(\rho).
\]  

**(6)**

We use the following expression for the Fourier transform \( \hat{f}_i(\lambda) \) of \( f(x) \in L_2^\alpha(\mathbb{R}) \) obtained in [1, Eq. (13)]:

\[
\hat{f}_i(t) = \hat{h}_i(t) \cdot (1 + t)^{-\alpha/2} + \frac{1}{\sqrt{t}} \int_{-\infty}^{\infty} h(x) \left\{ \int_{-\infty}^{\infty} \tau_{\alpha/2}(\rho) \cdot \sin \left[ \sqrt{\rho} (\rho - \xi) \right] \ d\rho \right\} \cdot \left[ q(x + \xi) u_i(x + \xi, t) + q(x - \xi) u_i(x - \xi, t) \right] \ d\xi \ dx.
\]  

**(7)**

As [1, Eqs. (16), (17)], it follows from (7), Parseval formula (4), and the obvious inequality \( t^\alpha(1+t)^{-\alpha} \leq 1 \) that

\[
\sum_{i=1}^{m} \int_{\lambda_0}^{\lambda} \left| \hat{f}_i(t) \right|^2 \ t^\alpha \ dp(t) \leq 2 \| h \|_{L_2^2(\mathbb{R})}^2
\]

\[
+ 4 \sum_{i=1}^{m} \int_{\lambda_0}^{\lambda} t^{\alpha-1} \left\{ \int_{0}^{\infty} \tau_{\alpha/2}(\rho) \cdot \sin \left[ \sqrt{\rho} (\rho - \xi) \right] \ d\rho \right\} \cdot \left[ \int_{-\infty}^{\infty} h(x) q(x + \xi) u_i(x + \xi, t) \ dx \right] \ d\xi \right\}^2 \ dp(t) +
\]

\[
+ 4 \sum_{i=1}^{m} \int_{\lambda_0}^{\lambda} t^{\alpha-1} \left\{ \int_{0}^{\infty} \tau_{\alpha/2}(\rho) \cdot \sin \left[ \sqrt{\rho} (\rho - \xi) \right] \ d\rho \right\} \cdot \left[ \int_{-\infty}^{\infty} h(x) q(x - \xi) u_i(x - \xi, t) \ dx \right] \ d\xi \right\}^2 \ dp(t).
\]  

**(8)**

\(^2\)In particular, for \( p = 2 \), the derivation order \( \alpha \) can be arbitrary from \( (0, 2) \).

\(^3\)See, e.g., [1, p.379].
For the second and third items on the right-hand side of (8) we apply the estimate obtained in [1, Eq. (21)]:

\[
\left| \int_{\xi}^{\infty} \tau_{\alpha/2}(\rho) \cdot \sin \left[ \sqrt{t}(\rho - \xi) \right] \, d\rho \right| \leq \frac{2}{\sqrt{t}} \cdot \tau_{\alpha/2}(\xi)
\]

which holds for all \( \xi > 0 \) and \( \alpha \in (0, 2) \). By this estimate, inequality (8) takes the form

\[
\sum_{i=1}^{m} \int_{\lambda_{0}}^{\infty} \left| \hat{f}_{i}(t) \right|^{2} \, t^{\alpha} \, dp(t) \leq 2\|h\|_{L_{2}(\mathbb{R})}^{2} + 16 \sum_{i=1}^{m} \int_{\lambda_{0}}^{\infty} \left| \int_{0}^{\infty} \tau_{\alpha/2}(\xi) \left| \int_{-\infty}^{\infty} h(x)q(x + \xi) \, u_{i}(x + \xi, t) \, dx \right| \, d\xi \right|^{2} \, dp(t) + \sum_{i=1}^{m} \int_{\lambda_{0}}^{\infty} \left( \int_{0}^{\infty} \tau_{\alpha/2}(\xi) \left| \int_{-\infty}^{\infty} h(x)q(x - \xi) \, u_{i}(x - \xi, t) \, dx \right| \, d\xi \right|^{2} \, dp(t) \equiv 2\|h\|_{L_{2}(\mathbb{R})}^{2} + I_{2} + I_{3}. \tag{9}
\]

We consider each of the cases \( p \geq 2, \, \alpha \in \left( \frac{1}{p}, 2 \right] \) and \( 1 \leq p \leq 2, \, \alpha \in \left( \frac{1}{p} - \frac{1}{2}, \frac{3}{2} \right] \) separately. Since both the second and third items on the right-hand side of (9) are estimated similarly, we restrict ourselves in both cases to the estimation of the second item \( I_{2} \).

Let \( q(x) \in L_{p}(\mathbb{R}) \) for some \( p \in [2, \infty) \) and \( \alpha \in \left( \frac{1}{p}, 2 \right] \).

First of all we note that it follows from the asymptotic behavior of McDonald function \( K_{(1-\alpha)/2}(\rho) \) as \( \rho \to 0 + 0 \) and as \( \rho \to +\infty \) that the integral \( \int_{0}^{+\infty} \left( \tau_{\alpha/2}(\xi) \right)^{\beta} \, d\xi \), for the function \( \tau_{\alpha/2}(\xi) \) defined by (6), converges for any \( \beta > 0 \) if \( 1 \leq \alpha \leq 2 \) and for any \( \beta \in \left( 0, \frac{1}{1 - \alpha} \right) \) if \( 0 < \alpha < 1 \).

Let us transform and estimate the integral

\[
\left\{ \int_{0}^{\infty} \left| \int_{-\infty}^{\infty} h(x)q(x + \xi) \, u_{i}(x + \xi, t) \, dx \right| \, d\xi \right\}^{2} = \left\{ \int_{0}^{\infty} \left[ \left( \tau_{\alpha/2}(\xi) \right)^{\frac{\beta}{p-1}} \right] \cdot \left[ \left( \tau_{\alpha/2}(\xi) \right)^{\frac{\beta-2}{p-2}} \right] \cdot \left| \int_{-\infty}^{\infty} h(y - \xi)q(y) \, u_{i}(y, t) \, dy \right| \, d\xi \right\}^{2}. \tag{10}
\]

We apply the Cauchy–Schwarz–Bunyakovskii inequality to the right-hand side of (10) regarded as the squared integral of the product of functions inside the square brackets:

\[
\left\{ \int_{0}^{\infty} \left| \int_{-\infty}^{\infty} h(x)q(x + \xi) \, u_{i}(x + \xi, t) \, dx \right| \, d\xi \right\}^{2} \leq \int_{0}^{\infty} \left( \tau_{\alpha/2}(\xi) \right)^{\frac{\beta}{p-1}} \, d\xi \cdot \int_{0}^{\infty} \left( \tau_{\alpha/2}(\xi) \right)^{\frac{\beta-2}{p-2}} \cdot \left| \int_{-\infty}^{\infty} h(y - \xi)q(y) \, u_{i}(y, t) \, dy \right| \, d\xi. \tag{11}
\]

Since for \( \alpha \in \left( \frac{1}{p}, 1 \right) \) we have

\[
\frac{p}{p - 1} \leq \frac{1}{1 - \alpha}, \tag{12}
\]
the first integral on the right-hand side of (11) converges (as mentioned above). Thus, the considered second item on the right-hand side of (9) satisfies the estimate
\[
I_2 = O(1) \sum_{i=1}^{m} t^{\alpha - 2} \int_{-\infty}^{\infty} (\tau_{\alpha/2}(\xi))^{\frac{p-2}{p}} \left| \int_{-\infty}^{\infty} h(y - \xi) q(y) u_i(y, t) dy \right|^2 d\xi dp(t). \tag{13}
\]

Changing the order of integration with respect to \(\xi\) and \(t\) on the right-hand side of (13) and taking into account that \(t^{\alpha - 2} \leq 1\) for \(t \geq 1\) and \(\alpha \leq 2\), we obtain
\[
I_2 = O(1) \int_{0}^{\infty} (\tau_{\alpha/2}(\xi))^{\frac{p-2}{p}} \left\{ \sum_{i=1}^{m} \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} h(y - \xi) q(y) u_i(y, t) dy \right|^2 dp(t) \right\} d\xi. \tag{14}
\]

By Parseval formula (4) applied to the function \(Q(y) \equiv h(y - \xi)q(y)\), for any \(\xi > 0\), the expression inside the braces on the right-hand side of (14) is equal to \(\int_{-\infty}^{\infty} |h(y - \xi)|^2 |q(y)|^2 dy\). The Young inequality [5, Chap.1, Sec 2.13] implies
\[
\int_{0}^{\infty} \left[ \int_{-\infty}^{\infty} |h(y - \xi)|^2 |q(y)|^2 dy \right]^{p/2} d\xi \leq \|q\|_{L_p(\mathbb{R})}^p \cdot \|h\|_{L_2(\mathbb{R})}^p. \tag{15}
\]

Therefore, applying the Hölder inequality to the right-hand side of (14) we obtain the estimate
\[
I_2 = O(1) \left\{ \int_{0}^{\infty} (\tau_{\alpha/2}(\xi))^{\frac{p-2}{p}} d\xi \right\} \cdot \left\{ \sum_{i=1}^{m} \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} |h(y - \xi)|^2 |q(y)|^2 dy \right]^{p/2} d\xi \right\}^{\frac{2}{p}}
\]
which is \(O(1)\|q\|_{L_p(\mathbb{R})}^2 \|h\|_{L_2(\mathbb{R})}^2\) by (12) and (15).

Thus, if \(q(x) \in L_p(\mathbb{R})\) with some \(p \in [2, \infty)\) then for all \(\alpha \in \left( \frac{1}{p}, 2 \right)\), inequality (9) implies the estimate
\[
\sum_{i=1}^{m} \int_{-\infty}^{\infty} \left| \hat{f}_i(t) \right|^2 \cdot t^\alpha dp(t) = O(1) \|h\|_{L_2(\mathbb{R})}^2. \tag{16}
\]

By the definition of the norm in the Sobolev–Liouville class [4, p.380] we have \(\|h\|_{L_2(\mathbb{R})} = \|f\|_{L_2^2(\mathbb{R})}\). Thus, estimate (16) implies the desired estimate (5).

Let potential \(q(x)\) belong to \(L_p(\mathbb{R})\) with some \(p \in [1, 2]\) and let \(\alpha \in \left( \frac{1}{p} - \frac{1}{2}, \frac{3}{2} \right)\). We rewrite the second item on the right-hand side of (9) in the following form:
\[
I_2 \leq 16 \sum_{i=1}^{m} \int_{-\infty}^{\infty} t^{\alpha - 2} \left\{ \int_{0}^{\infty} (\tau_{\alpha/2}(\xi))^{\frac{p-2}{p}} \left( \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| |u_i(y, t)| dy d\xi \right)^2 dp(t) \right\}^{\frac{2}{p}} \cdot \left\{ \int_{0}^{\infty} (\tau_{\alpha/2}(\eta))^{\frac{p-2}{p}} \left( \int_{-\infty}^{\infty} |h(z - \eta)| |q(z)| |u_i(z, t)| dz d\eta \right)^2 dp(t) \right\}^{\frac{2}{p}}. \tag{17}
\]
On changing the order of integration and summation we see that the right-hand side of (17) is

\[ 16 \int_{0}^{\infty} \tau_{\alpha/2}(\xi) \int_{0}^{\infty} \tau_{\alpha/2}(\eta) \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \int_{-\infty}^{\infty} |h(z - \eta)| \, |q(z)| \, dz \, dy \, d\eta \]

\[ \cdot \left\{ \sum_{i=1}^{\infty} \int_{x_{0}}^{\infty} t^{\alpha-2} |u_{i}(y, t)| |u_{i}(z, t)| \, dp(t) \right\} \, dz \, dy \, d\xi. \quad (18) \]

By the Cauchy–Schwarz–Bunyakovsky inequality, the quantity inside the braces in (18) is at most

\[ \left\{ \sum_{i=1}^{\infty} \int_{x_{0}}^{\infty} t^{\alpha-2} |u_{i}(y, t)|^{2} \, dp(t) \right\}^{1/2} \cdot \left\{ \sum_{i=1}^{\infty} \int_{x_{0}}^{\infty} t^{\alpha-2} |u_{i}(z, t)|^{2} \, dp(t) \right\}^{1/2}. \]

The Corollary 1 of Theorem 1 in [6], for \( \alpha < 3/2 \), both these factors are uniformly bounded: with respect to \( y \in \mathbb{R} \) and with respect to \( z \in \mathbb{R} \), respectively.

Thus, for any \( \alpha \in \left( 0, \frac{3}{2} \right) \), the second item on the right-hand side of (9) satisfies the estimate

\[ I_2 = O(1) \int_{0}^{\infty} \tau_{\alpha/2}(\xi) \int_{0}^{\infty} \tau_{\alpha/2}(\eta) \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \int_{-\infty}^{\infty} |h(z - \eta)| \, dz \, dy \, d\eta \]

\[ \cdot |q(z)| \, dz \, dy \, d\eta \, d\xi = O(1) \left[ \int_{0}^{\tau_{\alpha/2}(\xi)} \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \, dy \, d\xi \right]^{2}. \quad (19) \]

Since \( h(x) \in L_{2}(\mathbb{R}) \) and \( q(x) \in L_{p}(\mathbb{R}) \) for some \( p \in [1, 2] \), it follows from the Young inequality that the integral \( I(\xi) \equiv \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \, dy \) belongs to \( L_{\frac{2p}{2p-2}}(0, \infty) \) and

\[ \|I(\xi)\|_{L_{\frac{2p}{2p-2}}(0, \infty)} \leq \|q\|_{L_{p}(\mathbb{R})} \|h\|_{L_{2}(\mathbb{R})}. \quad (20) \]

Applying the Hölder inequality, and then inequality (20) to the integral inside the square brackets on the right-hand side of (19) we obtain

\[ \left[ \int_{0}^{\infty} \tau_{\alpha/2}(\xi) \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \, dy \, d\xi \right]^{2} \leq \]

\[ \leq \left[ \int_{0}^{\tau_{\alpha/2}(\xi)} \left( \int_{-\infty}^{\infty} |h(y - \xi)| |q(y)| \, dy \right) \frac{2p}{2p-2} \, d\xi \right] \frac{2p}{2p-2} \]

\[ \leq \left[ \int_{0}^{\infty} \left( \tau_{\alpha/2}(\xi) \right) \frac{2p}{2p-2} \, d\xi \right] \frac{2p}{2p-2} \cdot \|q\|_{L_{p}(\mathbb{R})} \cdot \|h\|_{L_{2}(\mathbb{R})}^{2}. \quad (20) \]

If \( \alpha \in \left( \frac{1}{p} - \frac{1}{2}, 1 \right) \) then \( \frac{2p}{3p-2} < \frac{1}{1 - \alpha} \) and the first factor on the right-hand side of (21) is finite.

Therefore, for \( \alpha \in \left( \frac{1}{p} - \frac{1}{2}, \frac{3}{2} \right) \), estimates (19) and (21), applied to inequality (9), imply estimate (16), and hence the desired estimate (5). ■
By the standard reasoning (see the proof of theorems in [1]), this lemma yields the following results on the convergence of the spectral expansion \( \sigma_\lambda(x, f) = \sum_{i=1}^{m} \int_{\Omega} \tilde{f}_i(t) u_i(x, t) \, dp(t) \) related to the selfadjoint extension \( \mathcal{H} \) of operator (2).

**Theorem 1 (the estimate for the rate of convergence).** Let \( \mathcal{H} \) be the strictly positive selfadjoint extension of the Schrödinger operator (2) with potential \( q(x) \) that belongs to \( L_p(R) \) for some \( p \in [1, \infty) \). Then for an arbitrary function \( f(x) \) from the Sobolev–Liouville class \( L^2_q(R) \) with the derivation order \( \alpha \) such that \( \alpha \in \left( \frac{1}{2}, 2 \right) \) for \( p \geq 2 \) and \( \alpha \in \left( \frac{1}{2}, \frac{3}{2} \right) \) for \( 1 \leq p < 2 \), the estimate

\[
|\sigma_\lambda(x, f) - f(x)| = \lambda^{\frac{1}{2} - \frac{p}{2}} \cdot o(1),
\]

holds uniformly with respect to \( x \in R \), where \( o(1) \) denotes the quantity that vanishes, as \( \lambda \to \infty \), uniformly with respect to \( x \in R \).

**Theorem 2 (the generalized Bernstein theorem).** Under the assumptions of Theorem 1, the spectral expansion, related to \( \mathcal{H} \), of an arbitrary function \( f(x) \) from the Sobolev–Liouville class \( L^2_q(R) \) with derivation order \( \alpha > 1/2 \) converges absolutely and uniformly\(^4\) over the entire line \( R \).

**Remark.** It follows from representation (7) for the generalized Fourier transforms of a function from the Sobolev–Liouville class that this paper’s lemma and hence Theorems 1 and 2, remain valid for the case in which the potential \( q(x) \) in (2) can be decomposed into the sum \( q(x) = \sum_{j=1}^{s} q_j(x) \) where \( q_j(x) \in L_{p_j}(R) \), \( j = 1, s \), with some \( 1 \leq p_1 < p_2 < \cdots < p_s \leq \infty \).

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**References**


\(^4\)The notion of absolute and uniform convergence of the spectral expansion \( \sigma_\lambda(x, f) \) is introduced in [1].