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ORDINARY DIFFERENTIAL EQUATIONS

Lower and Upper Bounds of the Instability Radii for Families of Polynomials with a Fixed Subset of Coefficients

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INTRODUCTION

The following problem arises in system stabilization by a feedback controller of given structure. Suppose that the characteristic polynomial of the closed system depends on parameters. Is it possible to choose parameter values ensuring that the polynomial is stable? The parameter range may be given in advance [1].

The papers [2, 3] deal with special cases of this problem in which the parameters are the $n+1$ or n (the higher coefficient is fixed to be equal to unity) coefficients of the characteristic polynomial (of degree n) of the closed system, which can vary in some domain of the real space. The main results of these papers are estimates of the distance from a given unstable polynomial to the nearest stable polynomial in some metric on the parameter space (the instability radius).

The present paper solves the more general problem of finding the instability radii of polynomials in the following setting. An arbitrary part of the coefficients is fixed, and the remaining coefficients can vary in some domain of the real space of the corresponding dimension. We describe a set of polynomials for which our estimate of the instability radius coincides with the exact value and obtain an upper bound of the instability radii for some classes of polynomials.

ROBUST INSTABILITY OF POLYNOMIALS

We take a subset $\{i_1, \ldots, i_m\} \subset \{0, \ldots, n\}$, $i_1 < \cdots < i_m$. Let

$$\{j_1, \ldots, j_{n-m+1}\} = \{0, \ldots, n\} \setminus \{i_1, \ldots, i_m\}, \qquad j_1 < \cdots < j_{n-m+1}.$$

We also take m arbitrary positive numbers $q_1, \ldots, q_m \in \mathbb{R}_+$. For the set $(i_1, \ldots, i_m; q_1, \ldots, q_m)$, consider the set $P_n(i_1, \ldots, i_m; q_1, \ldots, q_m)$ of polynomials of the form

$$p(s) = a_0 + a_1 s + \cdots + a_n s^n,$$
$$a_{i_k} = q_k, \qquad a_j \in \mathbb{R}, \qquad j \in \{j_1, \ldots, j_{n-m+1}\}, \qquad a_n \neq 0,$$

identified with vectors $b = (b_1, \ldots, b_{n-m+1})$ $(b_1 = a_{j_1}, \ldots, b_{n-m+1} = a_{j_{n-m+1}})$ of the space $\mathbb{R}^{n-m+1}$ equipped with the norm $\|b\| = \max\{|b_i|\}$.

To indicate the relationship of the polynomial $p(s)$ with a point of the space $\mathbb{R}^{n-m+1}$, we denote $p(s)$ by $p(s,b)$.

A polynomial $p(s,b) \in P_n$ is said to be *stable* if all of its roots in the complex plane lie strictly in the left half-plane and is said to be *unstable* otherwise.

Consider the set

$$\mathscr{P}_n(i_1, \ldots, i_m; q_1, \ldots, q_m; B) = \{p(s,b) = a_0 + a_1 s + \cdots + a_n s^n, \ b \in B\} \tag{1}$$

of polynomials in $P_n(i_1, \ldots, i_m; q_1, \ldots, q_m)$ such that the corresponding subset of coefficients varies in some admissible set $B \subset \mathbb{R}^{n-m+1}$. We say that this family is *robustly unstable* if the polynomials $p(s,b)$ are unstable for all $b \in B$.

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Consider the set

$$I(B)=\bigcup_{\omega=0}^{\infty} I(\omega;B)$$

of values of the polynomials in the family (1), where $I(\omega;B)=\{p(j\omega,b),\ b\in B\}$. The following assertion establishes sufficient conditions for the robust instability of the family (1).

Theorem 1. *Suppose that the set B is connected and, for some $b^0\in B$, the polynomial $p\left(s,b^0\right)$ is unstable and has no pure imaginary roots; moreover, suppose that $a_n\neq 0$ for all $b\in B$. Then the condition $0\notin I(B)$ is sufficient for the robust instability of the family* (1).

The proof is similar to that of Theorem 1 in [2].

ROBUST INSTABILITY OF AN INTERVAL FAMILY OF POLYNOMIALS

Consider the following interval family of polynomials:

$$\begin{aligned}
&\mathscr{P}_n\left(i_1,\ldots,i_m;q_1,\ldots,q_m;B\left(b^0\right)\right)\\
&\quad=\Big\{p(s,b)=a_0+a_1s+\cdots+a_ns^n,\ a_{j_k}=b_k,\ k=1,\ldots,n-m+1,\\
&\qquad a_{i_l}=q_l,\ l=1,\ldots,m,\ \left|a_{j_k}-b_k^0\right|\le\gamma\Big\},\\
&b^0=\left(b_1^0,b_2^0,\ldots,b_{n-m+1}^0\right),\qquad b=\left(b_1,b_2,\ldots,b_{n-m+1}\right).
\end{aligned}\tag{2}$$

Assumption 1. $\{i_1,\ldots,i_m\}\subset\{2,\ldots,n\}$ (i.e., a_0 and a_1 are not fixed).

Assumption 2. $b^0\in\mathbb{R}_+^{n-m+1}$ (i.e., $b_i^0>0$ for all i).

For the polynomial $p\left(s,b^0\right)$, we consider the Tsypkin–Polyak hodograph [1]

$$z\left(\omega,b^0\right)=x_0(\omega)+jy_0(\omega),\qquad 0\le\omega\le\infty,$$

where

$$\begin{aligned}
&x_0(\omega)=g_0(\omega)/r(\omega),\qquad y_0(\omega)=h_0(\omega)/q(\omega),\\
&g_0(\omega)=a_0^0-a_2^0\omega^2+a_4^0\omega^4-\cdots,\qquad h_0(\omega)=a_1^0-a_3^0\omega^2+a_5^0\omega^4-\cdots,\\
&a_0^0=b_1^0,\qquad a_1^0=b_2^0,\qquad a_{j_k}^0=b_k^0\quad(k=3,\ldots,n-m+1),\qquad a_{i_l}^0=q_l,\\
&r(\omega)=\alpha_0+\alpha_2\omega^2+\alpha_4\omega^4+\cdots,\qquad q(\omega)=\alpha_1+\alpha_3\omega^2+\alpha_5\omega^4+\cdots,\\
&\alpha_i=\begin{cases}1 & \text{for}\quad i\in\{j_1,\ldots,j_{n-m+1}\}\\ 0 & \text{for}\quad i\in\{i_1,\ldots,i_m\},\end{cases}\\
&\deg r(\omega)=\max\left(2j_1\left\{\frac{j_1+1}{2}\right\},\ldots,2j_{n-m+1}\left\{\frac{j_{n-m+1}+1}{2}\right\}\right),\\
&\deg q(\omega)=\max\left(2j_1\left\{\frac{j_1}{2}\right\},\ldots,2j_{n-m+1}\left\{\frac{j_{n-m+1}}{2}\right\}\right)-1,
\end{aligned}\tag{3}$$

and $\{\cdot\}$ is the fractional part of a number. In this notation, $p\left(j\omega,b^0\right)=g_0(\omega)+j\omega h_0(\omega)$.

Theorem 2. *Let Assumptions* 1 *and* 2 *be valid. Then the following conditions are sufficient for the robust instability of the family* (2):

(1°) *the polynomial $p\left(s,b^0\right)$ has no pure imaginary roots*;

(2°) $b_{n-m+1}^0>\gamma$ *if* $j_{n-m+1}=n$;

(3°) $b_1^0>\gamma$;

(4°) *the hodograph $z\left(\omega,b^0\right)$ does not meet the square with vertices $(\pm\gamma,\pm\gamma)$ as ω varies from* 0 *to* ∞.

Proof. Note that if we represent the value of an arbitrary polynomial in the family (2) at the point $j\omega$ in the form

$$p(j\omega, b) = g(\omega) + j\omega h(\omega), \qquad g(\omega) = a_0 - a_2\omega^2 + a_4\omega^4 - \cdots,$$
$$h(\omega) = a_1 - a_3\omega^2 + a_5\omega^4 - \cdots,$$

then the inequalities

$$\underline{g}(\omega) \le g(\omega) \le \bar{g}(\omega), \qquad \underline{h}(\omega) \le h(\omega) \le \bar{h}(\omega)$$

are satisfied for $\omega \ge 0$, where

$$\begin{aligned}
\underline{g}(\omega) &= \left(a_0^0 - \alpha_0\gamma\right) - \left(a_2^0 + \alpha_2\gamma\right)\omega^2 + \left(a_4^0 - \alpha_4\gamma\right)\omega^4 - \cdots,\\
\bar{g}(\omega) &= \left(a_0^0 + \alpha_0\gamma\right) - \left(a_2^0 - \alpha_2\gamma\right)\omega^2 + \left(a_4^0 + \alpha_4\gamma\right)\omega^4 - \cdots,\\
\underline{h}(\omega) &= \left(a_1^0 - \alpha_1\gamma\right) - \left(a_3^0 + \alpha_3\gamma\right)\omega^2 + \left(a_5^0 - \alpha_5\gamma\right)\omega^4 - \cdots,\\
\bar{h}(\omega) &= \left(a_1^0 + \alpha_1\gamma\right) - \left(a_3^0 - \alpha_3\gamma\right)\omega^2 + \left(a_5^0 + \alpha_5\gamma\right)\omega^4 - \cdots
\end{aligned}$$

Therefore, for each ω, the polynomial $p(j\omega, b)$ can vary in the rectangle with vertices

$$z_1 = \underline{g}(\omega) + j\omega\underline{h}(\omega), \qquad z_2 = \underline{g}(\omega) + j\omega\bar{h}(\omega),$$
$$z_3 = \bar{g}(\omega) + j\omega\bar{h}(\omega), \qquad z_4 = \bar{g}(\omega) + j\omega\underline{h}(\omega).$$

Therefore, the range of the family (2) for each ω is the rectangle

$$I\left(\omega; B\left(b^0\right)\right) = \left\{z = x + jy : \ |x - g_0(\omega)| \le \gamma r(\omega), \ |y - \omega h_0(\omega)| \le \gamma\omega q(\omega)\right\}.$$

Further, note that the fact that the hodograph $z\left(\omega, b^0\right)$ does not meet with the square with vertices $(\pm\gamma, \pm\gamma)$ is equivalent to the system of inequalities

$$\begin{aligned}
&|x_0(\omega)| > \gamma, \qquad |y_0(\omega)| > \gamma,\\
&|x_0(\omega)| > \gamma, \qquad |y_0(\omega)| \le \gamma,\\
&|x_0(\omega)| \le \gamma, \qquad |y_0(\omega)| > \gamma \quad \text{for all} \quad \omega \ge 0,
\end{aligned} \tag{4}$$

since $g_0(0) = b_1^0 > \gamma$. Furthermore, the condition $0 \in I\left(B\left(b^0\right)\right)$ is equivalent to the existence of an $\omega^* > 0$ such that $0 \in I\left(\omega^*; B\left(b^0\right)\right)$. [One has $0 \notin I\left(0; B\left(b^0\right)\right)$, since otherwise the inequality $|g_0(0)| < \gamma$ would be valid, while $g_0(0) = b_1^0 > \gamma$ by assumption.] But then ω^* satisfies the condition

$$\left|g_0\left(\omega^*\right)\right| \le \gamma r\left(\omega^*\right), \qquad \left|h_0\left(\omega^*\right)\right| \le \gamma q\left(\omega^*\right),$$

or

$$\left|x_0\left(\omega^*\right)\right| \le \gamma, \qquad \left|y_0\left(\omega^*\right)\right| \le \gamma.$$

Consequently, condition (4) is equivalent to $0 \notin I\left(B\left(b^0\right)\right)$. But then the family (2) is robustly unstable by Theorem 1. The proof of the theorem is complete.

A LOWER BOUND FOR THE INSTABILITY RADIUS

For a given set

$$\left(i_1, \ldots, i_m; q_1, \ldots, q_m\right),$$

by S and U we denote the set of points in $\mathbb{R}^{n-m+1}$ corresponding to stable and unstable polynomials of the set $P_n\left(i_1, \ldots, i_m; q_1, \ldots, q_m\right)$, respectively. The instability radius of an unstable polynomial $p(s, b)$ is defined as $R_U(b) = \inf_{c \in S} \|b - c\|$.

Theorem 2 not only establishes sufficient conditions for the robust instability of the family (2) but also permits one to estimate the instability radius $R_U\left(b^0\right)$ of the polynomial $p\left(s, b^0\right)$ with positive coefficients.

Let Assumption 1 be valid. For the polynomial $p(s, b^0)$, we consider the function

$$\psi(\omega) = \max\Big\{|x_0(\omega)|\,,\ \ |y_0(\omega)|\Big\}$$

defined for $\omega \geq 0$. [The functions $x_0(\omega)$ and $y_0(\omega)$ are given by (3).] By $\omega_1, \ldots, \omega_k$ we denote all positive roots of the equations

$$g_0(\omega)q(\omega) - h_0(\omega)r(\omega) = 0, \qquad g_0(\omega)q(\omega) + h_0(\omega)r(\omega) = 0, \tag{5a}$$

$$g_0'(\omega)r(\omega) - g_0(\omega)r'(\omega) = 0, \qquad h_0'(\omega)q(\omega) - h_0(\omega)q'(\omega) = 0. \tag{5b}$$

We set

$$\Phi\left(b^0\right) = \min_{1\leq i\leq k} \psi\left(\omega_i\right). \tag{6}$$

Theorem 3. *Let the polynomial*

$$p\left(s, b^0\right) = a_0^0 + a_1^0 s + \cdots + a_n^0 s^n \tag{7}$$

with positive coefficients, which belongs to the set $P_n\left(i_1, \ldots, i_m; q_1, \ldots, q_m\right)$ *under Assumption* 1, *be unstable and have no pure imaginary roots. Then the estimate*

$$R_U\left(b^0\right) \geq \gamma\left(b^0\right) \tag{8}$$

is valid, where

$$\gamma\left(b^0\right) = \min\left\{a_0^0, a_0^0 + \delta_{n,j_{n-m+1}}\left(a_n^0 - a_0^0\right), \Phi\left(b^0\right)\right\} \tag{9}$$

and δ_{ij} *is the Kronecker delta.*

Proof. Theorem 2 provides the following lower bound for the instability radius of the polynomial $p(s, b^0)$:

$$R_U\left(b^0\right) \geq \min\left\{\gamma^*, a_0^0 + \delta_{n,j_{n-m+1}}\left(a_n^0 - a_0^0\right), a_0^0\right\}, \tag{10}$$

where γ^* is the size (half-length of the side) of the maximum square $\{|x| \leq \gamma^*,\ |y| \leq \gamma^*\}$ inscribed in the Tsypkin–Polyak hodograph $z(\omega, b^0)$ for the polynomial $p(s, b^0)$.

On the complex plane, we introduce a metric as follows:

$$\varrho\left(z_1, z_2\right) = \max\left\{|\operatorname{Re} z_1 - \operatorname{Re} z_2|\,,\ \ |\operatorname{Im} z_1 - \operatorname{Im} z_2|\right\}.$$

Then γ^* can be found as the distance from the origin to the nearest point $z(\omega)$ of the hodograph; i.e., $\gamma^* = \min_{0\leq\omega\leq\infty} \varrho(0, z(\omega))$.

Since $z(\omega) = x(\omega) + iy(\omega)$, it follows that finding γ^* can be reduced to the minimization of the function $\psi(\omega) = \max\left\{|x_0(\omega)|, |y_0(\omega)|\right\}$ on the interval $[0, \infty)$. Note that the function $\psi(\omega)$ is positive and differentiable everywhere possibly except for the points such that $|x_0(\omega)| = |y_0(\omega)|$. Therefore,

$$\min_{0\leq\omega<\infty} \psi(\omega) = \min\left\{\psi(0), \psi(\infty), \psi\left(\omega_1\right), \ldots, \psi\left(\omega_k\right)\right\},$$

where $\psi(\infty) = \lim_{\omega\to\infty} \psi(\omega)$ and $\omega_1, \ldots, \omega_k$ are roots of Eq. (5). One of the following two conditions is satisfied at the points ω_i:

(1°) $x_0(\omega) = y_0(\omega)$ or $x_0(\omega) = -y_0(\omega)$ [i.e., $|x_0(\omega)| = |y_0(\omega)|$];

(2°) $x_0'(\omega) = 0$ or $y_0'(\omega) = 0$.

The first case corresponds to Eq. (5a) and the second case, to Eq. (5b).

Note that

$$\psi(\infty) = \lim_{\omega\to\infty} \psi(\omega) = \infty \quad \text{if} \quad n-1, n \in \left\{i_1, \ldots, i_m\right\};$$

otherwise, $\psi(\infty) = \max\left\{a_n^0, a_{n-1}^0\right\}$. Now, by taking into account the relation $\psi(0) = \max\left\{a_0^0, a_1^0\right\}$ and formula (6), we obtain the estimate (8), which completes the proof.

Now suppose that Assumption 1 is not satisfied for the set $P_n(i_1, \ldots, i_m; q_1, \ldots, q_m)$. Then the following cases are possible:

(1°) $i_1 = 0$, and the set $\{i_1, \ldots, i_m\}$ does not contain the entire set of odd or even indices;
(2°) $i_1 = 1$, and the set $\{i_1, \ldots, i_m\}$ does not contain the entire set of odd indices;
(3°) the set $\{i_1, \ldots, i_m\}$ contains the set of all even indices;
(4°) $i_1 = 0$, and the set $\{i_1, \ldots, i_m\}$ contains the set of all odd indices;
(5°) $i_1 = 1$, and the set $\{i_1, \ldots, i_m\}$ contains the set of all odd indices.

Theorem 4. *Let a polynomial* (7) *with positive coefficients belong to the set*

$$P_n(i_1, \ldots, i_m; q_1, \ldots, q_m),$$

be unstable, and have no pure imaginary roots. Furthermore, suppose that the set $\{i_1, \ldots, i_m\}$ *satisfies one of conditions* (1°)–(5°). *Then the estimate* (8) *is valid, where*

$$\gamma\left(b^0\right) = \min\left\{\Phi\left(b^0\right) + \delta_{n,j_{n-m+1}}\left(a_n^0 - \Phi\left(b^0\right)\right), \Phi\left(b^0\right)\right\}$$

in case (1°); $\gamma(b^0)$ *has the form* (9) *in case* (2°);

$$\gamma\left(b^0\right) = \min\left\{\left|\frac{h_0(\omega_1)}{q(\omega_1)}\right|, \ldots, \left|\frac{h_0(\omega_l)}{q(\omega_l)}\right|, \left|\frac{h_0(\omega_1)}{q(\omega_1)}\right| + \delta_{n,j_{n-m+1}}\left(a_n^0 - \left|\frac{h_0(\omega_1)}{q(\omega_1)}\right|\right)\right\}, \tag{11}$$

where $\omega_1, \ldots, \omega_l$ *are the positive real roots of the polynomial* $g_0(\omega)$, *and if the polynomial* $g_0(\omega)$ *has no real roots, then*

$$\gamma\left(b^0\right) = \begin{cases} \infty & \text{for } n \in \{i_1, \ldots, i_m\} \\ a_n^0 & \text{for } n \notin \{i_1, \ldots, i_m\} \end{cases} \tag{12}$$

in case (3°);

$$\gamma\left(b^0\right) = \min\left\{\left|\frac{g_0(\omega_1)}{r(\omega_1)}\right|, \ldots, \left|\frac{g_0(\omega_t)}{r(\omega_t)}\right|, \left|\frac{g_0(\omega_1)}{r(\omega_1)}\right| + \delta_{n,j_{n-m+1}}\left(a_n^0 - \left|\frac{g_0(\omega_1)}{r(\omega_1)}\right|\right)\right\},$$

where $\omega_1, \ldots, \omega_t$ *are the positive real roots of the polynomial* $h_0(\omega)$, *and if the polynomial* $h_0(\omega)$ *has no real roots, then* $\gamma(b^0)$ *is given by* (12) *in case* (4°);

$$\gamma\left(b^0\right) = \min\left\{\left|\frac{g_0(\omega_1)}{r(\omega_1)}\right|, \ldots, \left|\frac{g_0(\omega_v)}{r(\omega_v)}\right|, \left|\frac{g_0(\omega_1)}{r(\omega_1)}\right| + \delta_{n,j_{n-m+1}}\left(a_n^0 - \left|\frac{g_0(\omega_1)}{r(\omega_1)}\right|\right), a_0^0\right\},$$

where $\omega_1, \ldots, \omega_v$ *are the positive roots of the polynomial* $h_0(\omega)$ *in case* (5°).

Proof. We perform the proof for each of the above-mentioned cases (1°)–(5°).

In case (1°), the robust instability of the interval family (2) is provided by items (1°), (2°), and (4°) of Theorem 2, since the coefficient a_0^0 is fixed. It is only necessary to verify the condition $0 \notin I(0; B(b^0))$. Indeed,

$$I\left(0; B\left(b^0\right)\right) = \{z = x + jy : \ |x - g_0(0)| \le 0, \ |y| \le 0\}.$$

In this case, $0 \in I(0; B(b^0))$ if and only if $g_0(0) = 0$; but $g_0(0) = a_0^0 > 0$. Therefore, $0 \notin I(0; B(b^0))$. By taking into account the proof of Theorem 3 and the fact that, in this case, the hodograph $z(\omega, b^0)$ is defined for $\omega > 0$, we have the desired assertion.

In case (2°), the robust instability of the interval family (2) is provided by assumptions (1°)–(4°) of Theorem 2. Just as in case (1°) it is necessary only to verify the condition $0 \notin I(0; B(b^0))$. Indeed, $I(0; B(b^0)) = \{z = x + jy : \ |x - g_0(0)| \le \gamma, \ |y| \le 0\}$. In this case, we have $0 \in I(0; B(b^0))$ if and only if $|g_0(0)| \le \gamma$; but $g_0(0) = a_0^0 = b_1^0 > \gamma$. Therefore, $0 \notin I(0; B(b^0))$, and consequently, just as in case (1°), we have the desired assertion.

In case (3°), the set $I\left(B\left(b^0\right)\right)$ for the interval family (2) with $\omega > 0$ is a segment:

$$I\left(\omega; B\left(b^0\right)\right) = \{z = x + jy: \ |x - g_0(\omega)| \le 0, \ |y - \omega h_0(\omega)| \le \gamma\omega q(\omega)\}.$$

If $\omega = 0$, then the set $I\left(0; B\left(b^0\right)\right)$ is a singleton. Then the condition $0 \in I\left(\omega; B\left(b^0\right)\right)$ with $\omega > 0$ is equivalent to the system

$$|g_0(\omega)| = 0, \qquad |h_0(\omega)/q(\omega)| \le \gamma. \tag{13}$$

The inclusion $0 \in I\left(0; B\left(b^0\right)\right)$ is valid if and only if $g_0(0) = 0$; but $g_0(0) = a_0^0 > 0$. Consequently, $0 \notin I\left(0; B\left(b^0\right)\right)$. Therefore, if $\gamma < \gamma\left(b^0\right)$ is given by (11), where $\omega_1, \ldots, \omega_l$ are the positive real roots of the polynomial $g_0(\omega)$, then the interval family (2) is robustly instable by Theorem 1. Note that the polynomials $g_0(\omega)$ and $h_0(\omega)$ cannot have common roots, since, by assumption, the polynomial $p\left(s, b^0\right)$ has no pure imaginary roots.

If the polynomial $g_0(\omega)$ has no real roots, then condition (13) fails for any $\omega > 0$; consequently, relation (12) is valid.

In case (4°), the set $I\left(B\left(b^0\right)\right)$ for the interval family (2) with $\omega > 0$ is the segment

$$I\left(\omega; B\left(b^0\right)\right) = \{z = x + jy: \ |x - g_0(\omega)| \le \gamma r(\omega), \ |y - \omega h_0(\omega)| \le 0\}.$$

If $\omega = 0$, then the set $I\left(0; B\left(b^0\right)\right)$ is a singleton. Then the condition $0 \in I\left(\omega; B\left(b^0\right)\right)$ with $\omega > 0$ is equivalent to the system

$$|h_0(\omega)| = 0, \qquad |g_0(\omega)/r(\omega)| \le \gamma,$$

and, just as in case (3°), we obtain $0 \notin I\left(0; B\left(b^0\right)\right)$.

The subsequent consideration of this case is similar to that of case (3°) with h_0, q, g_0, and ω_l replaced by g_0, r, h_0, and ω_t, respectively.

Case (5°) can be considered by analogy with case (4°). The proof of the theorem is complete.

THE INSTABILITY RADII FOR A CLASS OF POLYNOMIALS

The following theorem shows that our estimate for the instability radii is attained for some sets of polynomials.

Theorem 5. *Let the assumptions of Theorem* 3 *be valid, and, in addition, let*

$$\gamma\left(b^0\right) = \psi\left(\omega^*\right) < \min\left\{a_0^0, a_0^0 + \delta_{n,j_{n-m+1}}\left(a_n^0 - a_0^0\right)\right\},$$

where $\omega^* = \arg\min\left(\psi\left(\omega_i\right)\right)$, *i.e.*, $\psi\left(\omega^*\right) = \Phi\left(b^0\right)$, *and moreover*,

(1°) $x\left(\omega^*\right) = y\left(\omega^*\right) > 0$;

(2°) *there exists an* $\varepsilon^* > 0$ *such that*

$$\Delta_{0\le\omega<\infty} \arg\left(z(\omega) - \left(\gamma\left(b^0\right) + \varepsilon\right)(1 + i)\right) = \pi n/2,$$
$$0 < \min\left\{a_0^0, a_0^0 + \delta_{n,j_{n-m+1}}\left(a_n^0 - a_0^0\right)\right\} - \gamma\left(b^0\right) - \varepsilon$$

for each $\varepsilon \in (0, \varepsilon^*)$.

Then $R_U\left(b^0\right) = \gamma\left(b^0\right)$.

Proof. Consider the polynomial

$$\begin{aligned} p\left(s, \tilde{b}\right) = {} & \left(a_0^0 - \alpha_0\gamma\left(b^0\right)\right) + \left(a_1^0 - \alpha_1\gamma\left(b^0\right)\right)s + \left(a_2^0 + \alpha_2\gamma\left(b^0\right)\right)s^2 \\ & + \left(a_3^0 + \alpha_3\gamma\left(b^0\right)\right)s^3 + \cdots + \left(a_{4k}^0 - \alpha_{4k}\gamma\left(b^0\right)\right)s^{4k} \\ & + \left(a_{4k+1}^0 - \alpha_{4k+1}\gamma\left(b^0\right)\right)s^{4k+1} + \left(a_{4k+2}^0 + \alpha_{4k+2}\gamma\left(b^0\right)\right)s^{4k+2} \\ & + \left(a_{4k+3}^0 + \alpha_{4k+3}\gamma\left(b^0\right)\right)s^{4k+3} + \cdots \end{aligned}$$

It follows from the assumptions of the theorem that the hodograph $z\left(\omega,\tilde{b}\right)$ can be obtained from the hodograph $z\left(\omega,b^0\right)$ by the parallel translation along the bisector of the first quadrant by the vector $\{-\gamma\left(b^0\right),-\gamma\left(b^0\right)\}$. The hodograph $z\left(\omega,\tilde{b}\right)$ satisfies the relations

$$\operatorname{Re} z\left(\omega,\tilde{b}\right)=\operatorname{Re} z\left(\omega,b^0\right)-\gamma\left(b^0\right),\qquad \operatorname{Re} z\left(\omega,\tilde{b}\right)=\operatorname{Re} z\left(\omega,b^0\right)-\gamma\left(b^0\right).$$

Let us show that $\tilde{b}\in\partial S$ [i.e., the polynomial $p\left(s,\tilde{b}\right)$ belongs to the boundary of the domain of stable polynomials]. Indeed, we set $\tilde{\gamma}=\gamma\left(b^0\right)+\varepsilon$ [where $\varepsilon>0$ is the arbitrary sufficiently small number occurring in assumption (2°) of the theorem] and consider the polynomial

$$\begin{aligned}p\left(s,b^*\right)=&\left(a_0^0-\alpha_0\tilde{\gamma}\right)+\left(a_1^0-\alpha_1\tilde{\gamma}\right)s+\left(a_2^0+\alpha_2\tilde{\gamma}\right)s^2+\left(a_3^0+\alpha_3\tilde{\gamma}\right)s^3+\cdots\\&+\left(a_{4k}^0-\alpha_{4k}\tilde{\gamma}\right)s^{4k}+\left(a_{4k+1}^0-\alpha_{4k+1}\tilde{\gamma}\right)s^{4k+1}+\left(a_{4k+2}^0+\alpha_{4k+2}\tilde{\gamma}\right)s^{4k+2}\\&+\left(a_{4k+3}^0+\alpha_{4k+3}\tilde{\gamma}\right)s^{4k+3}+\cdots\end{aligned}$$

The hodograph $z\left(\omega,b^*\right)$ satisfies the relations

$$\operatorname{Re} z\left(\omega,b^*\right)=\operatorname{Re} z\left(\omega,b^0\right)-\gamma\left(b^0\right)-\varepsilon,\qquad \operatorname{Re} z\left(\omega,b^*\right)=\operatorname{Re} z\left(\omega,b^0\right)-\gamma\left(b^0\right)-\varepsilon.$$

The hodograph $z\left(\omega,b^*\right)$ lies in some quadrant of the complex plane if and only if the Mikhailov hodograph $p\left(i\omega,b^*\right)$ lies in the same quadrant. This, together with the Mikhailov criterion [4, p. 106] and assumption (2°) of the theorem, implies that the polynomial $p\left(s,b^*\right)$ is stable. Since $\varepsilon>0$ is an arbitrarily small number, we have $p\left(s,\tilde{b}\right)\in\partial S$. But since $\varrho\left(b^0,\tilde{b}\right)=\gamma\left(b^0\right)\le R\left(b^0\right)$, it follows that $\gamma\left(b^0\right)=R\left(b^0\right)$, and the proof of the theorem is complete.

AN UPPER BOUND FOR THE INSTABILITY RADII

The following theorem permits one to find an upper bound for the instability radius of polynomials (7) in the set $\mathscr{P}_n\left(n;1;\mathbb{R}_+^n\right)$ [i.e., $a_i^0>0$, $i=0,1,\ldots,n-1$, $a_n^0=1$, and $b^0=\left(b_1^0,\ldots,b_n^0\right)=\left(a_0^0,\ldots,a_{n-1}^0\in\mathbb{R}_+^n\right)$].

By $\eta\left(b^0\right)$ we denote the maximum real part of the roots of the polynomial $p\left(s,b^0\right)$; i.e., $\eta\left(b^0\right)=\max\operatorname{Re}s_i$, $i=1,\ldots,n$, where the s_i are the roots of the polynomial $p\left(s,b^0\right)$.

Theorem 6. *Let $p\left(s,b^0\right)\in\mathscr{P}_n\left(n;1;\mathbb{R}_+^n\right)$ be an unstable polynomial; moreover, let $\eta\left(b^0\right)>0$. Then the instability radius of this polynomial can be estimated as*

$$R_U\left(b^0\right)\le\max\left\{\left|\frac{p^{(i)}\left(\eta\left(b^0\right),b^0\right)}{i!}-a_i^0\right|\right\},\qquad i=0,1,\ldots,n-1,$$

where $p^{(i)}\left(\eta\left(b^0\right),b^0\right)$ is the value of the ith derivative of the polynomial $p\left(s,b^0\right)$ with respect to s at the point $s=\eta\left(b^0\right)$.

Proof. We apply the transformation $s=\lambda+\eta\left(b^0\right)$ to the polynomial (7). Then we obtain

$$\tilde{p}\left(\lambda,\tilde{b}\right)=p\left(\lambda+\eta\left(b^0\right),b^0\right)=\tilde{a}_0+\tilde{a}_1\lambda+\cdots+\tilde{a}_{n-1}\lambda^{n-1}+\lambda^n,$$

where

$$\tilde{a}_i=\frac{1}{i!}\left.\frac{d^ip\left(s,b^0\right)}{ds^i}\right|_{s=\eta\left(b^0\right)}.$$

Since the transformation $s=\lambda+\eta\left(b^0\right)$ corresponds to the shift of the imaginary axis to the right by $\eta\left(b^0\right)$ and the transformed polynomial $\tilde{p}\left(\lambda,\tilde{b}\right)$ has roots lying on the imaginary axis and possibly in the left half-plane, we find that the polynomial $\tilde{p}\left(\lambda, \tilde{b}\right)$ belongs to the boundary ∂S of stable polynomials. Consequently, $R_U\left(b^0\right) \le \varrho\left(b^0, \tilde{b}\right)$, but

$$\varrho\left(b^0, \tilde{b}\right) = \max\left\{\left|p^{(i)}\left(\eta\left(b^0\right), b^0\right)/i! - a_i^0\right|\right\}, \qquad i = 0, 1, \ldots, n-1.$$

The proof of the theorem is complete.

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