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SHORT COMMUNICATIONS

Necessary Conditions for the Invertibility of Linear Discrete Dynamical Systems

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Abstract—We study conditions under which the inversion problem for a linear discrete dynamical system is well posed, i.e., its solution exists and is unique.

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Consider the linear discrete dynamical system with constant matrices

$$x^{t+1} = Ax^t + B\xi^t, \qquad y^t = Cx^t, \qquad t = 0, 1, 2, \ldots, \tag{1}$$

where the variable parameters x^t, y^t, and ξ^t and constant matrices A, B, and C have the dimensions

$$x \in \mathbb{R}^n, \qquad \xi \in \mathbb{R}^m, \qquad y \in \mathbb{R}^l, \qquad A \in \mathbb{R}^{n\times n}, \qquad B \in \mathbb{R}^{n\times m}, \qquad C \in \mathbb{R}^{l\times n}. \tag{2}$$

Below we use the standard control theory notation of the input ξ, the output y, and the state vector x. In addition, we assume that the matrices B and C have full rank; i.e., $\mathrm{Rk}\, B = m \le n$ and $\mathrm{Rk}\, C = l \le n$.

Consider the problem on the invertibility of that system. It is treated as follows: the problem is to obtain conditions for the inversion problem for the linear discrete dynamical system (1) to be well posed, i.e., for its solution to exist and be unique. The inversion problem is understood as the problem of reconstruction of the unknown input ξ on the basis of known output y for the case in which the matrices A, B, and C are known as well. The solution methods for the inversion problem are not considered in the present paper.

The considered problem can be posed as follows. Consider system (1) with the same matrices A, B, and C and with output $\hat{y}$,

$$\hat{x}^{t+1} = A\hat{x}^t + B\hat{\xi}^t, \qquad \hat{y}^t = C\hat{x}^t, \qquad t = 0, 1, 2, \ldots \tag{3}$$

The problem on invertibility conditions is stated for this system as follows: in what cases do the inputs of systems (1) and (3) with equal constant parameters and with possibly different state vectors coincide for coinciding values of outputs (i.e., for $y^t = \hat{y}^t$)? In this connection, we are interested either in the total coincidence of quantities starting from some time moment, i.e., $\xi^t - \hat{\xi}^t = 0$ for $t \ge t^*$, or in the fact that their difference asymptotically tends to zero, i.e., $\xi^t - \hat{\xi}^t \to 0$ as $t \to \infty$. Therefore, the considered problem naturally splits into two subproblems. Part of the argument is the same for both cases.

First, we note that it is reasonable to consider the invertibility problem only for systems for which the condition $m \le l$ is satisfied, since otherwise, i.e., in the case where the dimension of the input exceeds the dimension of the output, the inversion problem is necessarily ill posed and has a nonunique solution. Let us justify this fact. Suppose the contrary: $m > l$. Following a simple argument, we consider the homogeneous problem, which is understood as the inversion problem for a system with zero output. [In other words, we consider the difference of two systems (1) and (3) of the same structure with coinciding inputs.] By using the notation $x^t - \hat{x}^t = \Delta x^t$ and $\xi^t - \hat{\xi}^t = \Delta\xi^t$, we obtain the problem

$$\Delta x^{t+1} = A\Delta x^t + B\Delta\xi^t, \qquad C\Delta x^t = 0, \qquad t = 0, 1, 2, \ldots \tag{4}$$

If the homogeneous inversion problem has only the trivial solution, then the solution of the inhomogeneous inversion problem is unique. By multiplying Eq. (4) by C and by imposing the condition from the second equation in (4), we obtain an equation for the solution of the homogeneous inversion problem:

$$CA\Delta x^t + CB\Delta\xi^t = 0. \tag{5}$$

The solvability of this equation for $\Delta\xi^t$ is determined by the rank of the matrix CB. If $\operatorname{Rk} C = l < m = \operatorname{Rk} B$, then the number l of equations in system (5) is less than the number m of unknowns. Such a system cannot have a unique solution. Therefore, in this case, the homogeneous system either has no solution or has a nonunique solution; consequently, the requirement $m \le l$ is the first necessary condition of invertibility.

Let us return to the investigation of the invertibility problem for the case in which $m \le l$. We perform the Z-transformation for system (4) with zero initial conditions and rewrite it in the form

$$(zI - A)\Delta X - B\Delta\Xi = 0, \qquad C\Delta X = 0, \qquad t = 0, 1, 2, \ldots, \tag{6}$$

or in the matrix form

$$\begin{pmatrix} zI - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} \Delta X \\ \Delta\Xi \end{pmatrix} = 0. \tag{7}$$

Here ΔX and $\Delta\Xi$ stand for the Z-transforms of the state vector of the homogeneous system Δx and its unknown input $\Delta\xi$, respectively. A nontrivial solution of this system, if any, implies the nonuniqueness of the solution of the inversion problem. Note that, in the above-described case $l < m$, this system has infinitely many solutions among which there are unstable ones, which justifies the necessity of the condition $m \le l$.

An additional condition of the invertibility in the sense of the total coincidence of inputs (i.e., for the first subproblem) is given by the absence of nontrivial solutions of system (7), which is the case if the Rosenbrock matrix

$$R(z) = \begin{pmatrix} zI - A & -B \\ C & 0 \end{pmatrix} \tag{8}$$

of the system has no invariant zero [that is, a value of z at which the rank of the Rosenbrock matrix falls, $\operatorname{Rk} R(z) < n + m$].

This also follows from the fact that if $z = z^*$ is an invariant zero of the Rosenbrock matrix, then for a given $z = z^*$, a nonzero solution of the homogeneous inversion problem can be written out in the closed form

$$\Delta x^t = (z^*)^t \Delta x^0, \qquad \Delta\xi^t = (z^*)^t \Delta\xi^0,$$

where Δx^0, $\Delta\xi^0$ is a nontrivial solution of the homogeneous system with the Rosenbrock matrix for $z = z^*$:

$$\begin{pmatrix} z^*I - A & -B \\ C & 0 \end{pmatrix} \begin{pmatrix} \Delta x^0 \\ \Delta\xi^0 \end{pmatrix} = 0,$$

which exists by virtue of the incomplete rank of the Rosenbrock matrix for $z = z^*$. This can be verified by a straightforward substitution.

Let us return to the second subproblem. If there exist pairwise distinct zeros $z_1, z_2, \ldots, z_k$ diminishing the rank of the Rosenbrock matrix (one can show that $k \le n - m$ in this case), then for each of them, there exists a particular solution of the system, which, after the inverse Z-transformations, acquires the form (this can be verified by a straightforward computation) $\Delta\xi^t = \Delta\xi_i^0 z_i^t$.

It tends to zero if the values of the invariant zeros z_i, $i = 1, \ldots, k$, are stable, i.e., their absolute values are less than unity. This provides an additional necessary condition for the invertibility of the second subproblem, i.e., for the invertibility in the sense of the asymptotic convergence of inputs.

In the special case of a scalar input and a scalar output (i.e., for $l = m = 1$), this can be explained as follows. Let the invariant zeros of the Rosenbrock matrix exist but be stable. Since, in this case, the determinant of the Rosenbrock matrix coincides with the numerator of the transfer function of the system

$$W(z) = C(zI - A)^{-1}B = \beta(z)/\alpha(z),$$

i.e., $\det R(z) = \beta(z)$, which, in turn, is a characteristic polynomial of the zero dynamics [1, p. 67], it follows that the zero dynamics is stable. Hence it follows that the convergence $\Delta x^t \to 0$ as $t \to \infty$ takes place when moving along trajectories $C\Delta x^t = 0$. Since Δx tends to zero, and, by assumption, the matrix B has full rank, it follows from the first equation in system (4) that $\Delta\xi$ tends to zero as well. Hence it follows that the invertibility of the discrete system holds with asymptotic accuracy. We have thereby proved the following assertion.

Assertion. *The invertibility problem for the discrete system* (1) *is well posed only if* $m \le l$ *and the invariant zeros of the Rosenbrock matrix* (8) *of this system are either absent or stable. The absence of invariant zeros is necessary for the uniqueness of the solution of the inversion problem on the entire time interval, and the stability of invariant zeros of the Rosenbrock matrix is necessary for the convergence of all possible solutions of the inversion problem to each other as* $t \to \infty$.

Remark. The requirements of the assertion imply the controllability (stabilization) and observability (detectability) of the inverted system.

Indeed, since the first block column of the Rosenbrock matrix (8) has the same rank as that of the Rosenbrock observability matrix

$$R_{\text{obs}}(z) = \begin{pmatrix} zI - A \\ C \end{pmatrix}$$

of system (1), it follows that the requirement of absence of invariant zeros provides the full rank of the Rosenbrock observability matrix for all z, that is, the observability of system (1), and the requirement of the stability of invariant zeros implies the stability of the nonobserved subsystem of system (1), which provides its detectability. Similarly, the first block row of the Rosenbrock matrix (8) corresponds to the Rosenbrock controllability matrix

$$R_{\text{cont}}(z) = \begin{pmatrix} zI - A & | & B \end{pmatrix}$$

of system (1), and the absence of invariant zeros provides its full rank for all z and the controllability of the system. From the presence of stable invariant zeros, we obtain the stability of the noncontrolled subsystem of system (1), which implies the stabilization of the inverted system (1).

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