Narrow beams in scattering media: the advanced small-angle approximation

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The problem of the propagation of narrow radiation beams in a scattering medium is considered. The previously formulated small-angle approximation solution accounting for the path length spread is further developed. The numerical scheme for practical calculations is implemented, and the simulation results are presented and discussed. Applicability of the new solution to certain problems of optical communications and data transfer techniques is shown. © 2011 Optical Society of America

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1. INTRODUCTION

Investigation of narrow beams in scattering media [1–10] is important both from the theoretical and practical points of view.

The field of an infinitesimal point-directed (PD) source is an elementary fundamental solution (Green function) of the radiative transfer equation (RTE). Any RTE solution, i.e., the radiation field created in a scattering medium by an arbitrary set of the sources, is some linear combination of the elementary fields of the PD sources. Since the invention of laser sources, the problem has acquired large practical importance.

The radiation field of the PD source has a rather complicated structure even in a uniform homogeneous scattering medium. In particular, spatial and angular singularities are present in the scattered field. A practical solution of the PD problem for the RTE is not simple, and there have been few attempts at its rigorous solution. Tissendorf [8] applied the formalism of quantum mechanics and obtained the explicit expression for the evolutionary operator of the problem. However, numerical results presented there are limited to the relatively simple case of isotropic scattering.

Helliwell [6] attempted to solve the problem immediately with an implicit finite-difference scheme of the Carlson type [11]. Nothing was done for regularization of the singularities of the radiation field, and this probably was the cause of the low quality of the numerical results, which was pointed out by the author.

Most authors typically make some assumptions, more or less simplifying the problem statement and its numerical solutions. In optically thin media, the solution is well approximated by the single- or double-scattered radiation field [12], which can be found by immediate numerical integration.

Many important natural media, such as ocean water [2,13], clouds, fog, and biological liquids and tissues [14], are characterized by highly elongated phase scattering functions. The small-angle approximation (SAA) is effectively exploited in this case. The SAA solutions for narrow beams of elementary particles have been known since the middle of the 20th century [15]. For backscattered radiation, the cumulative-forward–single-back (CFSB) approximation [15] is used, in particular, for narrow beams in a scattering medium [17,18].

Many authors make additional simplifying assumptions, such as a Gaussian angular and spatial distribution of the radiances [5,7,19,20]. This is partly justified by the random nature of the multiple scattering, which could lead to the Gaussian solution in the statistical limit. These assumptions, with the model scattering functions of the Gaussian [7,21,22], Yukawa [2], and other [23] types, result in greatly simplified solutions, from which elementary estimates in the closed analytical form can often be derived [3,23].

On the other hand, the accuracy of the commonly used approximations is often insufficient. Many of the known SAA solutions of the RTE problem account for the path length spread [24,25] rather crudely or ignore it completely. For a complete review of most important papers, the reader is referred to [26]. This causes a systematic error in the solution, which grows with the optical path length of the radiation, and prevents the application of the SAA solution to certain nonstationary RTE problems, where the time dispersion is the principal effect [24,27]. If the SAA solution is used in the scheme of rigorous calculation of the radiation field as the sum of the anisotropic and diffuse parts $L = L_a + L_d$ [10], a large part of anisotropic radiances appears to be included in the diffuse component $L_d$. This increases the computational expense for its evaluation and the errors negatively influence the whole numerical scheme efficiency.

Recently, there have been suggested effective SAA solutions that quite accurately account for the path length spread. A stationary SAA solution for a flat unidirectional source in a flat layered medium was formulated in [28]. In [27], transient SAA solutions of the RTE have been found for femtosecond laser pulses and their polarization states (Stokes parameters). In [10], the solution for a narrow beam in a truly three-dimensional (3D) medium was formulated, and a numerical...
The solution for a PD source in a two-dimensional scattering medium [29] is presented. In the present paper, the stationary and transient solutions of the PD problem in a truly 3D medium are evaluated numerically and analyzed.

The rest of this paper is organized as follows. In Section 2, the basic equations of the suggested approach are derived in full detail. Numerical simulations according to the equations derived in Section 2 are presented in Section 3. In Section 4, the application of this approach to nonstationary radiation fields is discussed, and some numerical results are shown. In Section 5, the final conclusions of the paper are summarized.

2. CFSB RTE 2STREAM

Consider the radiative transfer equation

\[ (\Omega \cdot \nabla)I = -\varepsilon L + \frac{\lambda_\varepsilon}{4\pi} \int L(r, \Omega')x(\Omega', \Omega')d\Omega' + I(r, \Omega), \]  

where \( \Omega = (\mu_x, \mu_y, \mu_z) \) is the unit vector of the direction, \( r = (x, y, z) \), \( \mu_z = \cos \theta, \mu_x = \sin \theta \cos \phi, \mu_y = \sin \theta \sin \phi \), \( x(\Omega, \Omega') \) is the phase scattering function, \( L(r, \Omega) \) is the angular distribution of the radiance, and \( f(r, \Omega) \) is the arbitrary source function, highly anisotropic in the \( z \) direction (\( \mu_z \to 1 \)). Assume the extinction coefficient to be unity, \( \varepsilon = 1 \), without any restriction of generality.

Derive the separate equation for the forward and backward fluxes (\( L_1 \) and \( L_2 \), respectively) in the CFSB approximation. Divide the direction sphere into the front (\( \mu_z > 0 \)) and back (\( \mu_z < 0 \)) hemispheres and introduce the partition of unity \( M_1(\Omega) + M_2(\Omega) \equiv 1 \) such that \( M_1(\mu_z \to 1) \to 1, M_2(\mu_z \to 1) \to 0 \) and \( M_1(\mu_z \to -1) \to 0, M_2(\mu_z \to -1) \to 1 \), respectively. With this partition of unity, separate the scattering integral and the source function \( f(r) \) in the forward and back hemisphere contribution and account for the single backscattering of the forward flux \( L_1 \) into the back hemisphere in the equation for \( L_2 \).

We obtain a pair of coupled equations:

\[ (\Omega \cdot \nabla)L_1 = -L_1 + M_1(\Omega) \frac{\lambda_\varepsilon}{4\pi} \int L_1(r, \Omega')x(\Omega', \Omega')d\Omega' + M_1(\Omega)f(r), \]

\[ (\Omega \cdot \nabla)L_2 = -L_2 + M_2(\Omega) \frac{\lambda_\varepsilon}{4\pi} \int L_2(r, \Omega')x(\Omega', \Omega')d\Omega' + M_2(\Omega)f(r). \]

According to [10, 27, 28], we proceed in Eqs. (2) and (3) to the SAA with path length spread accounted for. Divide both sides of each equation by \( \mu_z \) and approximate the quantity \( \mu_z^{-1} \) by its Taylor expansion in \( \mu_z \) about the pole of the corresponding hemisphere \( \mu_z = \pm 1 \):

\[ \mu_z \equiv 1 + (1 - \mu_z) + (1 - \mu_z)^2 + \ldots + (1 - \mu_z)^n + o((1 - \mu_z)^n) \approx \frac{1}{\mu_z} \bigg|_{\mu_z \to \pm 1}. \]

We get the pair of SAA-CFSB equations:

\[ \frac{\partial}{\partial \Omega}L_1 + \frac{\lambda_\varepsilon}{4\pi} \int L_1(r, \Omega')x(\Omega', \Omega')d\Omega' + M_1(\Omega)f(r), \]

\[ \frac{\partial}{\partial \Omega}L_2 + \frac{\lambda_\varepsilon}{4\pi} \int L_2(r, \Omega')x(\Omega', \Omega')d\Omega' + M_2(\Omega)f(r). \]

The solution of this pair of equations [Eqs. (6) and (7)] with the proper boundary conditions is the solution of the RTE in the SSA-CFSB approximation. A rigorous solution of the problem can be written as the sum \( L = L_1 + L_2 + L_D \), where \( L_D \) is the unknown part of the field, neglected in the formulated approximation. Substituting this sum into the RTE [Eq. (1)], we get, for \( L_D \), the inhomogeneous Eq. (1) with the source function in the right-hand side:

\[ f_D = (\mu_z \mu_z - 1) \left( \mu_z \frac{\partial}{\partial x}L_1 + \mu_z \frac{\partial}{\partial y}L_1 + L_1 \right) + (\mu_z \mu_z - 1) \left( \mu_z \frac{\partial}{\partial x}L_2 + \mu_z \frac{\partial}{\partial y}L_2 + L_2 \right) + (1 - \mu_z \mu_z)M_1(\Omega). \]

As one can see, in the directions of the highest anisotropy \( \mu_z \to \pm 1 \), the source function \( f_D \) vanishes due to the factors \((1 - \mu_z \mu_z) \propto (\mu_z \mp 1)^{n+1}\). For this reason, \( L_D \) is a rather smooth function of \( \Omega \), which can be found with conventional numerical techniques without the difficulties caused by the singularities and high anisotropy of the solution.

In the framework of the spherical harmonics approach, the brightness distributions are expanded in the series

\[ L = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_{n,m} Y_n^m(\theta, \phi), \]

where \( Y_n^m(\theta, \phi) \) are the spherical harmonics [30]. The scattering phase function is also expanded in the series

\[ x(\Omega, \Omega') = 4\pi \sum_{n=0}^{\infty} \sum_{m=-n}^{n} x_n Y_n^m(\theta, \phi) Y_n^m(\theta', \phi'). \]
The factors $\mu_x, \mu_y,$ and $\mu_z$ in the spherical harmonics representation are matrix operators $\hat{\mu}_x, \hat{\mu}_y,$ and $\hat{\mu}_z$, respectively, so all the equations presented thus far are reduced to the systems of the ordinary differential equations. They can be solved, e.g., with the finite-difference scheme [11].

In the matrix formulation, the extension of the suggested approach to the vectorial RTE for polarized radiation is straightforward [27]. The angular distributions of the Stokes parameters and the scattering matrix of the polarized equation are expanded into a series of generalized spherical harmonics.

3. NUMERICAL SIMULATIONS

Consider the problem of the sounding of the cloudy layer by a narrow beam of continuous optical radiation [1]. The beam is normally incident onto the bottom $z = z_1$ of the flat horizontal layer of the uniform homogeneous scattering medium. The top of the layer is $z = z_2$. We set the extinction coefficient $\varepsilon = 1$ to be unity.

In the present paper, we focus on the CFSB solution of the scalar RTE, completely ignoring the polarization effects and diffuse component of the field $L_0$. Thus, we seek the solution as the sum of the unscattered beam $L_0$ and the forward and backward fluxes $L_1$ and $L_2$.

The unscattered flux $L_0$ is known, so $L_1$ and $L_2$ obey Eqs. (6) and (7) with the zero initial conditions $L_1(z_1) = 0$ and $L_2(z_2) = 0$, respectively. For the infinitesimal PD source, $L_0 = \delta(x)\delta(y)\delta(\Omega) \exp \left( \int \varepsilon dz \right)$,

the source function in Eqs. (6) and (7) equals

$$f (r, \Omega) = \frac{\Lambda}{4\pi} \int L_0 (r, \Omega') x(r, \Omega') d\Omega' = \delta(x)\delta(y) \frac{\Lambda}{4\pi} \delta(\Omega) \exp \left( - \int \varepsilon dz \right).$$

The radiation field is expanded into a series of spherical harmonics and the Fourier integral in the angular and spatial variables space, respectively:

$$L(x, y, z, \Omega) = \frac{1}{(2\pi)^2} \int \int \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \tilde{C}_{nm}(k_x, k_y, z) Y_m^n(\theta, \phi) \times \exp(ik_x x + ik_y y) dk_x dk_y,$$

The unknown expansion coefficients in Eq. (13) obey the matrix equations

$$\frac{\partial}{\partial z} \tilde{C}_1 + \mu_x \mu_y \mu_z \tilde{C}_1 + \mu_x \mu_y \mu_z \tilde{C}_1 = -\mu_x \tilde{C}_1 + \mu_x \tilde{M}_1 \Delta \tilde{C}_1 + \mu_x \tilde{M}_1 \Delta \tilde{G},$$

$$\frac{\partial}{\partial z} \tilde{C}_2 + \mu_x \mu_y \mu_z \tilde{C}_2 + \mu_x \mu_y \mu_z \tilde{C}_2 = -\mu_x \tilde{C}_2 + \mu_x \tilde{M}_2 \Delta \tilde{C}_2 + \tilde{C}_1 + \mu_x \tilde{M}_2 \Delta \tilde{G},$$

where $\tilde{C}_1$, $\tilde{C}_2$, and $\tilde{f}$ are the vectors of the expansion coefficients [Eq. (13)] for $L_1, L_2,$ and $f$, respectively; and $\mu_x, \mu_y,$ and $\mu_z$ are the matrix operators corresponding to the factors $\mu_x, \mu_y,$ and $\mu_z$. Explicit expressions for these matrices can be found, e.g., in [11]. Matrices $\mu_x^{\pm}$ are the matrix polynomials of $\mu_x$, defined by Eqs. (4) and (5). Matrices $M_1/2$ correspond to the functions $M_1/2(\Omega)$, constituting the partition of unity on the directions sphere. We choose the approximate numerical implementation of the partition of unity:

$$M_1(\mu_x) = -\frac{5}{32}(\mu_x + 1)^7 + \frac{35}{32}(\mu_x + 1)^6 - \frac{21}{8}(\mu_x + 1)^5 + \frac{35}{16}(\mu_x + 1)^4 = \sqrt{\pi} Y_{0,0}(\theta, \phi) + \frac{4}{3} \sqrt{\frac{\pi}{3}} Y_{1,0}(\theta, \phi) + \frac{4}{39} \sqrt{\frac{\pi}{3}} Y_{2,0}(\theta, \phi),$$

$$M_2(\mu_x) = -\frac{5}{32}(1 - \mu_x)^7 + \frac{35}{32}(1 - \mu_x)^6 - \frac{21}{8}(1 - \mu_x)^5 + \frac{35}{16}(1 - \mu_x)^4 = \sqrt{\pi} Y_{0,0}(\theta, \phi) - \frac{4}{3} \sqrt{\frac{\pi}{3}} Y_{1,0}(\theta, \phi) + \frac{2}{39} \sqrt{\frac{\pi}{3}} Y_{2,0}(\theta, \phi),$$

Matrix elements of all the mentioned operators are, in fact, the integrals of the products of three spherical harmonics, which can be expressed through the Wigner $3j$-symbols [31].

Because of the angular singularities of the radiance field, the numerical summation of the series is not stable. In practice, necessary precision of the calculations often is determined by the limit of the instrumental angular resolution of the experimental device used for the measurements [32–34]. For these reasons, the numerical solution can be apodized, i.e., convolved with some profile $M(\Omega)$, effectively regularizing the numerical simulation and limiting its angular resolution at the definite level.

The convenient apodization kernel is

$$M(\theta) = \frac{(s + 1)(\cos \theta + 1)^s}{2^{s+1} \pi},$$

where $s$ is the arbitrary parameter governing the effective angular width of the apodization profile. It can be shown that, in the spherical harmonics representation in Eqs. (9) and (10), the apodization kernel in Eq. (18) corresponds to the diagonal matrix operator $\tilde{M}$ with the coefficients $(s > n)$:

$$m_n = \frac{\Gamma(s + 1)\Gamma(s + 2)}{\Gamma(n + s + 1)\Gamma(n + s + 2)} = (s + 1)^{s+1/2}(s + 2)^{s+3/2}(s - n + 1)^{n-s-1/2} \times (n + s + 2)^{-n-s-3/2}.$$
For the calculations, the value of the mean scattering cosine \( g = 0.9 \), typical for the artificial suspensions prepared by Elliott [35] for his experiments, has been chosen. Since many realistic media have quite high single scattering albedos, the value \( \Lambda = 1 \) has been chosen for the calculation as a good representative value. The \( P_0 \) approximation [36] of the spherical harmonics method has been used in the numerical scheme, thus accounting for \( (99 + 1)^2 = 10,000 \) terms in the spherical harmonics expansions.

The angular radiance distribution of the backward flux \( L_2 \) (point \( A \) in Fig. 1; the distance from the beam axis \( R = 0.4 \)) is shown in Fig. 2. In Figs. 3 and 4, the angular radiance distributions of \( L_2 \) in the beam plane are shown with the solid curves, for heights of the top of the layer \( z_2 = 1.5 \) and \( z_2 = 5.0 \), respectively. The distances from the beam axis \( R \) to the observation point \( A \) are shown by the numerical labels. The angular radiance distributions of the singly and doubly scattered radiation are shown by the dashed and dashed–dotted curves, respectively. The apodization kernel \( M(\theta)(s = 5000) \) is shown in the separate graph at the left of each Figure.

4. PULSED BEAM PROPAGATION IN SCATTERING MEDIA

The generalization of the suggested approach to the nonstationary RTE,

\[
\frac{\partial}{\partial t} L + (\mathbf{\nabla} \cdot \mathbf{V}) L = -\varepsilon L + \frac{\Lambda\varepsilon}{4\pi} \int L(\mathbf{r}, \Omega') x(\Omega, \Omega') d\Omega' + f(\mathbf{r}, \Omega).
\]

is straightforward [27]. In this case, the frequency spectral components of the fluxes \( L_1, L_2, \) and \( L_D \) obey the corresponding equations [Eqs. (6), (7), and (1)] with the complex coefficients. The temporal profile of the pulse can be restored from its frequency spectrum. However, in the problems of optical communications and data transfer in the scattering media, an assessment of the available bandwidth is often enough [9,27].

For this reason, we focus our attention on the frequency spectrum of the forward flux \( L_1 \). Without restriction of generality, assume the radiation speed in the medium to be unity \( c = 1 \). The vectors of the coefficients \( c^{(1)}_{nm}(k_x, k_y, z, \omega) \) of the expansion

\[
L_1(x, y, z, \omega, t) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^\pi \exp(ik_x x + ik_y y + i\omega(t - z)) \sum_{n,m} \hat{c}^{(1)}_{nm}(k_x, k_y, z, \omega) Y^m_n(\theta, \phi) dk_x dk_y d\omega
\]

\( \hat{c}^{(1)}_{nm}(k_x, k_y, z, \omega) \) is shown in Fig. 3. (Color online) Radiance distributions of the backscattered radiation (point \( A \) in Fig. 1) in the beam plane at different distances \( R \) from the beam axis (shown by the labels). \( z_1 = 1.0, z_2 = 5.0, \varepsilon = 1, g = 0.9, \) and \( \Lambda = 1 \). Solid curves, total radiance; dashed curves, singly scattered radiance; dashed–dotted curves, doubly scattered radiance.
obey the matrix equation

\[ i\omega (1 - \mu) \tilde{C}_1 + \frac{\partial}{\partial z} \tilde{C}_1 + \tilde{\mu}_x ik_x \tilde{C}_1 + \tilde{\mu}_y ik_y \tilde{C}_1 = -\tilde{\mu}_z \tilde{M}_1 \Lambda \tilde{\alpha} \tilde{C}_1 + \tilde{\mu}_z \tilde{M}_1 \Lambda \tilde{\alpha} \tilde{f} \]  

(22)

Angular distribution of the frequency components of the spectrum \[ L_1(\omega, r, \Omega) \] at different distances from the beam axis are shown in Fig. 5. The optical thickness of the medium \( \tau_0 = 10 \). Despite the significant optical path length of the radiation, equal to one transport length \( \tau_0/(1 - \mu) = 1 \), the spectral bandwidth is relatively large \( (\omega \tau_0/(1 - \mu) > 1) \) even at notable distances from the beam axis. This is because the paths of the weakly scattered radiation with small dispersion of the path length are selectively concentrated in the vicinity of the beam axis. Strongly scattered radiation is distributed in the large volume within the medium, so its contribution in the central part of the beam cross-section area is negligible.

5. CONCLUSIONS AND REMARKS

The advanced SAA suggested by the authors in previous papers is adapted to the narrow beam problems in a truly 3D scattering medium. The solution of the problem is obtained without resorting to the Gaussian approximation for the radiation distribution of other common simplifying assumptions. The numerical results are presented for both stationary and transient problems.

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