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Subspace-stabilized sequential quadratic programming

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Abstract The stabilized sequential quadratic programming (SQP) method has nice local convergence properties: it possesses local superlinear convergence under very mild assumptions not including any constraint qualifications. However, any attempts to globalize convergence of this method indispensably face some principal difficulties concerned with intrinsic deficiencies of the steps produced by it when relatively far from solutions; specifically, it has a tendency to produce long sequences of short steps before entering the region where its superlinear convergence shows up. In this paper, we propose a modification of the stabilized SQP method, possessing better "semi-local" behavior, and hence, more suitable for the development of practical realizations. The key features of the new method are identification of the so-called degeneracy subspace and dual stabilization along this subspace only; thus the name "subspace-stabilized SQP". We consider two versions of this method, their local convergence properties, as well as a practical procedure for approximation of the degeneracy subspace. Even though we do not consider here any specific algorithms with theoretically justified global convergence properties, subspace-stabilized SQP can be a relevant substitute for the stabilized SQP in such algorithms using the latter at the "local phase". Some numerical results demonstrate that stabiliza-

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tion along the degeneracy subspace is indeed crucially important for success of dual stabilization methods.

Keywords sequential quadratic programming · degenerate solution; noncritical Lagrange multiplier · dual stabilization · superlinear convergence · global convergence

1 Introduction

The stabilized sequential quadratic programming (SQP) method was introduced in [29] for inequality-constrained optimization problems, and in the form of solving the min-max subproblems, as a tool to restore the superlinear convergence rate of the basic SQP, which is usually lost when the latter is applied to problems violating traditional constraint qualifications. It was recognized very soon though that this min-max subproblem is equivalent to the quadratic programming problem in the primal-dual space [23], and that the method is applicable to problems with equality constraints as well.

Convergence properties of the stabilized SQP method have been further studied in [11,30,31]. The sharpest known local convergence result for the stabilized SQP was obtained in [4], using the abstract iterative framework developed in [5]. Specifically, the local superlinear convergence of the method was established in [4] under the sole assumption of the second-order sufficient optimality condition. For purely equality-constrained problems, this assumption was further relaxed in [16] to the so-called noncriticality of the involved Lagrange multiplier (see below). The main point revealed by these results is the local dual stabilization property of the method in question, allowing to avoid (locally) the attraction of its dual sequences to critical Lagrange multipliers, the destructive phenomenon which is mainly responsible for the lack of superlinear convergence of the basic SQP when such multipliers do exist; see [12,15,22], [17, Chapter 7], the very recent discussion in [6,18,19,24,25,28], and many other related references therein.

However, by now, we are not aware of any evidently successful practical implementations of the stabilized SQP, and the reason for this is that this dual stabilization procedure is essentially local. Various attempts to globalize convergence of the stabilized SQP were undertaken in [3,7–10,13,20,21]. In particular, [8] reports on some relatively encouraging numerical results; less encouraging but reasonable results are reported in [20]. However, we believe that any approach to this problem would face principal difficulties when it comes to numerical performance. The reason for these difficulties is concerned with some undesirable global features of the stabilized SQP itself, recently exposed in [21], where a globalization based on the linesearch for the smooth exact penalty function from [27] has been investigated.

To be specific, consider the equality-constrained optimization problem

$$\begin{array}{ll} \text{minimize} & f(x) \\ \text{subject to} & h(x) = 0, \end{array} \tag{1}$$

where the objective function $f : \mathbb{R}^n \to \mathbb{R}$ and the constraint mapping $h : \mathbb{R}^n \to \mathbb{R}^l$ are at least twice differentiable. Let $L : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ be the usual Lagrangian of problem (1), i.e.,

$$L(x, \lambda) = f(x) + \langle \lambda, h(x) \rangle.$$

Given the current primal-dual iterate $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$ and the value $\sigma > 0$ of the stabilization parameter, the stabilized SQP method for problem (1) generates the direction $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^l$ by solving the linear system

$$\frac{\partial^2 L}{\partial x^2}(x, \lambda)\xi + (h'(x))^{\mathrm{T}}\eta = -\frac{\partial L}{\partial x}(x, \lambda), \quad h'(x)\xi - \sigma\eta = -h(x). \tag{2}$$

The next iterate of the basic stabilized SQP is then defined as $(x + \xi, \lambda + \eta)$.

Violation of constraint qualifications at some feasible point $\bar{x}$ of problem (1) means that $(\operatorname{im} h'(\bar{x}))^{\perp} = \ker(h'(\bar{x}))^{\mathrm{T}}$ is a nontrivial subspace in $\mathbb{R}^l$, which we will call the *degeneracy subspace*. (Here by im and ker we denote the range space and the null space of a linear operator, respectively.) In the cases of full degeneracy, i.e., when $h'(\bar{x}) = 0$, which means that the degeneracy subspace is equal to the entire $\mathbb{R}^l$, the stabilized SQP and its existing globalizations usually perform just fine. The problem, however, is that the stabilized SQP has a strong tendency to produce long series of short steps in cases of non-full degeneracy, i.e., when the degeneracy subspace is a nontrivial but *proper* subspace in $\mathbb{R}^l$. The stabilization mechanism of this method is intended to prevent it from moving along the set of Lagrange multipliers which is an affine manifold parallel to the degeneracy subspace. Moving along the set of multipliers is precisely the reason for slow convergence of the SQP method when applied to degenerate problems. However, when used not close enough to solutions, the specified stabilizing mechanism often results in "over-stabilization" enforcing the steps to be short in principle, rather than only in the directions of the degeneracy subspace. In such cases the performance of the stabilized SQP can be even much worse than the one observed for the usual SQP without any stabilization, and these difficulties are not concerned with local convergence rate, as they are encountered earlier than the superlinear convergence shows up. We emphasize that this behavior is not related to any deficiencies of particular globalizations; it rather appears to be an intrinsic property of the stabilized SQP itself, which must show up in any globalization attempting to use the full stabilized SQP step as often as possible. The specified effect has been observed already in [26]. See [21] and Section 4 below for details.

This gives rise to the following idea, realization of which is the main subject of this work: *what should be fixed is not any globalization strategy for the stabilized SQP but rather the stabilized SQP itself.* Specifically, the stabilization term $\sigma\eta$ in (2) should be replaced by a "smaller" one essentially affecting not the entire dual direction η but only its projection onto the degeneracy subspace, thus not blocking long steps in the directions orthogonal to this subspace. We refer to this approach as *subspace-stabilized SQP*. Of course, this approach can make practical sense only if equipped with a relatively cheap technique for approximation of the degeneracy subspace, since this subspace cannot be known exactly for an unknown $\bar{x}$. A surprisingly simple and natural technique of this kind will be suggested below, which is also shown to be quite reliable and cheap, especially when the number of constraints l is not too large (previous proposals of this kind employed the prohibitively expensive singular-value decomposition of the constraints Jacobian [14]).

Note that we do not consider in this paper any specific algorithms with theoretically justified global convergence properties; we rather propose to use the subspace-stabilized SQP as a relevant substitute for the stabilized SQP in such algorithms involving some kind of a "local phase" at which the stabilized SQP is applied to the equality-constrained problem obtained by identification of active inequality constraints, or to the original problem when there are no inequality constraints. One such algorithm has been developed in the series of papers [7,8,10], where the stabilized SQP is combined this way with a certain primal-dual augmented Lagrangian method. Another example is the algorithm from [20], where the standard augmented Lagrangian method is used instead as a tool for enforcing provable global convergence. This paper attempts to demonstrate that incorporating the newly developed subspace-stabilized SQP in these algorithms can significantly improve their performance on problems with degenerate but not fully degenerate constraints.

Some basic definitions and notation are in order. Stationary points and associated Lagrange multipliers of problem (1) are characterized by the Lagrange optimality system

$$\frac{\partial L}{\partial x}(x,\, \lambda) = 0, \quad h(x) = 0, \tag{3}$$

with respect to $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}^l$. Let $\mathcal{M}(\bar{x})$ stand for the set of Lagrange multipliers associated to $\bar{x}$, i.e., of those λ satisfying (3) with $x = \bar{x}$. Recall that $\bar{\lambda} \in \mathcal{M}(\bar{x})$ is called a critical Lagrange multiplier if there exists $\xi \in \ker h'(\bar{x}) \setminus \{0\}$ such that

$$\frac{\partial^2 L}{\partial x^2}(\bar{x},\, \bar{\lambda})\xi \in \mathrm{im}(h'(\bar{x}))^{\mathrm{T}},$$

and noncritical otherwise [12]. The multiplier is always noncritical if it satisfies the second-order sufficient optimality condition (SOSC):

$$\left\langle \frac{\partial^2 L}{\partial x^2}(\bar{x},\, \bar{\lambda})\xi,\, \xi \right\rangle > 0 \quad \forall\, \xi \in \ker h'(\bar{x}) \setminus \{0\}.$$

See [17, Chapters 1, 7] for details on critical and noncritical multipliers.

Throughout the paper $\bar{P}$ stands for the orthogonal projector onto the degeneracy subspace $(\mathrm{im}\, h'(\bar{x}))^{\perp}$ (for a fixed $\bar{x}$). By I we denote the identity mapping, and by $\mathcal{B}(u,\, \varepsilon)$ we denote the Euclidean ball of radius $\varepsilon > 0$, centered at u. All norms are Euclidean.

The rest of the paper is organized as follows. Section 2 presents two versions of the subspace-stabilized SQP and their local convergence properties. In

Section 3 we develop the technique for approximation of the degeneracy subspace, making the methods from Section 2 implementable. Section 4 presents some numerical experience with the newly developed algorithms, demonstrating their advantages over the usual SQP and the stabilized SQP. Concluding remarks summarizing our development are given in Section 5.

2 Subspace-stabilized SQP and its local convergence

We now describe the iteration of the subspace-stabilized SQP method. Let a mapping $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l\times l}$ and a function $\sigma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ be given. The linear operator $P(x,\, \lambda)$ is supposed to approximate a projector onto the degeneracy subspace (in some sense, to be specified below) as $(x,\, \lambda) \to (\bar{x},\, \bar{\lambda})$.

Given the current primal-dual iterate $(x,\, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$, we compute the direction $(\xi,\, \eta) \in \mathbb{R}^n \times \mathbb{R}^l$ by solving the linear system

$$\frac{\partial^2 L}{\partial x^2}(x,\, \lambda)\xi + (h'(x))^{\mathrm{T}}\eta = -\frac{\partial L}{\partial x}(x,\, \lambda), \quad h'(x)\xi - \sigma(x,\, \lambda)P(x,\, \lambda)\eta = -h(x) \tag{4}$$

(c.f. (2)). The next iterate is then defined as $(x+\xi,\, \lambda+\eta)$.

Take any stationary point $\bar{x}$ of problem (1) and any associated Lagrange multiplier $\bar{\lambda}$. The local convergence of our method will be justified by application of the local iterative framework from [5] to the Lagrange system of equations (3) and the iterative process the step of which is defined by (4). Therefore, one needs to verify Assumptions 1–3 of [5, Theorem 1]. As discussed in [16, Remark 1], Assumption 1 is equivalent to the error bound

$$\|x-\bar{x}\| + \mathrm{dist}(\lambda,\, \mathcal{M}(\bar{x})) = O\left(\left\|\left(\frac{\partial L}{\partial x}(x,\, \lambda),\, h(x)\right)\right\|\right) \tag{5}$$

as $(x,\, \lambda) \to (\bar{x},\, \bar{\lambda})$, and according to [16, Proposition 1], this is further equivalent to the assumption that $\bar{\lambda}$ is noncritical.

Verification of Assumptions 2 and 3 in [5] requires further assumptions regarding P and σ, and in the rest of this section we will consider separately two different sets of such assumptions. This gives rise to the two versions of subspace-stabilized SQP, with similar local convergence properties. Implementations of those versions require some specific practical rules for defining P and σ satisfying the needed assumptions. As for σ, natural ways of defining it arise from (5). Constructing appropriate P in a practical way is a more complicated issue; it will be the subject of Section 3.

2.1 Asymptotically vanishing stabilization

Suppose first that P and σ satisfy the following assumptions:

(P1) P is continuous at $(\bar{x},\, \bar{\lambda})$, and $(\mathrm{im}\, h'(\bar{x}))^{\perp}$ is an invariant subspace of $P(\bar{x},\, \bar{\lambda})$, i.e.,

$$P(\bar{x},\, \bar{\lambda})\eta = \eta \quad \forall\, \eta \in (\mathrm{im}\, h'(\bar{x}))^{\perp}.$$

(S1) σ is continuous at every point of $\{\bar{x}\} \times \mathcal{M}(\bar{x})$ close enough to $(\bar{x}, \bar{\lambda})$, $\sigma(\bar{x}, \lambda) = 0$ for all $\lambda \in \mathcal{M}(\bar{x})$ close enough to $\bar{\lambda}$, $\sigma(x, \lambda) \neq 0$ for all $(x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^l) \setminus (\{\bar{x}\} \times \mathcal{M}(\bar{x}))$ close enough to $(\bar{x}, \bar{\lambda})$, and

$$\|x - \bar{x}\| = O(|\sigma(x, \lambda)|) \tag{6}$$

as $(x, \lambda) \to (\bar{x}, \bar{\lambda})$.

We refer to the corresponding stabilization as asymptotically vanishing because the stabilizing second term in the left-hand side of the (4), distinguishing it from the basic SQP iteration system, tends to zero as (x, λ) tends to any point of $\{\bar{x}\} \times \mathcal{M}(\bar{x})$ near $(\bar{x}, \bar{\lambda})$.

Observe that assumption (P1) allows to take, e.g., $P(\cdot) \equiv I$, thus covering the usual stabilized SQP method. Other possible choices of P will be discussed below. Observe further that assumption (S1) is satisfied if $\bar{\lambda}$ is a noncritical multiplier, and σ is taken as

$$\sigma(x, \lambda) = \|\Phi(x, \lambda)\|^{\beta} \tag{7}$$

with any fixed $\beta \in (0, 1]$, where $\Phi : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^n \times \mathbb{R}^l$ is the operator of the Lagrange system:

$$\Phi(x, \lambda) = \left(\frac{\partial L}{\partial x}(x, \lambda),\, h(x) \right). \tag{8}$$

Assumption 2 in [5] characterizes the required quality of approximation of the original problem by the iteration subproblems. Define the mapping $\mathcal{A} : (\mathbb{R}^n \times \mathbb{R}^l) \times (\mathbb{R}^n \times \mathbb{R}^l) \to \mathbb{R}^n \times \mathbb{R}^l$,

$$\begin{aligned}\mathcal{A}((x, \lambda),\, (\xi, \eta)) = \biggl(&\frac{\partial L}{\partial x}(x, \lambda) + \frac{\partial^2 L}{\partial x^2}(x, \lambda)\xi + (h'(x))^{\mathrm{T}}\eta, \\ &h(x) + h'(x)\xi - \sigma(x, \lambda)P(x, \lambda)\eta \biggr).\end{aligned} \tag{9}$$

Observe that the iteration subproblem (4) can then be written as

$$\mathcal{A}((x, \lambda),\, (\xi, \eta)) = 0.$$

The required quality of approximation (of Φ by $\mathcal{A}$) is justified by the following

Proposition 1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ be twice differentiable in a neighborhood of $\bar{x} \in \mathbb{R}^n$, with their second derivatives continuous at $\bar{x}$. Let $\bar{x}$ be a stationary point of problem* (1)*, and let $\bar{\lambda} \in \mathcal{M}(\bar{x})$. Assume that $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l \times l}$ is bounded in a neighborhood of $(\bar{x}, \bar{\lambda})$, $\sigma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ is continuous at every point of the set $\{\bar{x}\} \times \mathcal{M}(\bar{x})$, close enough to $(\bar{x}, \bar{\lambda})$, and $\sigma(\bar{x}, \lambda) = 0$ for all $\lambda \in \mathcal{M}(\bar{x})$ close enough to $\bar{\lambda}$.*

Then there exists $\varepsilon > 0$ such that for any $C > 0$ there exists a function $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(t) = o(t)$ as $t \to 0$, and for Φ and $\mathcal{A}$ defined by (8) *and* (9) *it holds that*

$$\|\Phi(x + \xi,\, \lambda + \eta) - \mathcal{A}((x, \lambda),\, (\xi, \eta))\| \leq \omega(\|x - \bar{x}\| + \mathrm{dist}(\lambda,\, \mathcal{M}(\bar{x}))) \tag{10}$$

for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ *and all* $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^l$ *satisfying*

$$\|(\xi, \eta)\| \leq C(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))). \tag{11}$$

Proof Fix any $\varepsilon > 0$ such that f and h are twice differentiable in $\mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$, the value $\|P(x, \lambda)\|$ is bounded by some fixed constant for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$, σ is continuous at every point of $(\{\bar{x}\} \times \mathcal{M}(\bar{x})) \cap \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$, and $\sigma(\bar{x}, \lambda) = 0$ for all $\lambda \in \mathcal{M}(\bar{x}) \cap \mathcal{B}(\bar{\lambda}, \varepsilon)$.

Evidently,

$$\begin{aligned}\|\Phi(x + \xi, \lambda + \eta) - \mathcal{A}((x, \lambda), (\xi, \eta))\| \leq & \left\| \frac{\partial L}{\partial x}(x + \xi, \lambda + \eta) - \frac{\partial L}{\partial x}(x, \lambda) \right. \\ & \left. - \frac{\partial^2 L}{\partial x^2}(x, \lambda)\xi - (h'(x))^{\mathrm{T}}\eta \right\| \\ & + \|h(x + \xi) - h(x) - h'(x)\xi\| \\ & + |\sigma(x, \lambda)| \, \|P(x, \lambda)\eta\|.\end{aligned}$$

Therefore, it is sufficient to find functions $\omega_i \colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega_i(t) = o(t)$ for $i = 1, 2, 3$, and such that

$$\left\| \frac{\partial L}{\partial x}(x + \xi, \lambda + \eta) - \frac{\partial L}{\partial x}(x, \lambda) - \frac{\partial^2 L}{\partial x^2}(x, \lambda)\xi - (h'(x))^{\mathrm{T}}\eta \right\| \leq$$
$$\omega_1(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))), \tag{12}$$
$$\|h(x + \xi) - h(x) - h'(x)\xi\| \leq \omega_2(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))), \tag{13}$$
$$|\sigma(x, \lambda)| \, \|P(x, \lambda)\eta\| \leq \omega_3(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))) \tag{14}$$

for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and all (ξ, η) satisfying (11).

Set

$$R((x, \lambda), (\xi, \eta)) = \frac{\partial L}{\partial x}(x + \xi, \lambda + \eta) - \frac{\partial L}{\partial x}(x, \lambda) - \frac{\partial^2 L}{\partial x^2}(x, \lambda)\xi - (h'(x))^{\mathrm{T}}\eta,$$

and define the function ω_1 as

$$\omega_1(t) = \sup \left\{ \|R((x, \lambda), (\xi, \eta))\| \;\middle|\; \begin{array}{l} (x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon), \\ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^l, \\ \|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x})) \leq t, \\ \|(\xi, \eta)\| \leq C(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))) \end{array} \right\}. \tag{15}$$

Observe that the set over which supremum is taken in the right-hand side is nonempty and compact for any fixed $t \geq 0$. Therefore, ω_1 is well-defined, and for any $t \geq 0$ there exist some (x, λ) and (ξ, η) in this set such that $\omega_1(t) = \|R((x, \lambda), (\xi, \eta))\|$.

Evidently, the function ω_1 just defined satisfies (12) for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and all (ξ, η) satisfying (11). It remains to prove that $\omega_1(t) = o(t)$ as $t \to 0$. Suppose the contrary: there exist $\gamma > 0$ and a sequence $\{t_k\}$ of positive reals, convergent to zero and such that $\omega_1(t_k) \geq \gamma t_k$ for all k. The latter implies the existence of sequences $\{(x^k,\, \lambda^k)\} \subset \mathcal{B}((\bar{x},\, \bar{\lambda}),\, \varepsilon)$ and $\{(\xi^k,\, \eta^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ such that for all k

$$\|R((x^k,\, \lambda^k),\, (\xi^k,\, \eta^k))\| \geq \gamma t_k, \tag{16}$$

$$\|x^k - \bar{x}\| + \mathrm{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))) \leq t_k, \tag{17}$$

$$\|(\xi^k,\, \eta^k)\| \leq C(\|x^k - \bar{x}\| + \mathrm{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))). \tag{18}$$

From these relations and from the mean value theorem we obtain

$$\begin{aligned}
\gamma t_k &\leq \left\| \frac{\partial L}{\partial x}(x^k + \xi^k,\, \lambda^k + \eta^k) - \frac{\partial L}{\partial x}(x^k,\, \lambda^k) - \frac{\partial^2 L}{\partial x^2}(x^k,\, \lambda^k)\xi^k - (h'(x^k))^{\mathrm{T}}\eta^k \right\| \\
&\leq \left\| \frac{\partial L}{\partial x}(x^k + \xi^k,\, \lambda^k) - \frac{\partial L}{\partial x}(x^k,\, \lambda^k) - \frac{\partial^2 L}{\partial x^2}(x^k,\, \lambda^k)\xi^k \right\| \\
&\quad + \left\| (h'(x^k + \xi^k) - h'(x^k))^{\mathrm{T}}\eta^k \right\| \\
&\leq \sup_{\tau \in [0,\, 1]} \left\| \frac{\partial^2 L}{\partial x^2}(x^k + \tau\xi^k,\, \lambda^k) - \frac{\partial^2 L}{\partial x^2}(x^k,\, \lambda^k) \right\| \|\xi^k\| \\
&\quad + \sup_{\tau \in [0,\, 1]} \|h''(x^k + \tau\xi^k)\| \|\xi^k\| \|\eta^k\| \\
&= o(\|\xi^k\|) + O(\|\xi^k\|\|\eta^k\|) \\
&= o(t_k)
\end{aligned}$$

as $k \to \infty$, which is a contradiction.

Continuity of h'' at $\bar{x}$ implies the estimate $h''(x) \leq M$ for some $M > 0$ and for all x close enough to $\bar{x}$. Applying again the mean-value theorem, we have

$$\begin{aligned}
\|h(x + \xi) - h(x) - h'(x)\xi\| &\leq \sup_{\tau \in [0,\, 1]} \|h'(x + \tau\xi) - h'(x)\|\, \|\xi\| \\
&\leq \sup_{\tau \in [0,\, 1]} \|h''(x + \tau\xi)\|\, \|\xi\|^2 \\
&\leq \omega_2(\|x - \bar{x}\| + \mathrm{dist}(\lambda,\, \mathcal{M}(\bar{x})))
\end{aligned}$$

for all $(x,\, \lambda)$ close enough to $(\bar{x},\, \bar{\lambda})$, and for all $(\xi,\, \eta)$ satisfying (11), where the function ω_2 defined by $\omega_2(t) = C^2 M t^2$ obviously satisfies $\omega_2(t) = o(t)$ as $t \to 0$. Therefore, this function satisfies (13) for all $(x,\, \lambda) \in \mathcal{B}((\bar{x},\, \bar{\lambda}),\, \varepsilon)$ and all $(\xi,\, \eta)$ satisfying (11), where $\varepsilon > 0$ must be reduced, if necessary.

Finally, set

$$R((x,\, \lambda),\, \eta) = \sigma(x,\, \lambda) P(x,\, \lambda)\eta, \tag{19}$$

and consider ω_3 defined similarly to ω_1 in (15) but with $R((x,\, \lambda),\, (\xi,\, \eta))$ replaced by this $R((x,\, \lambda),\, \eta)$. Evidently, this ω_3 satisfies (14) for all $(x,\, \lambda) \in \mathcal{B}((\bar{x},\, \bar{\lambda}),\, \varepsilon)$ and all $(\xi,\, \eta)$ satisfying (11), and it remains to prove that $\omega_3(t) = o(t)$ as $t \to 0$. Supposing the contrary, we obtain the existence of $\gamma > 0$, a sequence $\{t_k\}$ of positive reals, convergent to zero, and sequences $\{(x^k,\, \lambda^k)\} \subset \mathcal{B}((\bar{x},\, \bar{\lambda}),\, \varepsilon)$ and $\{(\xi^k,\, \eta^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$, such that inequalities (16)–(18) hold for all k.

According to (17) and (18), passing to subsequences, if necessary, we may assume that $\{(x^k, \lambda^k)\}$ is convergent to $(\bar{x}, \lambda^*)$ with some $\lambda^* \in \mathcal{M}(\bar{x}) \cap \mathcal{B}(\bar{\lambda}, \varepsilon)$, and $\|\eta^k\| \leq Ct_k$ for all k. Since our choice of ε subsumes that $\{\|P(x^k, \lambda^k)\|\}$ is a bounded sequence, $\sigma(\bar{x}, \lambda^*) = 0$, and σ is continuous at $(\bar{x}, \lambda^*)$, we then obtain that

$$\gamma t_k \leq |\sigma(x^k, \lambda^k)| \|P(x^k, \lambda^k)\| \|\eta^k\| = o(t_k),$$

as $k \to \infty$, which is a contradiction. □

Evidently, the requirements on P and σ in this proposition are satisfied if assumptions (P1) and (S1) hold.

Finally, Assumption 3 in [5] consists of saying that for all $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$ close enough to $(\bar{x}, \bar{\lambda})$, system (4) has a solution (ξ, η) satisfying

$$\|(\xi, \eta)\| = O(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))) \tag{20}$$

as $(x, \lambda) \to (\bar{x}, \bar{\lambda})$. We establish this next.

Proposition 2 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ be twice differentiable in a neighborhood of $\bar{x} \in \mathbb{R}^n$, with their second derivatives continuous at $\bar{x}$. Let $\bar{x}$ be a stationary point of problem* (1)*, and let $\bar{\lambda} \in \mathcal{M}(\bar{x})$ be a noncritical multiplier. Assume that $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l \times l}$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ satisfy assumptions* (P1) *and* (S1)*.*

Then for all $(x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^l) \setminus (\{\bar{x}\} \times \mathcal{M}(\bar{x}))$ close enough to $(\bar{x}, \bar{\lambda})$, system (4) *has the unique solution (ξ, η), and this solution satisfies estimate* (20)*.*

Proof We first prove that for all $(x, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^l) \setminus (\{\bar{x}\} \times \mathcal{M}(\bar{x}))$ close enough to $(\bar{x}, \bar{\lambda})$, the matrix

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2}(x, \lambda) & (h'(x))^{\mathrm{T}} \\ h'(x) & -\sigma(x, \lambda) P(x, \lambda) \end{pmatrix}$$

of the linear system (4) is nonsingular.

Suppose not, then there exist sequences $\{(x^k, \lambda^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ and $\{(\xi^k, \eta^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ such that $\{(x^k, \lambda^k)\} \to (\bar{x}, \bar{\lambda})$, and for all k it holds that $(x^k, \lambda^k) \notin \{\bar{x}\} \times \mathcal{M}(\bar{x})$, $\|(\xi^k, \eta^k)\| = 1$,

$$\frac{\partial^2 L}{\partial x^2}(x^k, \lambda^k)\xi^k + (h'(x^k))^{\mathrm{T}}\eta^k = 0, \quad h'(x^k)\xi^k - \sigma(x^k, \lambda^k) P(x^k, \lambda^k)\eta^k = 0. \tag{21}$$

Without loss of generality suppose that $\{(\xi^k, \eta^k)\} \to (\xi, \eta)$. Then by passing onto the limit in the relations above it holds that $\|(\xi, \eta)\| = 1$ and

$$\frac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\lambda})\xi + (h'(\bar{x}))^{\mathrm{T}}\eta = 0, \quad h'(\bar{x})\xi = 0, \tag{22}$$

where the last equality is by continuity of P and σ at $(\bar{x}, \bar{\lambda})$, and by the equality $\sigma(\bar{x}, \bar{\lambda}) = 0$ (assumptions (P1) and (S1)).

If $\xi \neq 0$, then (22) contradicts the assumption that $\bar{\lambda}$ is noncritical. Therefore, $\xi = 0$, and hence, $\|\eta\| = 1$ and $(h'(\bar{x}))^{\mathrm{T}}\eta = 0$. Since by the assumption (P1) $(\operatorname{im} h'(\bar{x}))^{\perp} = \ker(h'(\bar{x}))^{\mathrm{T}}$ is an invariant subspace of $P(\bar{x},\, \bar{\lambda})$, we now conclude that $P(\bar{x},\, \bar{\lambda})\eta = \eta$.

From the second equality in (21) we derive that for all k

$$\begin{aligned}
\sigma(x^k,\, \lambda^k)\bar{P}P(x^k,\, \lambda^k)\eta^k &= \bar{P}h'(x^k)\xi^k \\
&= \bar{P}(h'(x^k) - h'(\bar{x}))\xi^k \\
&= O(\|x^k - \bar{x}\|\|\xi^k\|) \\
&= O(|\sigma(x^k,\, \lambda^k)|\|\xi^k\|)
\end{aligned}$$

as $k \to \infty$, where the last equality is by (6). Therefore, since $\sigma(x^k,\, \lambda^k) \neq 0$ for all k (assumption (S1)) and $\{\xi^k\} \to \xi = 0$, we have that

$$\eta = \bar{P}\eta = \bar{P}P(\bar{x},\, \bar{\lambda})\eta = \lim_{k\to\infty} \bar{P}P(x^k,\, \lambda^k)\eta^k = 0, \tag{23}$$

contradicting the equality $\|\eta\| = 1$.

It remains to prove that for all $(x,\, \lambda) \in (\mathbb{R}^n \times \mathbb{R}^l) \setminus (\{\bar{x}\} \times \mathcal{M}(\bar{x}))$ close enough to $(\bar{x},\, \bar{\lambda})$, the unique solution $(\xi,\, \eta)$ of system (4) satisfies estimate (20).

Suppose not, then there exist sequences $\{(x^k,\, \lambda^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ and $\{(\xi^k,\, \eta^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ such that $\{(x^k,\, \lambda^k)\} \to (\bar{x},\, \bar{\lambda})$, for all k it holds that $(x^k,\, \lambda^k) \notin \{\bar{x}\} \times \mathcal{M}(\bar{x})$, $(\xi^k,\, \eta^k) \neq 0$,

$$\begin{aligned}
\frac{\partial^2 L}{\partial x^2}(x^k,\, \lambda^k)\xi^k + (h'(x^k))^{\mathrm{T}}\eta^k &= -\frac{\partial L}{\partial x}(x^k,\, \lambda^k), \\
h'(x^k)\xi^k - \sigma(x^k,\, \lambda^k)P(x^k,\, \lambda^k)\eta^k &= -h(x^k),
\end{aligned} \tag{24}$$

and

$$\|x^k - \bar{x}\| + \operatorname{dist}(\lambda^k,\, \mathcal{M}(\bar{x})) = o(\|(\xi^k,\, \eta^k)\|) \tag{25}$$

as $k \to \infty$. Observe first that the latter implies

$$\begin{aligned}
\left\|\left(\frac{\partial L}{\partial x}(x^k,\, \lambda^k),\, h(x^k)\right)\right\| &= \left\|\left(\frac{\partial L}{\partial x}(x^k,\, \lambda^k) - \frac{\partial L}{\partial x}(\bar{x},\, \bar{\lambda}^k),\, h(x^k) - h(\bar{x})\right)\right\| \\
&= O(\|x^k - \bar{x}\| + \operatorname{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))) \\
&= o(\|(\xi^k,\, \eta^k)\|),
\end{aligned} \tag{26}$$

where $\bar{\lambda}^k$ is the metric projection of λ^k onto $\mathcal{M}(\bar{x})$.

Without loss of generality suppose that $\{(\xi^k,\, \eta^k)/\|(\xi^k,\, \eta^k)\|\} \to (\xi,\, \eta)$, $\|(\xi,\, \eta)\| = 1$. Then dividing (24) by $\|(\xi^k,\, \eta^k)\|$ and passing onto the limit, by (26) we obtain (22). Similarly to the argument above employing noncriticality of $\bar{\lambda}$ we derive that $\xi = 0$, $\|\eta\| = 1$, $(h'(\bar{x}))^{\mathrm{T}}\eta = 0$, and $P(\bar{x},\, \bar{\lambda})\eta = \eta$.

The second equality in (24) now implies that for all k

$$\begin{aligned}\sigma(x^k,\, \lambda^k)\bar{P}P(x^k,\, \lambda^k)\eta^k &= \bar{P}(h(x^k) + h'(x^k)\xi^k) \\ &= \bar{P}(h(x^k) - h(\bar{x}) - h'(\bar{x})(x^k - \bar{x}) \\ &\quad + (h'(x^k) - h'(\bar{x}))\xi^k) \\ &= O(\|x^k - \bar{x}\|^2) + O(\|x^k - \bar{x}\|\|\xi^k\|) \\ &= o(|\sigma(x^k,\, \lambda^k)|\|(\xi^k,\, \eta^k)\|) + O(|\sigma(x^k,\, \lambda^k)|\|\xi^k\|)\end{aligned}$$

as $k \to \infty$, where the second equality is by the mean-value theorem, and the last equality is by (6) and (25). Therefore, since $\sigma(x^k,\, \lambda^k) \neq 0$ for all k, $\sigma(x^k,\, \lambda^k) \to 0$ as $k \to \infty$ (assumption (S1)), and $\{\xi^k/\|(\xi^k,\, \eta^k)\|\} \to \xi = 0$, we again derive (23), contradicting the equality $\|\eta\| = 1$. □

By Propositions 1 and 2 and the discussion above, from [5, Theorem 1] (employing some relations in its proof) we now obtain the following local convergence result.

Theorem 1 *Under the assumptions of Proposition 2, the following assertions are valid:*

(a) *For some neighborhood $\mathcal{V}$ of $(\bar{x},\, \bar{\lambda})$ there exists the unique mapping $d(\cdot) = (\xi(\cdot),\, \eta(\cdot)) : \mathcal{V} \to \mathbb{R}^n \times \mathbb{R}^l$ with the following properties: $(\xi(x,\, \lambda),\, \eta(x,\, \lambda))$ satisfies* (4) *for every $(x,\, \lambda) \in \mathcal{V}$, and $d(\bar{x},\, \lambda) = 0$ if $\lambda \in \mathcal{M}(\bar{x})$.*

(b) *The neighborhood $\mathcal{V}$ can be chosen small enough, so that there exists $\ell > 0$ and a function $\chi : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\chi(t) = o(t)$ as $t \to 0$, and*

$$\|d(x,\, \lambda)\| \leq \ell(\|x - \bar{x}\| + \operatorname{dist}(\lambda,\, \mathcal{M}(\bar{x}))),$$

$$\|x + \xi(x,\, \lambda) - \bar{x}\| + \operatorname{dist}(\lambda + \eta(x,\, \lambda),\, \mathcal{M}(\bar{x})) \leq \chi(\|x - \bar{x}\| + \operatorname{dist}(\lambda,\, \mathcal{M}(\bar{x})))$$

for all $(x,\, \lambda) \in \mathcal{V}$.

(c) *There exists $M > 0$ such that for any $\varepsilon > 0$ there exists $\varepsilon_0 > 0$ such that for any $(x^0,\, \lambda^0) \in B((\bar{x},\, \bar{\lambda}),\, \varepsilon_0)$ the sequence $\{(x^k,\, \lambda^k)\}$ is correctly defined by the equality $(x^{k+1},\, \lambda^{k+1}) = (x^k,\, \lambda^k) + d(x^k,\, \lambda^k)$ for all k, and satisfies $\{(x^k,\, \lambda^k)\} \subset B((\bar{x},\, \bar{\lambda}),\, \varepsilon)$; this sequence converges to $(\bar{x},\, \lambda^*)$ for some $\lambda^* = \lambda^*(x^0,\, \lambda^0) \in \mathcal{M}(\bar{x})$, and for all k*

$$\|x^{k+1} - \bar{x}\| + \|\lambda^{k+1} - \lambda^*\| \leq M\chi(\|x^k - \bar{x}\| + \operatorname{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))). \tag{27}$$

Note that (27) immediately implies the following estimates for all k:

$$\begin{gathered}\|x^{k+1} - \bar{x}\| + \operatorname{dist}(\lambda^{k+1},\, \mathcal{M}(\bar{x})) \leq M\chi(\|x^k - \bar{x}\| + \operatorname{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))) \\ \|x^{k+1} - \bar{x}\| + \|\lambda^{k+1} - \lambda^*\| \leq M\chi(\|x^k - \bar{x}\| + \|\lambda^k - \lambda^*\|),\end{gathered}$$

and in particular, the rates of convergence of $\{(x^k,\, \lambda^k)\}$ to $(\bar{x},\, \lambda^*)$ and of $\{\|x^k - \bar{x}\| + \operatorname{dist}(\lambda^k,\, \mathcal{M}(\bar{x}))\}$ to zero are superlinear.

2.2 Nonvanishing stabilization

Suppose now that P and σ satisfy the following assumptions:

(P2) P is continuous at every point of $\{\bar{x}\} \times \mathcal{M}(\bar{x})$ close enough to $(\bar{x}, \bar{\lambda})$, $\operatorname{im} P(\bar{x}, \lambda) \cap \operatorname{im} h'(\bar{x}) = \{0\}$ for all $\lambda \in \mathcal{M}(\bar{x})$ close enough to $\bar{\lambda}$, and $\ker P(\bar{x}, \bar{\lambda}) \cap (\operatorname{im} h'(\bar{x}))^{\perp} = \{0\}$.

(S2) σ is continuous at $(\bar{x}, \bar{\lambda})$, and $\sigma(\bar{x}, \bar{\lambda}) \neq 0$.

The latter property is the reason why we call this stabilization nonvanishing.

We proceed with verifying Assumption 2 and 3 in [5] under (P2) and (S2). Observe that (P2) and (S2) are different from (P1) and (S1). With P and σ satisfying (P2) and (S2), the iterative process defined by (4) is related to the idea of the method from [14].

Regarding Assumption 2 in [5], observe that according to [5, Remark 2] it has to be verified only for solutions of subproblems. Specifically, the needed fact is established in the following

Proposition 3 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}^l$ be twice differentiable in a neighborhood of $\bar{x} \in \mathbb{R}^n$, with their second derivatives continuous at $\bar{x}$. Let $\bar{x}$ be a stationary point of problem* (1)*, and let $\bar{\lambda} \in \mathcal{M}(\bar{x})$. Assume that $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l\times l}$ and $\sigma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ satisfy assumptions* (P2) *and* (S2)*.*

Then there exists $\varepsilon > 0$ such that for any $C > 0$ there exists a function $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that $\omega(t) = o(t)$ as $t \to 0$, and for Φ and $\mathcal{A}$ defined by (8) *and* (9) *respectively, estimate* (10) *holds for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and all (ξ, η) satisfying* (4) *and* (11)*.*

Proof Fix any $\varepsilon > 0$ such that f and h are twice differentiable in $\mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$, P is continuous at every point of $(\{\bar{x}\} \times \mathcal{M}(\bar{x})) \cap \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$, $\operatorname{im} P(\bar{x}, \lambda) \cap \operatorname{im} h'(\bar{x}) = \{0\}$ and $\ker P(\bar{x}, \lambda) \cap (\operatorname{im} h'(\bar{x}))^{\perp} = \{0\}$ for all $\lambda \in \mathcal{M}(\bar{x}) \cap \mathcal{B}(\bar{\lambda}, \varepsilon)$ (assumption (P2)).

Arguing the same way as in proof of Proposition 1, we conclude that it is sufficient to prove the inequality (14) for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and all (ξ, η) satisfying (4) and (11), with some function $\omega_3\colon \mathbb{R}_+ \to \mathbb{R}_+$ satisfying $\omega_3(t) = o(t)$ as $t \to 0$.

For $R((x, \lambda), \eta)$ given by (19) set

$$\omega_3(t) = \sup \left\{ \|R((x, \lambda), \xi)\| \;\middle|\; \begin{array}{l} (x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon), \\ (\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^l \text{ satisfies (4)}, \\ \|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))) \le t, \\ \|(\xi, \eta)\| \le C(\|x - \bar{x}\| + \operatorname{dist}(\lambda, \mathcal{M}(\bar{x}))) \end{array} \right\}.$$

The set over which supremum is taken in the right-hand side is compact for any fixed $t \ge 0$. For those $t \ge 0$ for which this set is empty we set $\omega_3(t) = 0$, and with this convention the function ω_3 is well-defined. Moreover, for any $t \ge 0$ such that $\omega_3(t) \neq 0$ there exist some (x, λ) and (ξ, η) in this set such that $\omega_3(t) = \|R((x, \lambda), \eta)\|$.

The function ω_3 just defined satisfies (14) for all $(x, \lambda) \in \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and all (ξ, η) satisfying (4) and (11). It remains to prove that $\omega_3(t) = o(t)$ as $t \to 0$. Suppose the contrary: there exist $\gamma > 0$ and a sequence $\{t_k\}$ of positive reals, convergent to zero and such that $\omega_3(t_k) \geq \gamma t_k$ for all k. The latter implies the existence of sequences $\{(x^k, \lambda^k)\} \subset (\mathbb{R}^n \times \mathbb{R}^l) \cap \mathcal{B}((\bar{x}, \bar{\lambda}), \varepsilon)$ and $\{(\xi^k, \eta^k)\} \subset \mathbb{R}^n \times \mathbb{R}^l$ such that the inequalities (16)–(18) and the equalities in (24) hold for all k.

According to (17) and (18), passing to subsequences, if necessary, we may assume that $\{(x^k, \lambda^k)\}$ converges to $(\bar{x}, \lambda^*)$ with some $\lambda^* \in \mathcal{M}(\bar{x})$, and for all k it holds that $\|(\xi^k, \eta^k)\| \leq Ct_k$, implying that $\|x^k + \xi^k - \bar{x}\| \leq (1+C)t_k$. Hence, we may assume without loss of generality that

$$\left\{\frac{1}{t_k}\eta^k\right\} \to \eta, \quad \left\{\frac{1}{t_k}(x^k + \xi^k - \bar{x})\right\} \to \xi \tag{28}$$

for some $\xi \in \mathbb{R}^n$ and $\eta \in \mathbb{R}^l$. Finally, according to assumption (S2), we may assume without loss of generality that $\varepsilon > 0$ is chosen in such a way that $\sigma(x^k, \lambda^k) \to \bar{\sigma}$ as $k \to \infty$ for some real $\bar{\sigma} \neq 0$.

The second equation in (4) implies

$$\begin{aligned}
\sigma(x^k, \lambda^k)P(x^k, \lambda^k)\eta^k &= h(x^k) + h'(x^k)\xi^k \\
&= (h(x^k) - h(\bar{x}) - h'(\bar{x})(x^k - \bar{x})) + (h'(x^k) - h'(\bar{x}))\xi^k \\
&\quad + h'(\bar{x})(x^k + \xi^k - \bar{x}) \\
&= h'(\bar{x})(x^k + \xi^k - \bar{x}) \\
&\quad + O(\|x^k - \bar{x}\|^2) + O(\|x^k - \bar{x}\|\|\xi^k\|) \\
&= h'(\bar{x})(x^k + \xi^k - \bar{x}) + O(t_k^2),
\end{aligned} \tag{29}$$

where the next-to-last equality is by the mean-value theorem, and the last is by (17) and (18). Dividing (29) by t_k, passing onto the limit and employing (28) and assumption (P2), we obtain

$$\bar{\sigma}P(\bar{x}, \lambda^*)\eta = h'(\bar{x})\xi.$$

Since $\operatorname{im} P(\bar{x}, \lambda^*) \cap \operatorname{im} h'(\bar{x}) = \{0\}$ (which follows from assumption (P2)), this may only hold when $\bar{\sigma}P(\bar{x}, \lambda^*)\eta = 0$. On the other hand, (16) implies that $\|\bar{\sigma}P(\bar{x}, \lambda^*)\eta\| \geq \gamma$, giving a contradiction. $\square$

Finally, by the argument similar to the one in [14, Theorem 2.4], it can be seen that if assumptions (P2) and (S2) hold and $\bar{\lambda}$ is noncritical, then the matrix

$$\begin{pmatrix} \dfrac{\partial^2 L}{\partial x^2}(\bar{x}, \bar{\lambda}) & (h'(\bar{x}))^{\mathrm{T}} \\ h'(\bar{x}) & -\sigma(\bar{x}, \bar{\lambda})P(\bar{x}, \bar{\lambda}) \end{pmatrix}$$

is nonsingular, and this evidently implies Assumption 3 in [5]. Moreover, the solution of the iteration system (4) is necessarily unique for all $(x, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$ close enough to $(\bar{x}, \bar{\lambda})$.

Applying [5, Theorem 1] again, we obtain the following local convergence result.

Theorem 2 *Under the assumptions of Proposition 2, but with* (P1) *and* (S1) *replaced by* (P2) *and* (S2)*, respectively, the assertions of Theorem 1 remain valid.*

3 Approximation of the degeneracy subspace

In this section we are concerned with defining P with the needed properties in a practical way. We start with the following auxiliary problem setting. Let $\bar{A} \in \mathbb{R}^{l \times n}$ be some fixed but "unavailable" matrix, and let $r = \operatorname{rank} \bar{A}$ be known. The task is to construct a mapping $U \colon \mathbb{R}^{l \times n} \to \mathbb{R}^{l \times (l-r)}$ such that for any accumulation point $\bar{U}$ of $U(\cdot)$ at $\bar{A}$ it holds that $\operatorname{rank} \bar{U} = l - r$ and $\bar{A}^{\mathrm{T}} \bar{U} = 0$. We develop a simple Algorithm 31 accomplishing this task, which is a variant of the classical Gaussian elimination with full pivoting (the difference is that no actual permutation of rows and/or columns is performed). Specifically, for a given matrix $A \in \mathbb{R}^{l \times n}$ close enough to $\bar{A}$, this algorithm computes a matrix $U(A) \in \mathbb{R}^{l \times (l-r)}$ such that the mapping $U(\cdot)$ defined this way possesses the needed properties. Afterwards, the algorithm is extended to cover the case when r is not supposed to be known. The resulting Algorithm 32 produces matrices allowing to define the needed P.

Algorithm 31 Set $\mathcal{I}_0 = \emptyset$, $U^{(0)} = I \in \mathbb{R}^{l \times l}$, $A^{(0)} = A$, $s = 0$.

1. If $s = r$, go to step 4. Otherwise, find indices $i_s \in \{1, \ldots, l\} \setminus \mathcal{I}_s$ and $j_s \in \{1, \ldots, n\}$ such that
$$|A^{(s)}_{i_s j_s}| = \max_{\substack{i \in \{1, \ldots, l\} \setminus \mathcal{I}_s, \\ j \in \{1, \ldots, n\}}} |A^{(s)}_{ij}|. \tag{30}$$
If $A^{(s)}_{i_s j_s} = 0$, stop with failure. Otherwise, set $\mathcal{I}_{s+1} = \mathcal{I}_s \cup \{i_s\}$.
2. Define the transformation matrix $T^{(s)} \in \mathbb{R}^{l \times l}$,
$$T^{(s)}_{ij} = \begin{cases} -\frac{A^{(s)}_{j j_s}}{A^{(s)}_{i_s j_s}} & \text{if } i = i_s,\ j \notin \mathcal{I}_{s+1}, \\ 1 & \text{if } i = j, \\ 0 & \text{otherwise}, \end{cases} \qquad i, j = 1, \ldots, l. \tag{31}$$
Set $U^{(s+1)} = U^{(s)} T^{(s)}$ and $A^{(s+1)} = (T^{(s)})^{\mathrm{T}} A^{(s)}$.
3. Increase s by 1 and go to step 1.
4. Define $U(A)$ as the submatrix of $U^{(r)}$ with columns indexed by $i \in \{1, \ldots, l\} \setminus \mathcal{I}_r$.

In principle, $U(A)$ (and other products of Algorithm 31 and Algorithm 32 below) may depend on the choices of i_s and j_s on step 1 (when maximum in (30) is attained on several pairs of indices). However, the properties being established below are valid for any possible realization of $U(A)$. On the other hand, any specific implementation of these algorithms would always produce the same $U(A)$ for a given A.

By construction, for each s

$$U^{(s)} = U^{(s-1)}T^{(s-1)} = U^{(0)}T^{(0)} \dots T^{(s-1)} = T^{(0)} \dots T^{(s-1)}, \tag{32}$$

$$A^{(s)} = (T^{(s-1)})^{\mathrm{T}} A^{(s-1)} = (T^{(0)} \dots T^{(s-1)})^{\mathrm{T}} A^{(0)} = (U^{(s)})^{\mathrm{T}} A, \tag{33}$$

and $\det T^{(s)} = 1$, and therefore, $\det U^{(s)} = 1$ and $\operatorname{rank} A^{(s)} = \operatorname{rank} A = r$.

Suppose that Algorithm 31 stops with failure on step 1. In this case all rows of $A^{(s)}$ with indices in $\{1, \dots, l\} \setminus \mathcal{I}_s$ contain only zero entries, which implies that $\operatorname{rank} A^{(s)} \le s < r$. Therefore, if $\operatorname{rank} A \ge r$ (which is true for all A close enough to $\bar{A}$), then Algorithm 31 never stops with failure.

Furthermore, for each s it holds by construction that $|T_{ij}^{(s)}| \le 1$ for all $i, j = 1, \dots, l$. Therefore, from (32) it follows that $\|U^{(r)}\|$ (and hence $\|U(A)\|$) can be bounded by some constant that depends on r but does not depend on a specific matrix A.

Lemma 1 *If $A \in \mathbb{R}^{l \times n}$ is such that $\operatorname{rank} A = r$, then for a matrix $A^{(r)}$ produced by Algorithm 31 it holds that $A_{ij}^{(r)} = 0$ for all $i \in \{1, \dots, l\} \setminus \mathcal{I}_r$ and $j \in \{1, \dots, n\}$.*

Proof Merely for simplicity of presentation we will suppose that for each $s = 0, \dots, r-1$ it holds that $i_s = s+1$ and $j_s = s+1$. In this case matrix $A^{(s)}$ has the following form for each $s = 0, \dots, r$:

$$A^{(s)} = \begin{pmatrix} A_{11}^{(0)} & A_{12}^{(0)} & \cdots & A_{1s}^{(0)} & A_{1(s+1)}^{(0)} & \cdots & A_{1n}^{(0)} \\ 0 & A_{22}^{(1)} & \cdots & A_{2s}^{(1)} & A_{2(s+1)}^{(1)} & \cdots & A_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{ss}^{(s-1)} & A_{s(s+1)}^{(s-1)} & \cdots & A_{sn}^{(s-1)} \\ 0 & 0 & \cdots & 0 & A_{(s+1)(s+1)}^{(s)} & \cdots & A_{(s+1)n}^{(s)} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & A_{l(s+1)}^{(s)} & \cdots & A_{ln}^{(s)} \end{pmatrix}$$

Observe that $A_{i_s j_s}^{(s)} \ne 0$ for such s (otherwise the algorithm would terminate with failure, which is impossible since $\operatorname{rank} A = r$), and therefore, $A_{ii}^{(s)} \ne 0$ for $i \in \{1, \dots, s\}$. It further implies that the top-left $s \times s$-minor of $A^{(s)}$ is nonzero.

Therefore, if $A^{(r)}$ has nonzero entries in the rows from $r+1$ to l, then, evidently, $\operatorname{rank} A^{(r)} > r$, which is again impossible since $\operatorname{rank} A = r$. □

Lemma 2 *If $A \in \mathbb{R}^{l \times n}$ is such that $\operatorname{rank} A \ge r$, then for a matrix $U(A)$ produced by Algorithm 31 it holds that $\operatorname{rank} U(A) = l - r$. Moreover, if $\operatorname{rank} A = r$, then $A^{\mathrm{T}} U(A) = 0$.*

Proof The equality $\operatorname{rank} U(A) = l - r$ immediately follows from the definition of $U(A)$ in step 4 of the algorithm, since $\det U^{(r)} = 1$. The equality $A^{\mathrm{T}} U(A) = 0$ is again by the definition of $U(A)$, by (33), and by Lemma 1. □

We emphasize again that the properties asserted in Lemma 2 do not depend on the specific choices of i_s and j_s on step 1 of Algorithm 31, even though $U(A)$ itself may depend on these choices.

Proposition 4 *Let $\bar{A} \in \mathbb{R}^{l\times n}$ be such that* $\operatorname{rank} \bar{A} = r$, *and let $\{A_k\} \subset \mathbb{R}^{l\times n}$ be a sequence of matrices convergent to $\bar{A}$. For each k large enough, let $U(A_k)$ be produced by Algorithm 31.*

Then for any accumulation point $\bar{U}$ of the bounded sequence $\{U(A_k)\}$ it holds that $\operatorname{rank} \bar{U} = l - r$ *and* $\bar{A}^{\mathrm{T}}\bar{U} = 0$.

Proof Observe that $\operatorname{rank} A_k \geq r$ for all k large enough, and therefore, Algorithm 31 does not terminate with failure when applied to $A = A_k$ or $A = \bar{A}$.

In order to establish that $\operatorname{rank} \bar{U} = l - r$ and $\bar{A}^{\mathrm{T}}\bar{U} = 0$, we argue by contradiction: suppose that there exists an accumulation point $\bar{U}$ of $\{U(A_k)\}$ such that either $\operatorname{rank} \bar{U} < l - r$ or $\bar{A}^{\mathrm{T}}\bar{U} \neq 0$. Without loss of generality we assume that $\{U(A_k)\}$ converges to $\bar{U}$, and moreover, that for each $s = 0, \ldots, r-1$, the indices i_s and j_s chosen on step 1 of Algorithm 31 applied to $A = A_k$ are the same for all k (otherwise we pass to appropriate subsequences).

It is easy to see that for each $s = 0, \ldots, r-1$, Algorithm 31 applied to $A = \bar{A}$ may also pick up i_s and j_s specified above. Indeed, since (30) holds with $A = A_k$ for every k, and since $\{A_k\}$ converges to $\bar{A}$, passing onto the limit in (30) as $k \to \infty$, we obtain the equality

$$|\bar{A}^{(s)}_{i_s j_s}| = \max_{\substack{i\in\{1,\,\ldots,\,l\}\setminus\mathcal{I}_s,\\ j\in\{1,\,\ldots,\,n\}}} |\bar{A}^{(s)}_{ij}|,$$

and hence, (30) holds with $A = \bar{A}$ as well.

For each k and each $s = 0, \ldots, r-1$, let $T_k^{(s)}$, $U_k^{(r)}$ and $U(A_k)$ stand for the matrices produced by Algorithm 31 applied to $A = A_k$ (with the specified choices of i_s and j_s), and let $T^{(s)}$, $U^{(r)}$ and $U(\bar{A})$ stand for the matrices produced by Algorithm 31 applied to $A = \bar{A}$ (with the same choices of i_s and j_s).

From (31) and from convergence of $\{A_k\}$ to $\bar{A}$ it follows that $\{T_k^{(s)}\}$ converges to $T^{(s)}$ for all $s = 0, \ldots, r-1$, and hence, by (32), $\{U_k^{(r)}\}$ converges to $U^{(r)}$, implying that $\{U(A_k)\}$ converges to $U(\bar{A})$. Therefore, $U(\bar{A}) = \bar{U}$, and hence, any of the properties $\operatorname{rank} \bar{U} < l - r$ and $\bar{A}^{\mathrm{T}}\bar{U} \neq 0$ contradicts Lemma 2. □

Now we extend the constructions above to the case when the rank r of $\bar{A}$ is not supposed to be known. This construction relies on the following estimates.

Lemma 3 *Suppose that for $\bar{A} \in \mathbb{R}^{l\times n}$ it holds that* $\operatorname{rank} \bar{A} = r$.

Then there exist $\varepsilon > 0$ and $M > 0$ with the following properties: for any $A \in \mathbb{R}^{l\times n}$ close enough to $\bar{A}$, and for $R^{(s)}(A) \in \mathbb{R}^{(l-s)\times n}$ standing for the submatrix of $A^{(s)}$ with rows indexed by $i \in \{1, \ldots, l\} \setminus \mathcal{I}_s$, where $\mathcal{I}_s$ and $A^{(s)}$ are obtained by Algorithm 31, the following conditions hold:

$$\|R^{(s)}(A)\| \geq \varepsilon \quad \forall\, s = 0, \ldots, r-1, \tag{34}$$

$$\|R^{(r)}(A)\| \leq M\|A - \bar{A}\|. \tag{35}$$

Observe that according to Lemma 1, if $\operatorname{rank} A = r$, then $R^{(r)}(A) = 0$. Estimate (35) covers the case when $\operatorname{rank} A$ can be greater than r.

Proof We first prove the existence of $\varepsilon > 0$ satisfying (34) for all A close enough to $\bar{A}$. Suppose the contrary: there exists $q \in \{0, \ldots, r-1\}$ and a sequence $\{A_k\} \subset \mathbb{R}^{l\times n}$ convergent to $\bar{A}$ such that for matrices $R^{(q)}(A_k)$ it holds that

$$\lim_{k\to\infty} R^{(q)}(A_k) = 0. \tag{36}$$

We can assume without loss of generality that for each $s = 0, \ldots, q-1$ the indices i_s and j_s chosen on step 1 of Algorithm 31 applied to $A = A_k$ are the same for all k (otherwise we pass to appropriate subsequences). Repeating the same arguments as in proof of Proposition 4, we can easily show that for each $s = 0, \ldots, q-1$, Algorithm 31 applied to $A = \bar{A}$ may also pick up i_s and j_s specified above, and that the matrix $R^{(q)}(\bar{A})$ obtained accordingly satisfies

$$\lim_{k\to\infty} R^{(q)}(A_k) = R^{(q)}(\bar{A}).$$

Combining this condition with (36), we obtain $R^{(q)}(\bar{A}) = 0$, which means that for the matrix $\bar{A}^{(q)}$ obtained by Algorithm 31 applied to $A = \bar{A}$ it holds that $\operatorname{rank} \bar{A} = \operatorname{rank} \bar{A}^{(q)} \leq q < r$, giving a contradiction.

We next prove the existence of $M > 0$ satisfying (35) for all A close enough to $\bar{A}$. Observe that for $A = \bar{A}$ (35) is satisfied with any $M > 0$, since both sides of this inequality are equal to zero. Suppose that for some sequence $\{A_k\} \subset \mathbb{R}^{l\times n}\setminus\{\bar{A}\}$ convergent to $\bar{A}$, for matrices $R^{(r)}(A_k)$ generated according to Algorithm 31 it holds that

$$\lim_{k\to\infty} \frac{\|R^{(r)}(A_k)\|}{\|A_k - \bar{A}\|} = +\infty. \tag{37}$$

We again assume without loss of generality that for each $s = 0, \ldots, r-1$ the indices i_s and j_s chosen on step 1 of Algorithm 31 applied to $A = A_k$ are the same for all k, which implies that for each $s = 0, \ldots, r-1$, Algorithm 31 applied to $A = \bar{A}$ may also pick up specified indices i_s and j_s.

Note that for each $s = 0, \ldots, r-1$ the matrix $T^{(s)}$ defined by (31) for any fixed i_s and j_s, being considered as a function of A (mapping $\mathbb{R}^{l\times n}$ to $\mathbb{R}^{l\times l}$), is analytic near $\bar{A}$. Therefore, the matrix $A^{(r)} = A^{(r)}(A)$ defined according to (32), (33), is also analytic near $\bar{A}$ as a function of A, and hence, Lipschitz-continuous near $\bar{A}$. It then follows from definition of $R^{(r)}(\cdot)$ and from the equality $R^{(r)}(\bar{A}) = 0$ that for some $M > 0$

$$\|R^{(r)}(A_k)\| = \|R^{(r)}(A_k) - R^{(r)}(\bar{A})\| \leq \|A^{(r)}(A_k) - A^{(r)}(\bar{A})\| \leq M\|A_k - \bar{A}\|$$

for all k, which evidently contradicts (37). $\square$

This lemma allows us to modify Algorithm 31 so that it would be automatically estimating r. Suppose that along with $A \in \mathbb{R}^{l\times n}$, we have at hand some quantity $t \geq 0$ somehow measuring the distance from A to $\bar{A}$.

Algorithm 32 Set $\mathcal{I}_0 = \emptyset$, $U^{(0)} = I \in \mathbb{R}^{l\times l}$, $A^{(0)} = A$, $R^{(0)}(A,\, t) = A$, $s = 0$.

1. If $s = l$ or
$$\|R^{(s)}(A,\, t)\| \leq t, \tag{38}$$
set $r = s$ and go to step 4. Otherwise, find indices $i_s \in \{1,\, \dots,\, l\} \setminus \mathcal{I}_s$ and $j_s \in \{1,\, \dots,\, n\}$ such that (30) holds. Set $\mathcal{I}_{s+1} = \mathcal{I}_s \cup \{i_s\}$.
2. Define the transformation matrix $T^{(s)} \in \mathbb{R}^{l\times l}$ by (31). Set $U^{(s+1)} = U^{(s)}T^{(s)}$, $A^{(s+1)} = (T^{(s)})^{\mathrm{T}}A^{(s)}$, and define $R^{(s+1)}(A,\, t)$ as a submatrix of $A^{(s+1)}$ with rows indexed by $i \in \{1,\, \dots,\, l\} \setminus \mathcal{I}_{s+1}$.
3. Increase s by 1 and go to step 1.
4. Define $U(A,\, t)$ as the submatrix of $U^{(r)}$ with columns indexed by $i \in \{1,\, \dots,\, l\} \setminus \mathcal{I}_r$.

Note that the value $A^{(s)}_{i_s j_s}$ obtained on step 1 of Algorithm 32 is always distinct from zero, since otherwise the test (38) would be satisfied. Therefore, Algorithm 32 always terminates successfully.

Proposition 5 *Let $\tau > 0$ and $\theta \in (0,\, 1)$ be fixed. Let $\{A_k\} \subset \mathbb{R}^{l\times n}$ be a sequence of matrices convergent to some $\bar{A} \in \mathbb{R}^{l\times n}$. Let $\{\sigma_k\}$ be a sequence of reals such that $\sigma_k \to 0$ as $k \to \infty$, and there exists $M > 0$ such that for all k*
$$\|A_k - \bar{A}\| \leq M|\sigma_k|.$$

Then for all k large enough, Algorithm 32 applied to $A = A_k$ and $t = \tau|\sigma_k|^\theta$ terminates with $r = \operatorname{rank} \bar{A}$, and for any accumulation point $\bar{U}$ of the bounded sequence $\{U(A_k,\, \tau|\sigma_k|^\theta)\}$ it holds that $\operatorname{rank} \bar{U} = l - \operatorname{rank} \bar{A}$ and $\bar{A}^{\mathrm{T}}\bar{U} = 0$.

Proof Taking into account Proposition 4, we only need to show that for all k large enough Algorithm 32 stops with $r = \operatorname{rank} \bar{A}$, since in this case $U(A_k,\, \tau|\sigma_k|^\theta)$ coincides with $U(A_k)$ obtained by Algorithm 31 (assuming that both algorithms accept the identical choices of i_s and j_s for each $s = 0,\, \dots,\, r-1$).

Suppose first that there exist infinitely many indices k such that Algorithm 32 applied to $A = A_k$ and $t = \tau|\sigma_k|^\theta$ terminates with some $r < \operatorname{rank} \bar{A}$. Since $\operatorname{rank} \bar{A} \leq l$, and hence $r < l$, the test (38) is satisfied for these indices with $s = r$. Therefore, the corresponding subsequence of $\{R^{(r)}(A_k,\, \tau|\sigma_k|^\theta)\}$ tends to zero, which contradicts Lemma 3 according to which there exists $\varepsilon > 0$ such that $\|R^{(r)}(A_k)\| \geq \varepsilon$ for all k large enough, and hence, $\|R^{(r)}(A_k,\, \tau|\sigma_k|^\theta)\| \geq \varepsilon$ for all k large enough (since the matrices $R^{(s)}(A_k)$ defined by Lemma 3, and $R^{(s)}(A_k,\, \tau|\sigma_k|^\theta)$ produced by Algorithm 32 are the same for $s \leq \operatorname{rank} A_k$).

Suppose now that for infinitely many k Algorithm 32 stops with $r > \operatorname{rank} \bar{A}$. Evidently, for all such k it holds that
$$\|R^{(\operatorname{rank} \bar{A})}(A_k,\, \tau|\sigma_k|^\theta)\| > \tau|\sigma_k|^\theta \geq \frac{\tau\|A_k - \bar{A}\|^\theta}{M^\theta}.$$

On the other hand, from Lemma 3 we have that
$$\|R^{(\operatorname{rank} \bar{A})}(A_k)\| = O(\|A_k - \bar{A}\|)$$
as $k \to \infty$, giving a contradiction, since $\theta \in (0,\, 1)$. □

Getting back to the subspace-stabilized SQP, we can now define the mapping $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l\times l}$ as follows:

$$P(x,\, \lambda) = U(U^{\mathrm{T}}U)^{-1}U^{\mathrm{T}}, \tag{39}$$

where $U = U(h'(x),\, \tau|\sigma(x,\, \lambda)|^{\theta})$ is defined by Algorithm 32 applied to $A = h'(x)$ and $t = \tau|\sigma(x,\, \lambda)|^{\theta}$ for some fixed values $\tau > 0$, $\theta \in (0,\, 1)$.

It is standard and can be easily checked that

$$\bar{P} = \bar{U}(\bar{U}^{\mathrm{T}}\bar{U})^{-1}\bar{U}^{\mathrm{T}}$$

for any choice of a matrix $\bar{U} \in \mathbb{R}^{l\times(l-r)}$ satisfying $\operatorname{rank}\bar{U} = l - r$ and $(h'(\bar{x}))^{\mathrm{T}}\bar{U} = 0$. From Proposition 5 it then follows that if h is smooth enough at $\bar{x}$ and the function σ satisfies (S1), then for any $\lambda^* \in \mathcal{M}(\bar{x})$ close enough to $\bar{\lambda}$ the mapping P defined by (39) satisfies

$$\lim_{k\to\infty} P(x^k,\, \lambda^k) = \bar{P}$$

for any sequence $\{(x^k,\, \lambda^k)\}$ convergent to $(\bar{x},\, \lambda^*)$. The latter implies

Proposition 6 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable at $\bar{x} \in \mathbb{R}^n$, and let $h : \mathbb{R}^n \to \mathbb{R}^l$ be differentiable in a neighborhood of $\bar{x}$, with its derivative Lipschitz-continuous with respect to $\bar{x}$, that is,*

$$h'(x) - h'(\bar{x}) = O(\|x - \bar{x}\|)$$

as $x \to \bar{x}$. Let $\bar{x}$ be a stationary point of problem (1)*, and let $\bar{\lambda} \in \mathcal{M}(\bar{x})$. Assume that $\sigma : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}$ satisfies* (S1)*.*

Then for any $\tau > 0$ and $\theta \in (0,\, 1)$, the mapping $P : \mathbb{R}^n \times \mathbb{R}^l \to \mathbb{R}^{l\times l}$ defined by (39)*, where $U = U(h'(x),\, \tau|\sigma(x,\, \lambda)|^{\theta})$ is defined by Algorithm 32, satisfies both assumptions* (P1) *and* (P2)*.*

4 Numerical results

In this section we provide some numerical evidence that our stabilization along the appropriate subspace may indeed be an efficient improvement over the usual SQP and stabilized SQP methods. As commented above, in [21] it has been observed that the latter often demonstrates poor performance on problems with degenerate but not fully degenerate solutions: it has a strong tendency to generate long sequences of short primal-dual steps, thus drastically slowing down the convergence. Our primary interest is to demonstrate that stabilization along the subspace can indeed be a remedy for this bad behavior of the full-space stabilized SQP. Evidently, one crucial ingredient of this remedy is correct identification of the rank of the constraints' Jacobian.

Our computations were performed in Matlab environment, using its standard tools to solve linear systems. We recall that for a given current iterate $(x,\, \lambda) \in \mathbb{R}^n \times \mathbb{R}^l$, the usual SQP method generates the next iterate as

$(x+\xi,\, \lambda+\eta)$, where the step $(\xi,\, \eta)$ is computed as a solution of the system (2) with $\sigma = 0$. The stabilized SQP method generates the iterates the same way, but employing $\sigma = \sigma(x,\, \lambda)$ defined according to (7) with $\beta = 1$.

The same $\sigma(x,\, \lambda)$ is used in the subspace-stabilized SQP algorithm with asymptotically vanishing stabilization, for which the iteration system (2) is replaced by (4). As for $P(x,\, \lambda)$, it is defined according to Proposition 6, employing Algorithm 32 with the parameter values $\tau = 0.3$ and $\theta = 0.8$, which appear to be the best choice in our experiments. Recall that this procedure correctly identifies the rank of the constraints' Jacobian at the primal solution when $(x,\, \lambda)$ is close to a primal-dual solution of (3) with a noncritical dual part.

In the subspace-stabilized SQP algorithm with nonvanishing stabilization, we define $P(\cdot)$ exactly as above, but the algorithm itself employs $\sigma(\cdot) \equiv 1$.

In cases of convergence to a critical multiplier $\sigma(x^k,\, \lambda^k)$ usually becomes too small for large k, so the test (38) fails for $A = h'(x^k)$ and $t = \tau|\sigma(x^k,\, \lambda^k)|^\theta$, for any $s = 0, \ldots, l-1$, and therefore Algorithm 32 returns $r = l$ and $P(x^k,\, \lambda^k) = 0$. In this case, the subspace-stabilized SQP eventually turns into the pure SQP, and as discussed above, the rate of convergence in this case is only linear [17, Chapter 7].

We start with the following example.

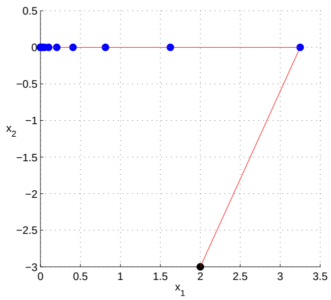
Example 1 This is problem 20204 from DEGEN test collection [1]. Let $n = l = 2$, $f(x) = (x_1^2 + x_2^2)/2$, $h(x) = ((x_1^2 + x_2^2)/2 - x_2,\, (x_1^2 + x_2^2)/2 + x_2)$. Then $\bar{x} = 0$ is the unique feasible point, and hence, the unique solution of problem (1). Furthermore, $\operatorname{rank} h'(\bar{x}) = 1$, and Lagrange multipliers associated with $\bar{x}$ are those $\lambda \in \mathbb{R}^2$ satisfying $\lambda_1 = \lambda_2$. The unique critical multiplier is $\bar{\lambda} = (-1/2,\, -1/2)$.

Figure 1 shows the primal sequences of the basic SQP method, the stabilized SQP method, and the subspace-stabilized SQP method with vanishing and nonvanishing stabilization, starting from $x^0 = (2,\, -3)$, $\lambda^0 = (-10,\, 15)$.

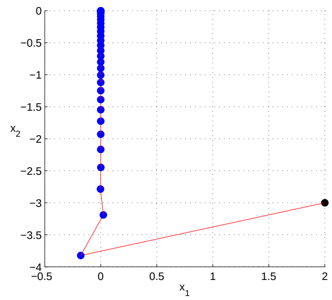
For the SQP method, the stopping criterion is satisfied after 17 iterations. One can see from Figure 1(a) that the primal sequence converges along $\ker h'(\bar{x})$, and the convergence rate is linear. The reason is that the dual sequence converges to the critical multiplier. We emphasize again that this is a typical behavior for the SQP method when applied to problems with degenerate constraints.

The stabilized SQP demonstrates quite a different behavior: the projections of the primal iterates onto $\ker h'(\bar{x})$ rapidly converge to zero, while their projections onto $(\ker h'(\bar{x}))^\perp$ move very slowly. The stopping criterion is satisfied only after 30 iterations despite the fact that the limiting multiplier is noncritical, and that eventually the superlinear convergence rate shows up (which cannot be seen from Figure 1(b)).

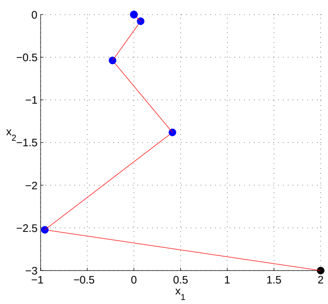
Poor behavior of the SQP and the stabilized SQP methods can be interpreted here as the effect of "inadequate dual stabilization". Dual sequences of the SQP method are not stabilized along the degeneracy subspace $\ker(h'(\bar{x}))^{\mathrm{T}}$, and as a result, these sequences converge to the critical multiplier along this



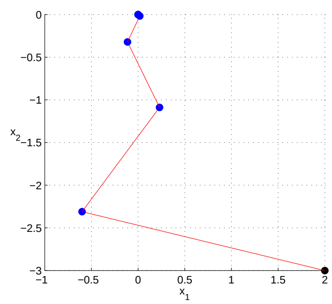
(a) SQP



(b) Stabilized SQP



(c) Subspace-stabilized SQP with vanishing stabilization



(d) Subspace-stabilized SQP with nonvanishing stabilization

Fig. 1 Primal sequences for Example 1; $x^0 = (2,\, -3)$, $\lambda^0 = (-10,\, 15)$.

subspace. On the other hand, dual sequences of the stabilized SQP are stabilized not only along the degeneracy subspace, but also along its complement $(\ker(h'(\bar{x}))^{\mathrm{T}})^{\perp} = \operatorname{im} h'(\bar{x})$, and such "over-stabilization" results in very short primal-dual steps.

At the same time, the subspace-stabilized SQP method correctly identifies the rank of $h'(\bar{x})$ and produces $P(x^k,\, \lambda^k)$ close to a $\bar{P}$. Hence, dual sequences are stabilized along the degeneracy subspace only, which results in fast convergence: the stopping criterion is satisfied after 7 and 6 iterations in cases of vanishing and nonvanishing stabilization, respectively. □

In the rest of this section we provide a comparison of the same methods but on randomly generated problems with quadratic objective functions and quadratic equality-constraints, using the generator from [15]. For each triple $(n,\, l,\, r)$ we generated 10 such problems with n variables and l constraints, such that 0 is a stationary point of each problem, with the rank of constraints' Jacobian at 0 equal to r, and with some associated Lagrange multiplier $\bar{\lambda}$ being also provided by the generator. We used all triples with n from 2 to 5, l from 1 to n, and r from 0 to l.

In these experiments, we were interested in "semi-local" behavior of the SQP and its stabilized versions. To that end, we generated starting points $(x^0, \lambda^0) \in \mathbb{R}^n \times \mathbb{R}^l$ satisfying $\|(x^0, \lambda^0 - \bar{\lambda})\| \leq R$, for four values $R = 0.01$, 0.1, 0.5, and 1, in order to figure how the iteration count depends on closeness of starting points to solutions. For each problem and each value of R, we performed 20 runs from such random starting points (all algorithms were initialized at the same starting points). Run was considered successful when the stopping criterion $\|\Phi(x^k, \lambda^k)\| \leq 10^{-8}$ was satisfied before the iteration count k exceeded 500.

The results are presented below in the form of performance profiles, which is a slightly modified version of the original proposal in [2]. For each algorithm the plotted function $\pi\colon [1, \infty) \to [0, 1]$ is defined as follows. Let k_p stand for the average iteration count of a given algorithm per one successful run for problem p. Let s_p denote the portion of successful runs on this problem. Let r_p be equal to the best (minimum) value of k_p over all algorithms. Then

$$\pi(\tau) = \frac{1}{P} \sum_{p \in S(\tau)} s_p,$$

where P is the number of problems in the test set, while $S(\tau)$ is the set of problems for which k_p is no more than τ times worse (larger) than the best result r_p:

$$S(\tau) = \{p = 1, \ldots, P \mid k_p \leq \tau r_p\}, \quad \tau \in [1, \infty).$$

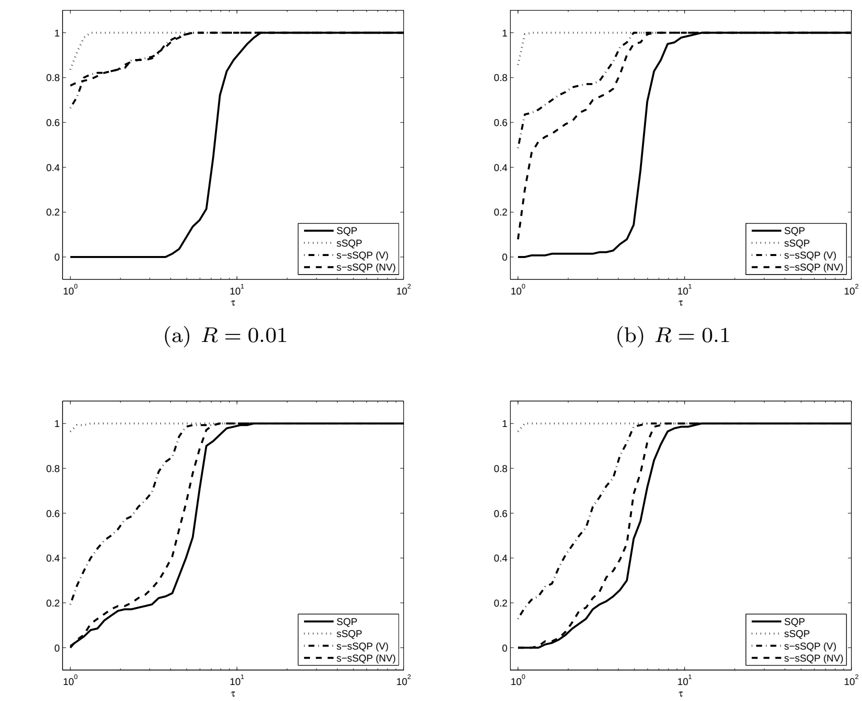
In particular, the value $\pi(1)$ corresponds to the portion of problems for which the given algorithm demonstrated the best result on the average. The values of $\pi(\tau)$ for large τ correspond to the portion of successful runs.

In the legend of performance profiles, the stabilized SQP is abbreviated as sSQP, while the subspace-stabilized SQP with vanishing and nonvanishing stabilization are abbreviated as s-sSQP (V) and s-sSQP (NV), respectively.

We consider separately the following three cases: the case of fully degenerate constraints ($r = 0$); the case when constraints are degenerate but not fully degenerate ($0 < r < l$); and the case of nondegenerate constraints ($r = l$). The reason is that in these three cases, the relative behavior of the methods in question is quite different.

Full degeneracy is of course a rather special kind of degeneracy; we consider it separately because, as discussed above, the stabilized SQP demonstrates very good performance in this case, and hardly requires any further improvements. This is confirmed by Figure 2: as expected, the stabilized SQP seriously outperforms the usual SQP. The subspace-stabilized SQP methods are somewhat less efficient on this kind of problems, but they are still evidently better than the usual SQP, especially when initialized close to solutions.

Now we turn our attention to the case when $0 < r < l$ which is a typical kind of constraints' degeneracy. The corresponding results are presented on Figure 3. As can be seen from Figure 3(a), and as expected from the local convergence theory presented above, if starting points are taken close enough to solutions, all stabilized methods behave well and significantly outperform the



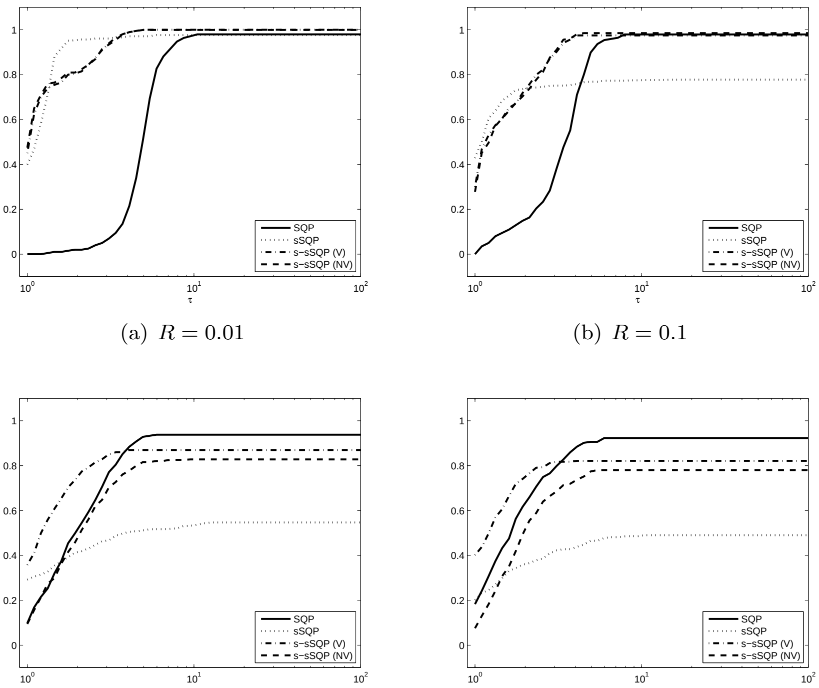
(a) $R = 0.01$ (b) $R = 0.1$

(c) $R = 0.5$ (d) $R = 1$

Fig. 2 Results for problems with fully degenerate constraints.

usual SQP. However, as can be seen from Figures 3(b)–3(d), with initialization farther from solutions, performance of the stabilized SQP degrades drastically: the method starts to produce long series of short steps, and in many cases this results in failures due to the iteration limit. On the contrary, performance of the subspace-stabilized SQP methods is not affected so seriously by moving starting points farther from solutions. In particular, the subspace-stabilized SQP with vanishing stabilization still requires less iterations on the average over successful runs than the SQP, and the portion of failures grows not much faster than for the SQP.

Finally we consider the problems with nondegenerate constraints. For such problems the SQP usually possesses local superlinear convergence, and hence, the theory presented above does not give reasons to expect that the stabilized methods would perform better than the SQP. This is confirmed by Figure 4, where the overall picture is similar to that in Figure 3 for the nonfully degenerate case, though the portion of failures for the subspace-stabilized SQP methods grows significantly faster as the starting points are being moved farther from the solution.



(a) $R = 0.01$ (b) $R = 0.1$

(c) $R = 0.5$ (d) $R = 1$

Fig. 3 Results for problems with nonfully degenerate constraints.

5 Concluding remarks

This paper is intended to demonstrate that stabilization along the degeneracy subspace is crucially important for a success of SQP methods supplied with dual stabilization. If stabilization is performed along a smaller subspace (or is not performed at all, as in the case of the pure SQP method), dual sequences typically converge to a critical multiplier, and the convergence rate is only linear. On the other hand, if stabilization is performed along a larger subspace (or the entire dual space, as in the stabilized SQP method), long sequences of short steps are typically encountered, inevitably degrading performance of any globalized algorithm attempting to accept the full stabilized SQP steps as often as possible. Loosely speaking, the subspace-stabilized SQP method is a blend of the conventional and stabilized SQP methods: it behaves like the stabilized SQP along the degeneracy subspace, and like the conventional SQP on the complement of this subspace. The subspace-stabilized SQP methods may also generate long sequences of short steps when not close enough to solutions (e.g., when the rank of the constraints' Jacobian at the solution is identified incorrectly), but the area where this does not typically happen is much larger than that for the stabilized SQP. Therefore, the transition from

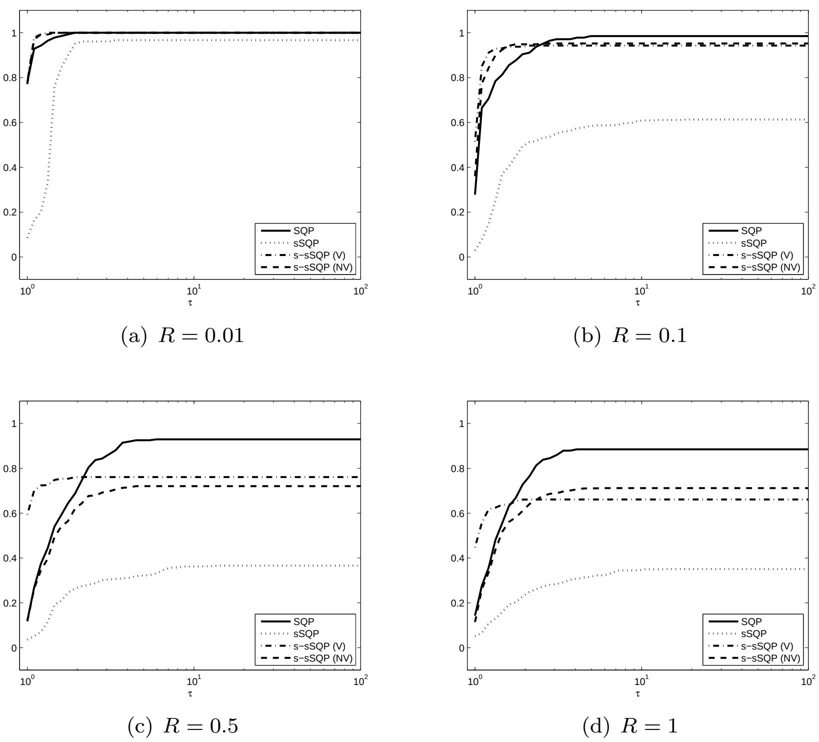


Fig. 4 Results for problems with nondegenerate constraints.

any outer phase algorithm to a local phase might be expected to pass much more smoothly if the latter would be the subspace-stabilized SQP rather than the stabilized SQP.

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