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MECHANICS

# Liouvillian Solutions in the Problem of Rolling of a Heavy Homogeneous Ball on a Surface of Revolution

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**Abstract**—The problem of a heavy homogeneous ball rolling without slipping on a surface of revolution is a classical problem of the nonholonomic system dynamics. Usually, when considering this problem, following the E.J. Routh approach, it is convenient to define explicitly the equation of the surface on which the ball's center is moving. This surface is equidistant from the surface on which the contact point is moving. It is known from the classic works by Routh and F. Noether that, if a ball rolls on a surface such that its center moves along a surface of revolution, then the problem is reduced to solving the second-order linear differential equation. Therefore, it is of interest to study for which surfaces of revolution the corresponding second-order linear differential equation admits a general solution expressed by Liouvillian functions. To solve this problem, it is possible to apply the Kovacic algorithm to the corresponding second-order linear differential equation. In this paper, we present our own method to derive the corresponding second-order linear differential equation. In the case in which the center of the ball moves along an ellipsoid of revolution, we prove that the general solution of the equation is expressed through Liouvillian functions.

**Keywords:** rolling without slipping, homogeneous ball, surface of revolution, Kovacic algorithm, Liouvillian solutions.

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## 1. PROFLEM FORMULATION

Let us consider a problem of rolling of a homogeneous ball on an arbitrary convex perfectly rough surface under the action of the forces such that their resultant force passes through the ball center [1, 2]. Let $G$ be the ball's center of gravity and moving axes $GC$, $GA$, and $GB$ designate, respectively, the normal to the bearing surface and two perpendicular straight lines lying on the tangent plane that passes through the contact point of the ball and the surface. The directions of the $GA$ and $GB$ axes will be specified later. Let us denote by $\mathbf{e}_1$, $\mathbf{e}_2$, and $\mathbf{e}_3$ the unit basis vectors of the $GA$, $GB$, and $GC$ axes, respectively. Let $\mathbf{\Omega} = \theta_1\mathbf{e}_1 + \theta_2\mathbf{e}_2 + \theta_3\mathbf{e}_3$ be the angular velocity of the chosen moving coordinate system, $\mathbf{v}_G = u\mathbf{e}_1 + v\mathbf{e}_2 + w\mathbf{e}_3$ be the velocity vector of point $G$ (it is obvious that $w = 0$, since the moving ball is not separated from the bearing surface during its motion), and $\boldsymbol{\omega} = \omega_1\mathbf{e}_1 + \omega_2\mathbf{e}_2 + \omega_3\mathbf{e}_3$ be the angular velocity of the ball. Let us denote by $\mathbf{N} = F_1\mathbf{e}_1 + F_2\mathbf{e}_2 + N\mathbf{e}_3$ the reaction force acting on the ball from the bearing surface. Let us denote by $\mathbf{P} = X\mathbf{e}_1 + Y\mathbf{e}_2 + P\mathbf{e}_3$ the resultant force applied to the ball center of gravity. Let $m$ be the ball mass; $R$, its radius; and $J$, the moment of inertia of the ball with respect to any axis that passes through its center of gravity. Assuming that the ball is rolling along the convex side of the unmoving surface and the positive direction of the $GC$ axis is directed outward towards the convexity, we write down the equation of motion of the ball in the vector form as follows:

$$m\dot{\mathbf{v}}_G + [\mathbf{\Omega}\times\mathbf{v}_G] = \mathbf{P} + \mathbf{N}, \tag{1}$$

$$J\dot{\boldsymbol{\omega}} + [\mathbf{\Omega}\times J\boldsymbol{\omega}] = [\mathbf{GK}\times\mathbf{N}]. \tag{2}$$

Equations (1) and (2) express the laws of change in the momentum and angular momentum of the ball with respect to the chosen moving coordinate system, respectively. Here, $\mathbf{GK} = -R\mathbf{e}_3$ is the radius vector outgoing from center of gravity $G$ of the ball to the point of its contact with the bearing surface.

Since the velocity of the ball's point of contact with the bearing surface is zero at each time instant, we obtain

$$\mathbf{v}_G + [\boldsymbol{\omega} \times \mathbf{GK}] = 0. \quad (3)$$

Equations (1)—(3) can be written in the scalar form as follows:

$$m\dot{u} - m\theta_3 v = X + F_1, \quad m\dot{v} + m\theta_3 u = Y + F_2, \quad m\theta_1 v - m\theta_2 u = P + N, \quad (4)$$

$$J\dot{\omega}_1 + J\theta_2\omega_3 - J\theta_3\omega_2 = F_2 R, \quad J\dot{\omega}_2 + J\theta_3\omega_1 - J\theta_1\omega_3 = -F_1 R, \quad \dot{\omega}_3 + \theta_1\omega_2 - \theta_2\omega_1 = 0, \quad (5)$$

$$u - R\omega_2 = 0, \quad v + R\omega_1 = 0. \quad (6)$$

Excluding $F_1$, $F_2$, $\omega_1$, and $\omega_2$ from Eqs. (4)–(6), we obtain

$$\dot{u} - \theta_3 v = \frac{R^2 X}{J + mR^2} + \frac{JR\theta_1\omega_3}{J + mR^2}, \quad \dot{v} + \theta_3 u = \frac{R^2 Y}{J + mR^2} + \frac{JR\theta_2\omega_3}{J + mR^2}. \quad (7)$$

Ball's center of gravity $G$ moves on the surface obtained from the given surface by shifting along the normal by a distance identical to ball radius $R$. Let the $GA$ and $GB$ axes be directed along the tangent lines to the curvature lines of this surface. Now, let us derive the second-order linear differential equation to integration of which this problem is reduced.

## 2. DERIVATION OF THE MAIN EQUATION

Let us find first the expression for angular velocity $\boldsymbol{\Omega}$ of the chosen moving coordinate system $GA$, $GB$, and $GC$ depending on the $u$ and $v$ components of the velocity of ball center of mass $G$. Let the surface on which the ball's center of mass moves be defined relative to some unmoving coordinate system by the equation

$$\mathbf{r} = \mathbf{r}(q_1, q_2), \quad (8)$$

where $q_1$ and $q_2$ are the Gaussian coordinated on the surface. Let us assume that the coordinate grid on the surface (8) consists of the lines of curvature. The directions of these lines at each point are indicated by the orthogonal unit vectors

$$\mathbf{e}_1 = \frac{1}{h_1}\frac{\partial \mathbf{r}}{\partial q_1}, \quad \mathbf{e}_2 = \frac{1}{h_2}\frac{\partial \mathbf{r}}{\partial q_2}, \quad (\mathbf{e}_i \cdot \mathbf{e}_j) = \delta_{ij}. \quad (9)$$

Here, $h_1$ and $h_2$ denote the Lame parameters

$$h_i(q_1, q_2) = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|, \quad i = 1, 2.$$

Vector $\mathbf{e}_3 = [\mathbf{e}_1 \times \mathbf{e}_2]$ is the vector normal to surface (8) at a point $(q_1, q_2)$. The ball's center of mass $\mathbf{v}_G$ can be determined by using the formula

$$\mathbf{v}_G = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathbf{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{r}}{\partial q_2}\dot{q}_2 = u\mathbf{e}_1 + v\mathbf{e}_2.$$

It follows from this expression that velocity components $u$ and $v$ are linked with coordinates $q_1$ and $q_2$ and their derivatives by the formulas

$$u = h_1\dot{q}_1, \quad v = h_2\dot{q}_2. \quad (10)$$

Denoting the principal curvatures of surface (8) by $k_i(q_1, q_2)$, $i = 1, 2$, we obtain

$$\frac{\partial \mathbf{e}_3}{\partial q_1} = -h_1 k_1 \mathbf{e}_1, \quad \frac{\partial \mathbf{e}_3}{\partial q_2} = -h_2 k_2 \mathbf{e}_2. \quad (11)$$

Formulas (11) are consequence of the Rodrigues theorem [3] known in differential geometry, in which it is also necessary to take into account that the coordinate grid we have chosen on surface (8) is orthogonal and consists of curvature lines. Using formulas (9) and (11), we can obtain the following relationships:

$$\frac{\partial \mathbf{e}_1}{\partial q_1} = -\frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\mathbf{e}_2 + h_1 k_1 \mathbf{e}_3, \quad \frac{\partial \mathbf{e}_1}{\partial q_2} = \frac{1}{h_1}\frac{\partial h_2}{\partial q_1}\mathbf{e}_2,$$
$$\frac{\partial \mathbf{e}_2}{\partial q_1} = \frac{1}{h_2}\frac{\partial h_1}{\partial q_2}\mathbf{e}_1, \quad \frac{\partial \mathbf{e}_2}{\partial q_2} = -\frac{1}{h_1}\frac{\partial h_2}{\partial q_1}\mathbf{e}_1 + h_2 k_2 \mathbf{e}_3. \tag{12}$$

The angular velocity of coordinate system *GA*, *GB*, *GC* can be found by using the standard formula

$$\mathbf{\Omega} = (\dot{\mathbf{e}}_2 \cdot \mathbf{e}_3)\mathbf{e}_1 + (\dot{\mathbf{e}}_3 \cdot \mathbf{e}_1)\mathbf{e}_2 + (\dot{\mathbf{e}}_1 \cdot \mathbf{e}_2)\mathbf{e}_3,$$

where we use the following denotations:

$$\dot{\mathbf{e}}_i = \frac{d\mathbf{e}_i}{dt} = \frac{\partial \mathbf{e}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \mathbf{e}_i}{\partial q_2}\dot{q}_2, \quad i = 1,2,3.$$

Taking into account formulas (11) and (12), we obtain the following expression for angular velocity $\mathbf{\Omega}$:

$$\mathbf{\Omega} = h_2 k_2 \dot{q}_2 \mathbf{e}_1 - h_1 k_1 \dot{q}_1 \mathbf{e}_2 + \left(\frac{\dot{q}_2}{h_1}\frac{\partial h_2}{\partial q_1} - \frac{\dot{q}_1}{h_2}\frac{\partial h_1}{\partial q_2}\right)\mathbf{e}_3.$$

Taking into account formulas (10), let us rewrite the expression for angular velocity $\mathbf{\Omega}$ in the form

$$\mathbf{\Omega} = k_2 v \mathbf{e}_1 - k_1 u \mathbf{e}_2 + \frac{1}{h_1 h_2}\left(\frac{\partial h_2}{\partial q_1} v - \frac{\partial h_1}{\partial q_2} u\right)\mathbf{e}_3.$$

So, we get the following expressions for components $\theta_1$, $\theta_2$, and $\theta_3$ of angular velocity $\mathbf{\Omega}$ of the moving coordinate system *GA*, *GB*, and *GC*:

$$\theta_1 = k_2 v, \quad \theta_2 = -k_1 u, \quad \theta_3 = \frac{1}{h_1 h_2}\left(\frac{\partial h_2}{\partial q_1} v - \frac{\partial h_1}{\partial q_2} u\right). \tag{13}$$

Now, let us assume that the surface on which the ball's center of mass *G* moves is a surface of revolution defined with respect to some unmoving coordinate system by the equation

$$\mathbf{r} = (\rho(q_1)\cos q_2, \rho(q_1)\sin q_2, \zeta(q_1)). \tag{14}$$

In this case, Lame parameters $h_1$ and $h_2$ have the form

$$h_1 = h_1(q_1) = \sqrt{\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2}, \quad h_2 = h_2(q_1) = \rho(q_1). \tag{15}$$

Principal curvatures $k_1$ and $k_2$ of the surface are calculated by using the formulas

$$h_1 = k_1(q_1) = \frac{\left(\frac{d^2\zeta}{dq_1^2}\frac{d\rho}{dq_1} - \frac{d\zeta}{dq_1}\frac{d^2\rho}{dq_1^2}\right)}{\left(\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2\right)^{\frac{3}{2}}}, \quad k_2 = k_2(q_1) = \frac{\frac{d\zeta}{dq_1}}{\rho\sqrt{\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2}}. \tag{16}$$

The lines of curvature on the surface of revolution are its meridians and parallels. Let vertical axis *Z* be the symmetry axis of the considered surface of revolution. Along with coordinates $q_1$ and $q_2$, let us introduce Euler angles $\theta$, $\psi$, and $\varphi$ such that the angle between the *GC* and *Z* axes is $\theta$; $\psi$ is the angle between the plane containing the *Z* and *GC* axes and some fixed vertical plane. Let us assume that components $\theta_1$, $\theta_2$, and $\theta_3$ of angular velocity $\mathbf{\Omega}$ of the coordinate system *GA*, *GB*, and *GC* are defined by means of the standard kinematic Euler formulas

$$\theta_1 = \dot{\psi}\sin\theta\sin\varphi + \dot{\theta}\cos\varphi, \quad \theta_2 = \dot{\psi}\sin\theta\cos\varphi - \dot{\theta}\sin\varphi, \quad \theta_3 = \dot{\psi}\cos\theta + \dot{\varphi},$$

in which the value of angle $\varphi$ is set to be $-\pi/2$. Therefore, we obtain

$$\theta_1 = -\dot{\psi}\sin\theta, \quad \theta_2 = \dot{\theta}, \quad \theta_3 = \dot{\psi}\cos\theta. \tag{17}$$

On the other hand, comparing (17) with formulas (13), we find

$$-\dot{\psi}\sin\theta = k_2 h_2 \dot{q}_2, \quad \dot{\theta} = -k_1 h_1 \dot{q}_1, \quad \dot{\psi}\cos\theta = \frac{1}{h_1}\frac{dh_2}{dq_1}\dot{q}_2. \tag{18}$$

From the second equation of system (18), we can find the relationship between variables $\theta$ and $q_1$. Therefore, we can consider that surface (14) is set depending on $\theta$ and $q_2$; i.e.,

$$\rho\big|_{q_1=q_1(\theta)} = \sigma(\theta), \quad \zeta\big|_{q_1=q_1(\theta)} = \tau(\theta). \tag{19}$$

Lame parameters $h_1$ and $h_2$ and principal curvatures $k_1$ and $k_2$ are now defined by the formulas of the form (15), (16), in which we should take $\sigma$ and $\tau$ instead of $\rho$ and $\zeta$; all the derivatives are taken with respect to independent variable $\theta$. Next, we take into account the second formula from (18), represent it in the form $\dot{\theta} = -k_1 u$, and find from it that

$$u = -\frac{\dot{\theta}}{k_1}. \tag{20}$$

Next, we consider the third formula from (5) and formulas (6). It follows from (6) that

$$\omega_1 = -\frac{v}{R}, \quad \omega_2 = \frac{u}{R} = -\frac{\dot{\theta}}{Rk_1}.$$

Taking into account these formulas, as well as (13) and (18), from the third equation of system (5), we obtain

$$\dot{\omega}_3 = \theta_2\omega_1 - \theta_1\omega_2 = \frac{v\dot{\theta}}{Rk_1}(k_2 - k_1). \tag{21}$$

Finally, from formula (21) we get the equation

$$\frac{d\omega_3}{d\theta} = \frac{v}{Rk_1}(k_2 - k_1). \tag{22}$$

Now, let us assume that the ball rolls under the action of the force of gravity. Then,

$$Y = 0, \quad \theta_3 = -\frac{k_1}{h_2}\frac{dh_2}{d\theta}v,$$

and, with (20) taken into account, the second equation from (7) takes the form

$$\frac{dv}{d\theta} + \frac{v}{h_2}\frac{dh_2}{d\theta} = \frac{JR}{J + mR^2}\omega_3. \tag{23}$$

Differentiating formula (23) for the second time and taking into account Eq. (22), we obtain

$$\frac{d}{d\theta}\left(\frac{dv}{d\theta} + \frac{v}{h_2}\frac{dh_2}{d\theta}\right) = \frac{J}{J + mR^2}\cdot\frac{v}{k_1}(k_2 - k_1). \tag{24}$$

So, the problem of rolling of a ball on an unmoving convex surface under the action of the force of gravity on the assumption that ball's center $G$ is moving on the surface of revolution is reduced to integrating second-order linear differential equation (24). The coefficients of this equation are determined by the shape of the surface of revolution on which the ball center is moving. One can pose the question of which the surfaces of revolution are for which Eq. (24) is integrable in an explicit form, for example, for which its general solution can be expressed through the Liouvillian functions. To answer the question of the existence of the Liouvillian solutions of a second-order linear differential equation, the Kovacic algorithm is usually used [4–6]. It is proved below that, if the center of a rolling ball is moving on an ellipsoid of revolution, then the problem is integrable in the Liouvillian functions.

## 3. ROLLING ON AN ELLISPOID OF REVOLUTION

Let the absolutely rough surface on with a ball is rolling be such that the ball center belongs to the ellipsoid of revolution with semiaxes $a$ and $b$. The equation for it can be written in the form (14):

$$\mathbf{r} = (a\sin q_1\cos q_2, a\sin q_1\sin q_2, b\cos q_1).$$

In the considered case, second-order linear differential equation (24) can be represented in the form

$$\frac{d}{d\theta}\left(\frac{dv}{d\theta}+\frac{b^2\cos\theta}{\sin\theta(a^2\sin\theta+b^2\cos^2\theta)}v\right)=\frac{J}{J+mR^2}\cdot\frac{(b^2-a^2)\sin^2\theta}{a^2\sin^2\theta+b^2\cos^2\theta}v. \quad (25)$$

Therefore, the problem of rolling of a ball on an absolutely rough surface such that, during rolling, the ball center moves on an ellipsoid of revolution, is reduced to integrating second-order linear differential equation (25). Let us substitute the independent variable in Eq. (25) according to the formula $x=\cos^2\theta$ and introduce the designations

$$\frac{J}{J+mR^2}=n^2<1,\quad \frac{b^2}{a^2}=k^2.$$

Then, Eq. (25) can be transformed into an equation with rational coefficients:

$$\frac{d^2v}{dx^2}+p_1(x)\frac{dv}{dx}+p_2(x)v=0, \quad (26)$$

where

$$p_1(x)=\frac{2(k^2-1)x^2+3x-1}{2x(x-1)((k^2-1)x+1)},$$

$$p_2(x)=\frac{1}{4x(x-1)((k^2-1)x+1)}\left[\frac{2k^2}{((k^2-1)x+1)}-\frac{(2x-1)k^2}{x-1}-n^2(k^2-1)(x-1)\right].$$

To reduce linear differential equation (26) to a simpler form, we make the substitution

$$y=v\sqrt{x-1}\left(\frac{x}{(k^2-1)x+1}\right)^{\frac{1}{4}}.$$

Then, second-order linear differential equation (26) can be written as follows:

$$\frac{d^2y}{dx^2}=\frac{4(n^2-1)(k^2-1)^2x^2+4(n^2-1)(k^2-1)x-3}{16x^2((k^2-1)x+1)^2}y. \quad (27)$$

Equation (27) has exactly the form that is necessary to apply the Kovacic algorithm [4–6] to this equation. Application of the Kovacic algorithm [4–6] to Eq. (27) shows that the general solution of this equation can be represented in the form

$$y=(x((k^2-1)x+1))^{\frac{1}{4}}\left(C_1(\Phi(x))^{\frac{n}{2}}+C_2(\Phi(x))^{-\frac{n}{2}}\right),$$

$$\Phi(x)=1+2(k^2-1)x+2\sqrt{(k^2-1)x((k^2-1)x+1)},$$

where $C_1$ and $C_2$ are arbitrary constants. So, we can come to the conclusion that the general solution of initial equation (25) can be expressed through the Liouvillian functions.

## FUDING

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## ADDITIONAL INFORMATION

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