Report by Professor Oleg German
on the Ph.D. Thesis by Farrokh Razavinia
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"Applications of the quantization in Physics and Noncommutative algebras"

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Report

The area of the thesis under review, understood in broad sense, is Noncommutative Algebra related with its arithmetic aspects.

A noncommutative algebra has, along with addition and multiplication, an important operation called the "commutator", defined by $[a, b] = ab - ba$. Noncommutativity says precisely that this operation is nontrivial. It might happen that for the functions we care about, $[a, b]$ may be very small. Let A be the algebra, and consider the $A/\hbar A$. If $\hbar$ is supposed to be a "very small number", then taking this quotient should only throw away fine-grained information, but some sort of "classical" geometry should still survive (notice that since $[a, b]$ is divisible by $\hbar$, it goes to 0 in the quotient, so the quotient is commutative and corresponds to a classical geometric space). We can make this precise by demanding that there is a vector-space lift $(A/\hbar A) \to A$, and that A is generated by the image of this lift along with the element $\hbar$. Anyway, so with this whole set up, the quotient $A/\hbar A$ actually has a little more structure than just being a commutative algebra. In particular, since $[a, b]$ is divisible by $\hbar$, let's consider the element $\{a, b\} = \hbar^{-1}[a, b]$. Then quantization is the process of reversing the above quotient.

In deformation quantization, we start with a classical theory given by a Poisson manifold. Then, (by definition) the algebra of functions forms a Poisson algebra. A quantization of this algebra is a noncommutative algebra with operators X_f for f a function. There is also a formal parameter $\hbar$ as before. This algebra satisfies

$$X_f \; X_g = X_{fg} + \mathcal{O}(\hbar) \; .$$

The idea of quantization is that the Poisson bracket becomes a commutator, or

$$[X_f, X_g] = \hbar X_{\{f,g\}} + \mathcal{O}(\hbar^2) \; .$$

Thus, we have a noncommutative version of classical mechanics. The existence of such an algebra is a theorem of Kontsevich.

The subject has various arithmetics combinatorial and geometrical aspects

The thesis consists of the Conventions and the organization of the thesis, Introduction, and five more Chapters. The Introduction is, on the one hand, detailed enough and some of the main problems and results of the thesis are clearly formulated, and, on the other hand, contains an informal and conceptual discussion on the background, motivating problems, relevant literature, and brief account of employed machinery. Chapter 3 is devoted to Feigin's homomorphisms, local integral of motions and lattice Virasoro algebra associated to $\widehat{\mathfrak{sl}_2}$ and $\mathfrak{sl}_2$. Chapter 4 gives a very brief introduction to the history of the creation of generalized Heisenberg algebras and of the quantum generalized Heisenberg algebras. Chapter 5 is technically the most involved, which is concerned with a class of algebras $\mathcal{H}_q(f,g)$ over an arbitrary field $\mathbb{F}$ with three generators h, x and y, three parameters $q \in \mathbb{F}$, $f(h) \in \mathbb{F}[h]$, $g(h) \in \mathbb{F}[h]$, and three relations $hx = xf(h), yh = f(h)y, yx - qxy = g(h)$.

Chapter 6 is devoted to the famous centralizer theorem for free associative algebras. This theorem, proved by George Bergman in his PhD thesis in 1968 and confirming a conjecture by Paul Cohn, states that the centralizer of any nonscalar element of the algebra is the polynomial ring in one variable. Since then, it remains an important working tool in the area. Note that the proof includes some tricky combinatorics around noncommutative versions of the Euclidian division algorithm. The author of this thesis revisited this theorem and discovered a new proof. The proof combines several different ingredients: going over to generic matrices (the approach introduced to noncommutative algebra by Shimshon Amitsur), a certain version of quantization method, some original results of Paul Cohn on the noncommutative integral closure, and also some theory of invariants (Claudio Procesi, Alexandr Zubkov, Stephen Donkin) that allow one to make some parts of the proof characteristic-free.

The last chapter treats noncommutative analogues of linearization problems for group actions. The author's focus is on the torus actions where the starting point is a seminal work by Andrzej Bialynicki-Birula.

I found the thesis to be interesting, clearly written. There are several remarks of expositional matter that had no influence on my positive evaluation of the thesis. It contains several important, new, original results and definitely meets the standards applied for PhD theses. To summarise, this is a nice thesis on the ring theory side of the interface between mathematical physics and traditional noncommutative ring theory and I recommend to approve it.

With kind regards,

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Нач. отдела кадров