

## Noether's theorem for dissipative quantum dynamical semi-groups

John E. Gough, Tudor S. Ratiu, and Oleg G. Smolyanov

Citation: *Journal of Mathematical Physics* **56**, 022108 (2015); doi: 10.1063/1.4907985

View online: <http://dx.doi.org/10.1063/1.4907985>

View Table of Contents: <http://scitation.aip.org/content/aip/journal/jmp/56/2?ver=pdfcov>

Published by the [AIP Publishing](#)

---

### Articles you may be interested in

[Quantum dynamical semigroups for finite and infinite Bose systems](#)

*J. Math. Phys.* **48**, 012106 (2007); 10.1063/1.2406053

[Hudson's theorem for finite-dimensional quantum systems](#)

*J. Math. Phys.* **47**, 122107 (2006); 10.1063/1.2393152

[Dynamical symmetry group and quantum splittings for a free particle on the group manifold AdS 3](#)

*J. Math. Phys.* **46**, 083501 (2005); 10.1063/1.1982767

[Construction of a family of quantum Ornstein–Uhlenbeck semigroups](#)

*J. Math. Phys.* **45**, 609 (2004); 10.1063/1.1641150

[Subharmonic projections for a quantum Markov semigroup](#)

*J. Math. Phys.* **43**, 1074 (2002); 10.1063/1.1424475

---

Did your publisher get  
**18 MILLION DOWNLOADS** in 2014?

AIP Publishing did.



THERE'S POWER IN NUMBERS. Reach the world with AIP Publishing.



# Noether's theorem for dissipative quantum dynamical semi-groups

John E. Gough,<sup>1,a)</sup> Tudor S. Ratiu,<sup>2,b)</sup> and Oleg G. Smolyanov<sup>3,c)</sup>

<sup>1</sup>*Aberystwyth University, Aberystwyth SY23 3BZ, Wales, United Kingdom*

<sup>2</sup>*Section de Mathématiques and Bernoulli Center, École Polytechnique Fédérale de Lausanne, Lausanne CH 1015, Switzerland*

<sup>3</sup>*Mechanics and Mathematics Faculty, Moscow State University, Moscow 119991, Russia*

(Received 27 October 2014; accepted 2 February 2015; published online 18 February 2015)

Noether's theorem on constants of the motion of dynamical systems has recently been extended to classical dissipative systems (Markovian semi-groups) by Baez and Fong [J. Math. Phys. **54**, 013301 (2013)]. We show how to extend these results to the fully quantum setting of quantum Markov dynamics. For finite-dimensional Hilbert spaces, we construct a mapping from observables to completely positive maps that leads to the natural analogue of their criterion of commutativity with the infinitesimal generator of the Markov dynamics. Using standard results on the relaxation of states to equilibrium under quantum dynamical semi-groups, we are able to characterise the constants of the motion under quantum Markov evolutions in the infinite-dimensional setting under the usual assumption of existence of a stationary strictly positive density matrix. In particular, the Noether constants are identified with the fixed point of the Heisenberg picture semi-group. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4907985>]

## I. INTRODUCTION

Noether's Theorem has traditionally been formulated within the framework of closed systems where it has been of central importance in formulating the concept of constants of the motion for Lagrangian dynamical systems; however, the widespread belief is that it does not apply to dissipative systems. Recently, a version of the theorem has been formulated by Baez and Fong<sup>1</sup> for classical Markovian models. Here, a constant of the motion is a random variable on the state space of a classical Markov process whose probability distribution is time invariant under the Markov semi-group. While entirely classical, their investigation draws on analogies to closed quantum dynamics: indeed, random variables may be identified with the operators of pointwise multiplication by the random functions, and they establish that constancy of the motion equates with commutativity of these operators with the Markov transition mechanism operators (or their infinitesimal generator).

In this paper, we extend these results to the setting quantum dynamical semi-groups (QDSs).<sup>2</sup>

## II. CLASSICAL MARKOV PROCESSES WITH FINITE STATE SPACES

This section reviews the results of Baez and Fong.<sup>1</sup> Our only contribution in this section is to present the results in a fashion that makes it easier to see extensions to quantum Markov models in Secs. III and IV.

---

a)Electronic mail: [jug@aber.ac.uk](mailto:jug@aber.ac.uk)

b)Electronic mail: [tudor.ratiu@epfl.ch](mailto:tudor.ratiu@epfl.ch)

c)Electronic mail: [smolyanov@yandex.ru](mailto:smolyanov@yandex.ru)

Let  $\Gamma$  be a finite set, say,  $\Gamma = \{1, \dots, d\}$  for definiteness. We denote by  $\Sigma(\Gamma)$  the set of all probability vectors  $\mathbf{p}$ , that is,  $d$ -tuples  $\mathbf{p} = [p_x]$ , with  $p_x \geq 0$ , and  $\sum_x p_x = 1$ .

A *classical dynamical semi-group* (CDS) with state space  $\Gamma$  is determined by a transition mechanism  $(T_t)_{t \geq 0}$  of the form  $T_t = [T_{xy}(t)]$  that forms a matrix semi-group:  $T_t T_s = T_{t+s}$ , with  $T_0 = I_d$ , the  $d \times d$  identity matrix, such that  $T(t)$  maps  $\Sigma(\Gamma)$  to itself. The entries  $T_{xy}(T)$  give the conditional probabilities for a transition from state  $y$  to state  $x$  in time  $t$ . An initial probability vector  $\mathbf{p}$  evolves under the transition mechanism as  $\mathbf{p}(t) = T_t \mathbf{p}$ . Note that we require the identities

$$\sum_x T_{xy}(t) = 1, \quad \text{for all } y \in \Gamma. \tag{1}$$

It follows from the semi-group law that

$$T_t = e^{tM}, \tag{2}$$

where  $M \in \mathbb{R}^{d \times d}$  is called the infinitesimal generator of the transition mechanism. To lowest order in  $t$ , we have  $0 \leq p_x(t) = p_x + t \sum_y M_{xy} p_y + O(t^2)$  and to be true for all  $t \geq 0$ , we require that

$$M_{xy} \geq 0, \quad \text{for all } x \neq y. \tag{3}$$

We also require that

$$\sum_{x \in \Gamma} M_{xy} = 0, \quad \text{for all } y \in \Gamma, \tag{4}$$

which is the infinitesimal form of (1). Requirements (3) and (4) characterise the infinitesimal generators of Markov transition matrices.

Now, let  $A$  be a random variable on  $\Gamma$ , its expectation for a fixed probability vector  $\mathbf{p} \in \Sigma(\Gamma)$  is

$$\mathbb{E}[A] = \sum_{x \in \Gamma} A(x) p_x.$$

This may be written as  $\mathbb{E}[A] = \sum_a a \mathbb{K}^A(a)$ , where

$$\mathbb{K}^A(a) = \sum_{\{x \in \Gamma: A(x)=a\}} p_x = \Pr\{A = a\}. \tag{5}$$

$\mathbb{K}^A$  is the probability distribution of  $A$  determined by  $\mathbf{p}$ . The probability distribution determined by  $\mathbf{p}(t) = T(t) \mathbf{p}$  is similarly denoted as  $\mathbb{K}_t^A$ , and for any Borel function  $f$ , we have that

$$\mathbb{E}_t[f(A)] = \sum_{x,y \in \Gamma} f(A(x)) T_{xy}(t) p_y \equiv \sum_a f(a) \mathbb{K}_t^A(a).$$

*Definition 1.* A random variable  $A$  with sample space  $\Gamma$  is said to be a constant for the CDS with Markov mechanism  $T$  on  $\Gamma$  if, for each initial  $\mathbf{p} \in \Sigma(\Gamma)$ , its probability distribution  $\mathbb{K}_t^A$  does not depend on  $t \geq 0$ .

This means that for each  $\mathbf{p} \in \Sigma(\Gamma)$  and any polynomial function  $f$ , we have for all  $t \geq 0$

$$\mathbb{E}_t[f(A)] = \mathbb{E}[f(A)].$$

The transition mechanism furnishes an equivalence relation on  $\Gamma$  as follows: we say that  $x \sim y$  if and only if we have the matrix element  $[M^n]_{xy} \neq 0$  for some positive integer power  $n$ . We shall denote the  $\sigma$ -algebra<sup>3</sup> generated by the equivalence classes determined by transition mechanism by  $\mathcal{M}$ .

**Theorem (Baez-Fong).**<sup>1</sup> For each random variable  $A$  on  $\Gamma$ , we define the  $d \times d$  matrix

$$\hat{A} = \text{diag}(A(1), \dots, A(d)) = \begin{bmatrix} A(1) & 0 & \dots & 0 \\ 0 & A(2) & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & A(d) \end{bmatrix}. \tag{6}$$

Let  $M$  be the infinitesimal generator for a Markov transition mechanism on  $\Gamma$ . Then, the following are equivalent:

1.  $A$  is a constant of the CDS;
2. the mean and variance of the probability distribution  $\mathbb{K}_t^A$  are constant in time;
3.  $A$  is measurable with respect to the  $\sigma$ -algebra  $\mathcal{M}$ ;
4.  $[\hat{A}, M] = 0$ .

The proof can be found in Ref. 1, and we sketch the arguments for completeness in the Appendix.

More generally, let  $(\Gamma, \mathcal{G}, \mu)$  be a  $\sigma$ -finite measure space and set

$$\mathfrak{A} = L^\infty(\Gamma, \mathcal{G}, \mu).$$

The space of densities  $L^1(\Gamma, \mathcal{G}, \mu)$  is the dual to  $\mathfrak{A}$ , and a semi-group  $(T_t)_{t \geq 0}$  on the densities leads to a dual semi-group  $(J_t)_{t \geq 0}$  on  $\mathfrak{A}$  given by

$$\int_\Gamma A(x) (T_t S)(x) \mu[dx] = \int_\Gamma (J_t A)(x) S(x) \mu[dx],$$

for all  $A \in \mathfrak{A}$ ,  $S \in L^1(\Gamma, \mathcal{G}, \mu)$ . With an obvious abuse of notation, we write  $(T_t \mathbb{P})[dx]$  for  $(T_t \rho)(x) \mu[dx]$  whenever  $\mathbb{P}$  is absolutely continuous with respect to  $\mu$  with Radon-Nikodym derivative  $\rho$ . The probability distribution of a random variable  $A \in \mathfrak{A}$  at time  $t$  for a given initial distribution  $\mathbb{P}[dx] = \rho(x) \mu[dx]$  is then

$$\mathbb{K}_t^{A, \rho} = (T_t \mathbb{P}) \circ A^{-1},$$

and the random variable  $A$  is a constant under the CDS if  $\mathbb{K}_t^{A, \rho}$  is independent of  $t$  for all fixed probability densities  $\rho \in L^1(\Gamma, \mathcal{G}, \mu)$ .

Baez and Fong establish the more general result for a continuous CDS  $(T_t)_{t \geq 0}$  on  $L^1(\Gamma, \mathcal{G}, \mu)$  that a random variable  $A$  satisfies  $[\hat{A}, T_t] = 0$  (for all  $t \geq 0$ ) if and only if the first two moments of  $\mathbb{K}_t^{A, \rho}$  are independent of  $t$  for all fixed probability densities  $\rho \in L^1(\Gamma, \mathcal{G}, \mu)$ .

It is instructive to express these results in a more algebraic language so as to anticipate the quantum version. The collection of random variables  $\mathfrak{A} = L^\infty(\Gamma, \mathcal{G}, \mu)$  is, in fact, a commutative von Neumann algebra. The random variables that are the constants under a given Markovian dynamics are, in the case of finite sample space  $\Gamma$  at any rate, identified as the  $\mathcal{M}$ -measurable ones. Mathematically, there is a one-to-one correspondence between  $\sigma$ -algebras  $\mathcal{M}$  and the algebras of bounded  $\mathcal{M}$ -measurable functions  $\mathfrak{M}$ . In fact,  $\mathfrak{M}$  will then be a commutative von Neumann sub-algebra, so an alternative statement of the Baez-Fong result is that a bounded random variable will be a constant of a classical Markov semi-group if and only if it belongs to the von Neumann algebra  $\mathfrak{M}$  determined by the transition mechanism (this applies to their result on general Markov semi-groups).

### III. NOETHER'S THEOREM FOR QUANTUM MARKOV DYNAMICS (FINITE STATE SPACE)

We now move to the setting of a finite-dimensional Hilbert space  $\mathfrak{h} = \mathbb{C}^d$ . The space of states becomes the set  $\Sigma_d$  of density matrices, that is, matrices  $\varrho \in \mathbb{C}^{d \times d}$  satisfying the properties  $\varrho \geq 0$  and  $\text{tr} \{ \varrho \} = 1$ . Each  $\varrho \in \Sigma_d$  determines the expectation (or state)

$$\mathbb{E}[\cdot] = \text{tr} \{ \varrho \cdot \}.$$

Observable quantities are described by observables, that is, Hermitian  $d \times d$  matrices. Each observable  $A$  admits a spectral decomposition  $A = \sum_a a P_a$ , where the sum is over the spectrum of  $A$  and  $P_a$  is the orthogonal projection onto the eigenspace corresponding to eigenvalue  $a$ . We find that  $\mathbb{E}[f(A)] = \sum_a f(a) \mathbb{K}^A(a)$  where the distribution  $\mathbb{K}^A$  is now given by Born's rule  $\mathbb{K}^A(a) = \text{tr} \{ \varrho P_a \}$ .

Let  $\mathfrak{A} = \mathfrak{B}(\mathfrak{h})$  be the algebra of bounded operators on the Hilbert space, here just represented as  $d \times d$  complex valued matrices. We are interested in linear maps  $\mathcal{T} : \mathfrak{A} \mapsto \mathfrak{A}$  which take density matrices to density matrices. Furthermore, we impose the standard requirement that the maps are

completely positive (CP), see, e.g., Ref. 2. They should also be trace-preserving

$$\text{tr} \{ \mathcal{T}(S) \} = \text{tr} \{ S \}, \quad \text{for all } S \in \mathfrak{A}.$$

As is well-known, such CP maps (also called quantum communication channels) may be represented as<sup>2</sup>

$$\mathcal{T}(S) = \sum_k V_k S V_k^*,$$

where  $\{V_k\}_k$  is a set of operators in  $\mathfrak{A}$  called the Kraus maps. The trace preserving property is ensured if  $\sum_k V_k^* V_k = I_d$ .

A family  $(\mathcal{T}_t)_{t \geq 0}$  of CP trace-preserving maps forming a semi-group ( $\mathcal{T}_0 = id, \mathcal{T}_t \circ \mathcal{T}_s = \mathcal{T}_{t+s}$ ) is said to be a QDS.

*Definition 2.* An observable  $A$  is a constant of QDS if, for any initial state  $\varrho \in \Sigma_d$ , we have  $\mathbb{K}_t^A(a) = \text{tr} \{ \mathcal{T}_t(\varrho) P_a \}$  independent of  $t \geq 0$ .

Let  $A \in \mathfrak{A}$  be an observable with spectral decomposition  $\sum_a a P_a$ . For any polynomial  $f$ , we define a map  $\widehat{f(A)} : \mathfrak{A} \mapsto \mathfrak{A}$  by

$$\widehat{f(A)}(S) \triangleq \sum_a f(a) P_a S P_a. \tag{7}$$

This will be our quantum mechanical analogue of the Baez-Fong map (6): where they convert random variables on a  $d$ -dimensional sample space into a diagonal matrix, we convert a Hermitian matrix into a CP map. We, in fact, see that

$$\begin{aligned} \text{tr} \{ \widehat{f(A)}(\varrho) \} &= \sum_a f(a) \text{tr} \{ P_a \varrho P_a \} \\ &= \sum_a f(a) \text{tr} \{ \varrho P_a \} = \text{tr} \{ f(A) \varrho \} \\ &= \mathbb{E}[f(A)]. \end{aligned}$$

Now, let us consider the criterion

$$[\widehat{f(A)}, \mathcal{T}_t] = 0, \quad \text{for all } t \geq 0, \tag{8}$$

where the commutator of maps is now understood as  $[\mathcal{T}_1, \mathcal{T}_2] \equiv \mathcal{T}_1 \circ \mathcal{T}_2 - \mathcal{T}_2 \circ \mathcal{T}_1$ .

*Proposition.* Condition (8) implies that  $A$  is a constant for the QDS (see Definition 2).

*Proof:* If condition (8) is satisfied, then we have that

$$\begin{aligned} \mathbb{E}_t[f(A)] &= \text{tr} \{ f(A) \mathcal{T}_t(\varrho) \} \\ &= \text{tr} \{ \widehat{f(A)} \circ \mathcal{T}_t(\varrho) \} \\ &= \text{tr} \{ \mathcal{T}_t \circ \widehat{f(A)}(\varrho) \} \\ &= \text{tr} \{ \widehat{f(A)}(\varrho) \} \quad (\text{as } \mathcal{T}_t \text{ is trace-preserving!}) \\ &= \mathbb{E}[f(A)]. \quad \square \end{aligned}$$

For the class of quantum Markov dynamics considered here, it is well-known<sup>7,8</sup> that they are generated by infinitesimal maps  $\mathcal{M} : \mathfrak{A} \mapsto \mathfrak{A}$  so that

$$\mathcal{T}_t = e^{t\mathcal{M}},$$

where we have the specific form

$$\mathcal{M}(S) \equiv \sum_k \left\{ L_k S L_k^* - \frac{1}{2} S L_k^* L_k - \frac{1}{2} L_k^* L_k S \right\} + i[S, H], \tag{9}$$

with  $\{L_k\}$  a collection of operators in  $\mathfrak{A}$  and  $H \in \mathfrak{A}$  Hermitian. The state  $\varrho_t = \mathcal{T}_t(\varrho)$  then satisfies the master equation  $\frac{d}{dt}\varrho_t = \mathcal{M}(\varrho_t)$ .

The infinitesimal form of condition (8) is then

$$[\widehat{f(A)}, \mathcal{M}] = 0. \tag{10}$$

We remark that  $\mathcal{T}_t$  is the Schrödinger picture form of the channel; however, we can also work with the Heisenberg picture form  $\mathcal{J}_t$ . These are CP maps determined by the duality  $\text{tr}\{\mathcal{T}_t(S)A\} \equiv \text{tr}\{S\mathcal{J}_t(A)\}$ , for all  $S, A \in \mathfrak{A}$ . The trace-preserving property of  $\mathcal{T}_t$  is equivalent to the property  $\mathcal{J}_t(I_d) = I_d$ . The generator of the semi-group  $\mathcal{J}_t$  is then the Gorini-Kossakowski-Sudarshan-Lindblad generator  $\mathcal{L}$  adjoint to  $\mathcal{M}$  which takes the form<sup>2</sup>

$$\mathcal{L}(A) \equiv \sum_k \left\{ L_k^* A L_k - \frac{1}{2} A L_k^* L_k - \frac{1}{2} L_k^* L_k A \right\} - i[A, H]. \tag{11}$$

An observable  $A$  is said to be a fixed point of a quantum Markov dynamics if it is a fixed point of the corresponding Heisenberg maps, that is,  $\mathcal{J}_t(A) = A$  for all  $t \geq 0$ . Clearly, if  $A$  is a fixed point of  $\mathcal{J}_t$ , then  $\text{tr}\{\mathcal{T}_t(\varrho)A\} \equiv \text{tr}\{\varrho\mathcal{J}_t(A)\} = \text{tr}\{\varrho A\}$ , so it is a constant of the quantum Markov dynamics.

*Proposition.* An observable  $A$  satisfies condition (8) if and only if it is a fixed point of the corresponding Heisenberg maps.

*Proof:* We begin by noting that

$$\begin{aligned} \text{tr}\{[\widehat{f(A)}, \mathcal{T}_t](S)\} &= \text{tr}\{f(A)\mathcal{T}_t(S)\} - \text{tr}\{\widehat{f(A)}(S)\} \\ &= \text{tr}\{S\mathcal{J}_t(f(A))\} - \text{tr}\{Sf(A)\}, \end{aligned}$$

and so if  $A$  is a fixed point of  $\mathcal{J}_t$ , then  $[\widehat{f(A)}, \mathcal{T}_t] = 0$  as  $S$  was arbitrary.

Conversely, suppose that condition (8) is satisfied. Let us take the Kraus form for map at time  $t$ :  $\mathcal{T}_t(S) \equiv \sum_k V_k(t) S V_k^*$ . (Note that  $V_k(t)$  is not required to depend continuously on  $t$ .) The Heisenberg map will then have the adjoint form  $\mathcal{J}_t(A) = \sum_k V_k(t)^* A V_k(t)$ . Then, the condition implies that

$$\sum_{k,a} V_k(t) P_a S P_a V_k(t)^* f(a) = \sum_{k,a} f(a) P_a V_k(t) S V_k(t)^* P_a.$$

As the choice of  $f$  was arbitrary, this implies that

$$\sum_k V_k(t) P_a S P_a V_k(t)^* = \sum_k P_a V_k(t) S V_k(t)^* P_a. \tag{12}$$

If we set  $S = P_b$  where  $a \neq b$ , then we find that (12) reduces to  $0 = \sum_k (P_a V_k(t) P_b) (P_b V_k(t)^* P_a)$  and so we see that  $P_a V_k(t) P_b \equiv 0$ . On the other hand, setting  $S = P_a$  leads to the conclusion that if  $Y_a(t) = \sum_k V_k(t) P_a V_k(t)^*$ , then  $Y_a(t) = P_a Y_a(t) P_a$ . It follows that

$$[V_k(t), P_a] = 0$$

for each eigenvalue  $a$  of  $A$  and therefore  $[V_k(t), A] = 0$ . As  $A$  commutes with all the Kraus operators, we get that

$$\mathcal{J}_t(A) = \sum_k V_k(t)^* A V_k(t) \equiv \sum_k V_k(t)^* V_k(t) A = A.$$

So condition (8) implies that  $A$  is a fixed point of the quantum Markov dynamics. □

#### IV. CONSTANTS OF QUANTUM MARKOV SEMI-GROUPS (GENERAL CASE)

In this section, we investigate the question of characterising the constant observables for a given quantum dynamic semi-group in the general setting where the observables belong to the set  $\mathfrak{A}$  of bounded operators on a fixed separable Hilbert space  $\mathfrak{h}$ . Let  $\mathfrak{T}(\mathfrak{h})$  denote the set of trace-class operators on  $\mathfrak{h}$ , then the constants of a QDS  $(\mathcal{T}_t)_{t \geq 0}$  are the elements of the set

$$\mathfrak{M} = \left\{ A \in \mathfrak{A} : \text{tr}\{(\mathcal{T}_t \varrho) f(A)\} = \text{tr}\{\varrho f(A)\}, \forall \varrho \in \mathfrak{T}(\mathfrak{h}), t \geq 0, f : \mathbb{C} \mapsto \mathbb{C} \text{ bounded measurable} \right\}.$$

Note that  $\mathfrak{M}$  may contain non-self-adjoint elements and may be a non-commuting set of operators.

Transferring to the Heisenberg picture, we likewise define the *fixed points* of the QDS  $(\mathcal{J}_t)_{t \geq 0}$  to be the collection of operators

$$\mathfrak{F} \triangleq \left\{ A \in \mathfrak{A} : \mathcal{J}_t(A) = A, \forall t \geq 0 \right\}.$$

Intuitively, one suspects for any reasonably defined QDS the constants of the QDS  $\mathfrak{M}$  and the fixed points of the QDS  $\mathfrak{F}$  are the same. To make any progress in the general case, however, we need to meet certain technical requirements. The following assumption will turn out to be sufficient.

**Postulate (P)** *There exists a strictly positive density matrix  $\hat{\varrho} > 0$  on  $\mathfrak{h}$  that is stationary with respect to QDS, that is,  $\mathcal{T}_t(\hat{\varrho}) = \hat{\varrho}$  for all  $t > 0$ , where  $\mathcal{T}_t$  is the Schrödinger picture form of the QDS.*

We can now state a Noether theorem for quantum dynamical semi-groups.

**Theorem** *Suppose we are given a QDS of maps  $(\mathcal{T}_t)_{t \geq 0}$  continuous in the trace-norm topology. If the postulate (P) holds then the collection of fixed points  $\mathfrak{F}$  forms a von Neumann algebra and this is the algebra  $\mathfrak{M}$  of constants for the QDS. Moreover, the infinitesimal generator then takes form (9) with  $H, L_k, \sum_k L_k^* L_k \in \mathfrak{A}$  and we have*

$$\mathfrak{M} \equiv \left\{ A \in \mathfrak{A} : [A, H] = 0, [A, L_k] = 0, [A, L_k^*] = 0, \forall k \right\}. \quad (13)$$

*Proof:* We first observe that the dual Heisenberg semi-group  $(\mathcal{J}_t)_{t \geq 0}$  will be norm continuous. For  $A$  to be a constant under the QDS, we require that  $\text{tr}\{\varrho \mathcal{J}_t(f(A))\} = \text{tr}\{\varrho f(A)\}$  for each  $t > 0$  and each density matrix  $\varrho$ , and any bounded continuous function  $f$ , say. Consequently, we must have that  $f(A)$  will also be a fixed point of the QDS. What is not immediate at present is the property that  $A \in \mathfrak{F}$  necessarily implies that  $f(A) \in \mathfrak{F}$ , or in other words that the fixed points form an algebra.

However, the existence of  $\hat{\varrho}$ , satisfying the conditions of the postulate (P), implies that the state  $\hat{\mathbb{E}}[\cdot]$  it generates is a normal, faithful stationary state for the QDS.<sup>5</sup> This ensures that the fixed points  $\mathfrak{F}$  now form a von Neumann algebra.<sup>4</sup> Consequently, if  $A$  is a fixed point, then any continuous bounded function  $f(A)$  is also a fixed point, and therefore,  $A$  is a constant of the QDS.

The generators of norm-continuous QDS have Lindblad generators of form (9). The characterisation of the fixed points as given in (13) then follows from Theorem 3.3, p. 281, of Frigerio and Verri.<sup>6</sup>  $\square$

Note that  $\mathfrak{M}$  may be non-commutative. The requirement that the Schrödinger semi-group  $(\mathcal{T}_t)_{t \geq 0}$  be continuous in the trace-norm topology implies that the infinitesimal generator  $\mathcal{M}$  still takes form (9). This is the classic result of Lindblad.<sup>7</sup> This restriction is not essential, and sufficient conditions for when form (9) holds but with the operators  $H, L_k$  are unbounded have been derived by Fagnola and Rebolledo,<sup>4</sup> see also Ref. 9.

Postulate (P), however, is more essential as it ensures that the fixed points  $\mathfrak{M}$  form a von Neumann algebra. We may explain characterisation (13) of  $\mathfrak{M}$  in terms of the infinitesimal generator as follows: for a closed Hamiltonian evolution, the commutativity requirement with the Hamiltonian  $H$  is straightforward; for  $A$  self-adjoint to be a constant, we would require  $\mathcal{J}_t(A^2) \equiv \mathcal{J}_t(A)^2$  which at the infinitesimal level implies  $\mathcal{L}(A^2) - \mathcal{L}(A)A - A\mathcal{L}(A) = 0$ , but this last expression takes the form  $\sum_k [A, L_k]^* [A, L_k] \equiv 0$  and this requires  $[A, L_k] = 0$  for each  $k$ . Every non-self-adjoint Noether constant should then be the sum  $A_1 + iA_2$  with  $A_1, A_2$  self-adjoint Noether constants.

Postulate (P) deserves some further comment as it implies that  $\mathfrak{M}$  is invariant under the modular group  $(\sigma_t^{\hat{\mathbb{E}}})_{t \in \mathbb{R}}$  from the Tomita-Takesaki theory, in fact,

$$[\sigma_t^{\hat{\mathbb{E}}}, \mathcal{J}_t] = 0,$$

and moreover that there exists a conditional expectation  $\hat{\mathbb{E}}[\cdot|\mathfrak{M}]$  from  $\mathfrak{A}$  onto  $\mathfrak{M}$ . In the present situation,  $\hat{\mathbb{E}}[\cdot|\mathfrak{M}]$  will be the (unique) faithful, normal conditional expectation with  $\hat{\mathbb{E}}[\cdot]$  as invariant state,

$$\hat{\mathbb{E}}[M\hat{\mathbb{E}}[A|\mathfrak{M}]] = \hat{\mathbb{E}}[MA],$$

for all  $M \in \mathfrak{M}$ . We remark that conditional expectations from a von Neumann algebra onto a sub-algebra do not generally exist.

The original motivation for quantum dynamical semi-groups was to study convergence to equilibrium of arbitrary initial states for general quantum open systems. The results in this area may be rephrased using the Noether theorem as saying that all states converge to the (unique) equilibrium state  $\hat{\rho}$  if and only if the only constants of the QDS are multiples of identity, that is,  $\mathfrak{M} \equiv \mathbb{C}I$ . More generally, a non-trivial algebra  $\mathfrak{M}$  of observables that are constant under a QDS implies that we do not have convergence to a unique stationary state.

A general condition for the existence of a stationary state  $\hat{\rho} > 0$  is given by Fagnola and Rebollo, <sup>4</sup> including their Theorem 4.2 which gives the condition on the infinitesimal generator.

We finally remark that  $\hat{\mathbb{E}}[\mathcal{J}_t(A)|\mathfrak{M}] = \mathcal{J}_t(\hat{\mathbb{E}}[A|\mathfrak{M}])$  as both sides equal  $\hat{\mathbb{E}}[A|\mathfrak{M}]$  due to the invariance and stationarity of the state  $\hat{\mathbb{E}}$ . This may be written as

$$\left[ \hat{\mathbb{E}}[\cdot|\mathfrak{M}], \mathcal{J}_t \right] = 0, \quad \forall t \geq 0. \quad (14)$$

This is the quantum analogue of the criterion established by Baez and Fong for classical Markov semi-groups.

## ACKNOWLEDGMENTS

J.G. acknowledges support from EPSRC Grant No. EP/L006111/1 Quantum Stochastic Analysis for Nano-photonic Circuit Design. O.G.S. acknowledges Russian Foundation for Basic Research Grant No. 14-01-00516. Both J.G. and O.G.S. are grateful to the hospitality of the Bernoulli Inter-faculty Centre, EPF Lausanne, where this research was carried out. TSV is partially supported by NCCR SwissMAP and Grant No. 200021-140238, both of the Swiss National Science Foundation.

## APPENDIX: SKETCH OF THE PROOF OF THE BAEZ-FONG THEOREM

(1) *implies* (2). Clearly constancy of the distribution implies constancy of the first and second moments.

(2) *implies* (3). If (2) holds, then the time-derivative of the expectation  $\sum_{x,y} A(x)^m T_{xy}(t)p(y)$  vanishes for every probability vector and for  $m = 0, 1, 2$ . This implies that  $\sum_{x \in \Gamma} A(x)^m M_{xy} = 0$  for all  $y \in \Gamma$  and for  $m = 0, 1, 2$ . In turn (by expanding the bracket), the expression  $\sum_{x,y} [A(x) - A(y)]^2 M_{xy}$  vanishes identically. As  $M_{xy} \geq 0$  for  $x \neq y$ , we see that if  $M_{xy} \neq 0$  and  $x \neq y$ , then  $A(x) - A(y) = 0$ . By transitivity of the equivalence relation, we get that  $A(x) = A(y)$  for all  $x \sim y$ .

(3) *implies* (4). We note that the matrix  $C = [\hat{A}, M]$  has components  $C_{xy} = A(x)M_{xy} - M_{xy}A(y)$ . Evidently, we have  $C_{xy} = 0$  whenever  $x$  and  $y$  belong to different equivalence classes, since we then have  $M_{xy} = 0$ . If we additionally assume (3) that  $A(x) = A(y)$  whenever  $x \sim y$ , we see immediately that  $C_{xy} = [A(x) - A(y)]M_{xy} \equiv 0$  whenever  $x \sim y$ .

(4) *implies* (1). We see that  $\frac{d}{dt} \sum_x f(A(x))p_x(t) = \sum_{x,y} f(A(x))M_{xy}p_y(t)$ ; however, if (4) holds, then we have the identity  $f(A(x))M_{xy} = M_{xy}f(A(y))$ , and so  $\frac{d}{dt} \sum_x f(A(x))p_x(t) = \sum_{x,y} f(A(y))M_{xy}p_y(t)$ , but this will vanish identically due to (4).

<sup>1</sup> J. C. Baez and B. Fong, *J. Math. Phys.* **54**, 013301 (2013).

<sup>2</sup> R. Alicki and M. Fannes, *Quantum Dynamical Systems* (Oxford University Press, Oxford, 2000).

<sup>3</sup> Actually since  $\Omega$  is finite at present, we only need the algebra of subsets: specifically  $\mathcal{M}$  is the collection of sets consisting of the empty set, each of the equivalence classes, and all of their possible unions. However, the general result requires a  $\sigma$ -algebra.



- <sup>4</sup> F. Fagnola and R. Rebolledo, *Open Quantum Systems III. Recent Developments*, Lecture Notes in Mathematics Vol. 1882 (Springer, 2006), pp. 161–205.
- <sup>5</sup> The terminology here is standard in operator theory: A state  $\mathbb{E}$  is normal if it takes the form  $\text{tr}\{\rho \cdot\}$  and faithful if the density matrix  $\hat{\rho}$  is strictly positive (that is,  $\mathbb{E}[X] = 0$  for a positive operator  $X$  if and only if  $X = 0$ ).
- <sup>6</sup> A. Frigerio and M. Verri, *Math. Z.* **180**(2), 275–286 (1982).
- <sup>7</sup> G. Lindblad, *Commun. Math. Phys.* **48**, 119–130 (1976).
- <sup>8</sup> V. Gorini, A. Kossakowski, and E. C. G. Sudarshan, *J. Math. Phys.* **17**(5), 821 (1976).
- <sup>9</sup> F. Fagnola and R. Rebolledo, *Infinite Dimens. Anal., Quantum Probab. Relat. Top.* **1**(4), 1–12 (1998).