Two Approaches to the Solution of Coefficient Inverse Problems for Wave Equations

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Abstract—Two approaches to solving coefficient inverse problems for wave equations are compared. One approach is based on integral representations obtained with the help of the Green's function for the wave equation. In the other approach, the gradient of the error functional is directly computed in terms of the solution of the adjoint problem for a partial differential equation. The methods developed are intended for finding inhomogeneities in homogeneous media and can be applied in medicine diagnostics, acoustic and seismic near surface exploration, engineering seismics, etc.

DOI: 10.1134/S0965542512020078
This a preprint of the Materials accepted for publication in «Computational Mathematics and Mathematical Physics», 2012, Vol. 52, No. 2, pp. 245–251.
http://link.springer.com/article/10.1134/S0965542512020078

Keywords: coefficient inverse problems, wave equation, Helmholtz equation, computer simulation, tomography, parallel computations.

1. INTRODUCTION

At present tomographic research (x-ray, MRT) is a necessary part of most medical studies. The high resolution of x-ray and magnetic resonance tomographs is determined by simple adequate-to-reality mathematical models and by the high accuracy of input data. In this case, 3D data interpretation is reduced to a set of independent 2D linear problems, which can be solved on a personal computer. By applying special software, any cross section of reconstructed 3D functions can be produced, the structure of blood vessels can be distinguished, etc. [1].

The results achieved in ultrasonic, acoustic, and seismic areas are much less impressive, because inverse data interpretation leads to complicated nonlinear problems even in the simplest models. The wave nature of radiation generates 3D nonlinear problems, which cannot be represented as a set of 2D problems.

Even more complicated is the interpretation of acoustic and seismic measurements in elastic media [2], in which single wave propagation (scalar case) is replaced by several waves (longitudinal, transverse, and surface). In the latter case, tensor models have to be formally used [3]. However, if longitudinal and transverse waves propagate at different velocities, inhomogeneities located at sufficiently large depths are reached at different times. As a result, we can again use a scalar wave equation, which describes physical processes, such as diffraction, refraction, and wave interference.

Mathematical aspects of solving coefficient inverse problems for wave equations have been examined in numerous publications. The first works in this area appeared in the early 1970s [4–6]. Classical uniqueness problems were studied in [7, 8]. 2D coefficient inverse problems were considered in [9, 10]. Carleman estimates were addressed in [11, 12], where numerical methods were proposed for solving a coefficient inverse problem with a source at infinity (plane wave). Coefficient inverse problems in integral representations of the Green's function were investigated in [13–18].

The goal of this paper is to assess the possibility of solving coefficient inverse problems for wave equations on modern computers. Although we consider the simplest scalar model, the solution of an inverse nonlinear coefficient problem is a complicated task. Below, the capabilities of approximate methods for solving coefficient inverse problem for wave equations are estimated in the framework of differential and integral approaches.

2. MATHEMATICAL STATEMENT OF COEFFICIENT INVERSE PROBLEMS FOR WAVE EQUATIONS

Consider the wave equation describing an acoustic or electromagnetic field $u(r, t)$ in the scalar approximation:

$$c^{-2}(r)u_{tt}(r,t) - \Delta u(r,t) = \delta(r-q)\cdot f(t), \quad (2.1))$$
$$u(r,t=0) = u_t(r,t=0) = 0,$$

where $c(r)$ is usually the wave velocity in the medium, $r \in R^3$ is the position of a point in space, Δ is the Laplacian with respect to r, $q \in Q$ is the point at which the source is located, and $f(t)$ is the function describing the momentum generated by the source. Assume that the inhomogeneity of the medium is caused only by variations in the velocity. For simplicity, the velocity outside the inhomogeneity is defined as $c(r) \equiv c_0 = const$, where c_0 is given.

Assume that the field $u(r, t)$ is measured in a domain P ($r \in P$); i.e., radiation sensors run over P; point radiation sources run over a domain Q ($q \in Q$); the inhomogeneity occupies a bounded domain R; P and Q may coincide; and the intersections of R and P and of R and Q are empty.

In the inverse problem under study, the unknowns are $c(r)$ and $u(r, t)$. Given the values of $u(r, t)$ in P, the goal is to reconstruct the function $c(r)$ describing the inhomogeneity of the medium.

In this formulation, impulsive sources are used for diagnostics. By taking the Fourier transform in time of the left- and right-hand sides of Eq. (2.1), the problem can be reduced to Helmholtz's equation.

3. SOLUTION METHODS FOR COEFFICIENT INVERSE PROBLEMS FOR THE WAVE EQUATION BASED ON INTEGRAL REPRESENTATIONS

Assume that the source is a point harmonic oscillator. Then the problem can be described by the Helmholtz equation

$$\Delta u(r,q,\omega) + \kappa^2(r,\omega)u(r,q,\omega) = f(r,q,\omega), \quad (3.1)$$

where Δ is the Laplacian with respect to r. In the scalar approximation, this equation describes the acoustic or electromagnetic field $u(r, q, \omega)$ generated by a source described by the function $f(r, q, \omega)$. For a source located at a point $q \in Q$, this function has the form $f(r,q,\omega) = -\delta(r-q)$, where ω is the circular frequency of the radiation source. The medium inhomogeneity is caused only by variations in the phase velocity $c(r)$, and $\kappa(r,\omega) = \omega / c(r)$. Outside the inhomogeneity, $\kappa(r,\omega) = \kappa_0 = \omega / c_0$, where $c_0 = const$ is given.

It is well known that the Green's function for Eq. (3.1) in a homogeneous medium satisfies the equation

$$\Delta G(r,q,\omega) + \frac{\omega^2}{c_0{}^2} G(r,q,\omega) = -\delta(r-q)$$

and has the form

$$G(r,g,\omega) = \frac{1}{4\pi\|r-q\|}\cdot \exp(i\frac{\omega}{c_0}\|r-q\|).$$

As before, we assume that the field $u(r, q, \omega)$ is measured in P ($r \in P$); i.e., the radiation sensors run over P; the point radiation sources run over Q ($q \in Q$); and the inhomogeneity occupies a bounded domain R. Writing the equations for R and P separately, we obtain a nonlinear system of equations (see [13, 15])

$$\begin{cases} u(r,q,\omega) = u_0(r,q,\omega) + \omega^2 \int\limits_R G(r',r,\omega)\xi(r')u(r',q,\omega)dr', \quad r \in R \\ U(p,q,\omega) = \omega^2 \int\limits_R G(r',p,\omega)\xi(r')u(r',q,\omega)dr', \quad p \in P \end{cases} \quad (3.2)$$

Here,

$$U(p,q,\omega) = u(p,q,\omega) - u_0(p,q,\omega),\ u_0(r,q,\omega) = -\int\limits_Q G(r',r,\omega) f(r',q,\omega)dr',\ \xi(r) = c_0{}^{-2} - c^{-2}(r).$$

In the inverse problem, the unknowns are the medium properties $\xi(r)$ and the field $u(r, q, \omega)$.

Given the function $U(p, q, \omega)$ measured in the sensor domain, the goal is to reconstruct the medium inhomogeneity $\xi(r)$.

Define the vector $X = \begin{pmatrix} \xi \\ u \end{pmatrix}$. System (3.2) can be rewritten as

$$F(X) = 0, \tag{3.3}$$

where

$$F(X) = \begin{cases} u(r,q,\omega) - u_0(r,q,\omega) - \omega^2 \int_R G(r',r,\omega)\xi(r')u(r',q,\omega)dr' & r \in R \\ \omega^2 \int_R G(r',p,\omega)\xi(r')u(r',q,\omega)dr' - U(p,q,\omega) & p \in P \end{cases}.$$

Equation (3.3) is a nonlinear equation of the first kind in a Hilbert space and can be solved using the regularized iterative Gauss–Newton method (see [13, 19])

$$X_{p+1} = X_p - (F_p'^* F_p' + \alpha_p E)^{-1}(F_p'^* F(X_p) + \alpha_p (X_p - \zeta)) \tag{3.5}$$

where $F'_p = F'(X_p)$ is the Fréchet derivative for (3.4), $F_p'^*$ is the adjoint of the operator F_p', p is iteration number. The regularizing properties of the algorithm are ensured by a suitable choice of the sequence α_p and the element ζ [13]. The formulas for $F'_p = F'(X_p)$ and $F_p'^* F_p'$ can be written out explicitly (see [17, 18]). Here, $F_p'^* F_p'$ is a positive definite Hermitian matrix of size $N(t+1) \times N(t+1)$ having the form

$$F_p'^* F_p' = \begin{bmatrix} *(NxN) & 0 & 0 & *(NxN) \\ 0 & \ddots & 0 & \vdots \\ 0 & 0 & *(NxN) & *(NxN) \\ *(NxN) & \cdots & *(NxN) & *(NxN) \end{bmatrix} = \begin{bmatrix} C^{1*}C^1 & 0 & \cdots & 0 & C^{1*}D^1 \\ 0 & C^{2*}C^2 & \cdots & 0 & C^{2*}D^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & C^{t*}C^t & C^{t*}D^t \\ D^{1*}C^1 & D^{2*}C^2 & \cdots & D^{t*}C^t & D^{1*}D^1 + D^{2*}D^2 + \ldots + D^{t*}D^t \end{bmatrix} \tag{3.6}$$

where N is the number of grid points in the computational domain R, $t = q \cdot d$, q is the number of sounding frequencies used, d is the number of source locations, and C and D are matrices [17, 18].

Computationally, the inverse problem in question is extremely time-consuming. The number of unknowns in the problem increases as $O(tn^3)$, where n is the number of grid points in a single direction of a 3D grid ($N = n^3$). The most time-consuming operations in Gauss–Newton procedure (3.5) are the computation of matrix $F_p'^* F_p'$ (3.6) and the inversion of the operator $(F_p'^* F_p' + \alpha_p E)$ which occur at each iteration of the procedure. The computation of the matrix requires $O(tn^9)$ addition and multiplication operations. The inversion of the operator by iterative methods requires $O(tn^6)$ operations at each iteration step (the number of iterations is, as a rule, several hundreds [18]).

The above algorithm leads to the computation of the vector $X = \begin{pmatrix} \xi \\ u \end{pmatrix}$. In fact, we are interested only in the component $\xi(r)$ describing the inhomogeneity. This feature of the problem can be used as follows. Nonlinear system (3.2) is rewritten as

$$\begin{cases} F_1(u,\xi) = 0 \\ F_2(u,\xi) = 0 \end{cases} \tag{3.7}$$

Here, F_1 and F_2 are mappings of Hilbert spaces H_1 and H_2 (in the general case) to other spaces H_1' and H_2' and are assumed to have the following properties: F_1 and F_2 are continuous; the partial derivatives F_{1u}, F_{2u}, $F_{1\xi}$, and $F_{2\xi}$ exist in H_1 and H_2, respectively; and the operator F_{1u} has a bounded inverse. Eliminating the unknown field $u(r, q, \omega)$ from system (3.7), we obtain an operator equation for determining $\xi(r)$:

$$F_2(\varphi(\xi), \xi) = 0 \tag{3.8}$$

where $\varphi(\xi)$ defines a continuous mapping $u=\varphi(\xi)$. Under the above conditions on F_1 and F_2, the mapping $\varphi(\xi)$ is differentiable and

$$\frac{d\varphi}{d\xi}=-\frac{\partial F_1}{\partial \xi}\cdot\left(\frac{\partial F_1}{\partial u}\right)^{-1}.$$

Note that

$$F_2^{'}=\frac{\partial F_2}{\partial u}\cdot\left(-\frac{\partial F_1}{\partial \xi}\cdot\left(\frac{\partial F_1}{\partial u}\right)^{-1}\right)+\frac{\partial F_2}{\partial \xi}$$

Therefore, the Gauss–Newton iteration for Eq. (3.8) has the form

$$\xi_{p+1}=\xi_p-(F_{2p}^{'*}F_{2p}^{'}+\alpha_p E)^{-1}(F_{2p}^{'*}F_{2p}(\varphi(\xi_p),\xi_p)+\alpha_p(\xi_p-\zeta_p)) \tag{3.9}$$

Here, α_p are positive numerical parameters and ζ_p is an element of H_2.

Computationally, iterative process (3.9) is also time-consuming. In contrast to (3.5), the number of unknowns is somewhat smaller, namely, $O(n^3)$. The inversion of the operator $(F_{2p}^{'*}F_{2p}^{'}+\alpha_p E)$ by iterative methods requires $O(n^6)$ operations at each iteration step, which is also less than in (3.5). However, the computation of $(F_{2p}^{'*}F_{2p}^{'}+\alpha_p E)$ still requires $O(tn^9)$ addition and multiplication operations.

4. METHODS FOR SOLVING COEFFICIENT INVERSE PROBLEMS FOR THE WAVE EQUATION BASED ON DIFFERENTIAL REPRESENTATIONS

Consider a problem described by the wave equation in a three-dimensional domain $\Omega\subset R^3$ bounded by a surface S on a time interval $[0, T]$ with a point source located at a point r_0:

$$c(r)u_{tt}(r,t)-\Delta u(r,t)=\delta(r-r_0)\cdot f(t), \tag{4.1}$$

$$u(r,t=0)=u_t(r,t=0)=0, \tag{4.2}$$

$$\partial_n u|_{ST}=p(r,t). \tag{4.3}$$

Here, for simplicity, in contrast to (2.1), $c^{-0.5}(r)$ is the wave velocity in the medium, while $\partial_n u|_{ST}$ is the derivative along the normal to the surface S in the domain $S\times T$. The inverse problem is to determine the inhomogeneity-describing function $c(r)$ from measurements of $U(s, t)$ on the domain boundary S over the time $(0, T)$ for various source locations r_0.

As is known, problem (4.1)–(4.3) defines $u(r, t)$ as an implicit function of $c(r)$. Consider the inverse problem as the minimization of the quadratic functional (see [20])

$$\Phi(u(c))=\frac{1}{2}\left\|u|_{ST}-U\right\|^2=\frac{1}{2}\left\|Mu-U\right\|^2=\frac{1}{2}\int_0^T\int_S(u(s,t)-U(s,t))^2 dsdt. \tag{4.4}$$

Here, M is a linear projector onto the boundary, $\|\cdot\|^2$ is the squared norm in $L_2(S\times T)$, and $U(s, t)$ are wave measurements on the domain boundary S over the time $(0, T)$.

The functional is minimized using gradient iterative methods (see [19]). The mathematical problem of calculating the gradient of functional (4.4) is as follows. We find the part of the increment of functional (4.4) that is linear with respect to an arbitrary variation dc:

$$d\Phi'(u(c))=\int_{ST}(u-U)(u_c dc)dsdt \quad . \tag{4.5}$$

where u_c is the Fréchet derivative. Since $u(r, t)$ solves problem (4.1)–(4.3) with some $c(r)$ (i.e., $u(r, t)$ is an implicit function of $c(r)$), taking the total derivative with respect to $c(r)$ in (4.1), we have

$$c(r)(u_c dc)_{tt}-\Delta(u_c dc)+u_{tt}(r,t)dc=0. \tag{4.6}$$

Moreover, introducing the linear restriction operator P to $t = 0$, we find from (4.2) that $u(r,t=0)\equiv P(u(r,t))=0=const$. Differentiation with respect to $c(r)$ yields $(P(u(r,t)))_c^{'}dc=P(u_c dc)=(u_c dc)(r,t=0)=0$. Similarly, it follows from (4.2) and (4.3) that

$$(u_c dc)(r,t=0)=(u_c dc)_t(r,t=0)=0\,, \tag{4.7}$$

$$\partial_n(u_c dc)|_{ST}=0\,. \tag{4.8}$$

Thus, for any variation *dc*, the function (u_c *dc*)(*r*,*t*) is a solution of problem (4.6)–(4.8). Define the operator *A*: $Au=c(r)u_{tt}(r,t)-\Delta u(r,t)$.

Consider the following problem, which is called "adjoint" to basic problem (4.1)–(4.3):

$$Bw=c(r)w_{tt}(r,t)-\Delta w(r,t)=0\,, \tag{4.9}$$

$$w(r,t=T)=w_t(r,t=T)=0\,, \tag{4.10}$$

$$\partial_n w|_{ST}=u|_{ST}-U\,, \tag{4.11}$$

where *u* is a solution of problem (4.1)–(4.3). Define $(u_c dc)(r,t)=\tilde{u}(r,t)$ for some variation dc. Consider the scalar product $(Bw,\tilde{u})$. Using relations (4.7), (4.10), and (4.11), we obtain

$$0=(Bw,\tilde{u})=\int_{\Omega T}(Bw)\tilde{u}\,drdt=\int_{\Omega}\int_0^T c(r)w_{tt}(r,t)\tilde{u}(r,t)dtdr-\int_0^T\int_{\Omega}\Delta w(r,t)\tilde{u}(r,t)drdt=$$

$$=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr-\int_0^T\int_S\partial_n w(s,t)\tilde{u}(s,t)dsdt+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt=$$

$$=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr-\int_{ST}(u(s,t)-U(s,t))\tilde{u}(s,t)dsdt+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt\,.$$

Thus, we have

$$\int_{ST}(u(s,t)-U(s,t))\tilde{u}(s,t)dsdt=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt \tag{4.12}$$

Consider the scalar product $(w,A\tilde{u})$. Combining (4.2), (4.6), and (4.10) produces

$$(w,A\tilde{u})=\int_{\Omega T}w(r,t)A(\tilde{u}(r,t))drdt=$$

$$=-\int_{\Omega}\int_0^T w(r,t)u_{tt}(r,t)dc(r)dtdr=\int_{\Omega}\int_0^T w_t(r,t)u_t(r,t)dc(r)dtdr \tag{4.13}$$

On the other hand, using (4.7), (4.8), and (4.10), we have

$$(w,A\tilde{u})=\int_{\Omega}\int_0^T w(r,t)c(r)\tilde{u}_{tt}(r,t)dtdr-\int_0^T\int_{\Omega}w(r,t)\Delta\tilde{u}(r,t)drdt=$$

$$=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr-\int_0^T\int_S w(s,t)\partial_n\tilde{u}(s,t)dsdt+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt=$$

$$=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt\,. \tag{4.14}$$

Combining (4.13) and (4.14) yields

$$\int_{\Omega}\int_0^T w_t(r,t)u_t(r,t)dc(r)dtdr=-\int_{\Omega}\int_0^T c(r)w_t(r,t)\tilde{u}_t(r,t)dtdr+\int_0^T\int_{\Omega}\nabla w(r,t)\nabla\tilde{u}(r,t)drdt\,. \tag{4.15}$$

Then it follows from (4.5), (4.12), and (4.15) that

$$d\Phi'(u(c)) = \int_{ST} (u-U)(u_c dc) ds dt = \int_{\Omega} \left[\int_0^T w_t(r,t) u_t(r,t) dt \right] dc(r) dr$$

Extracting the linear part with respect to *dc*, for the gradient of the functional $\Phi(u(c))$, we finally have

$$\Phi'_C(u(c)) = \int_0^T w_t(r,t) u_t(r,t) dt, \tag{4.16}$$

here $u(r, t)$ is a solution of problem (4.1)–(4.3) and $w(r, t)$ is the solution of “adjoint” problem (4.9)–(4.11) with the given $c(r)$. Thus, to compute the gradient of the functional, we have to solve the basic and “adjoint” problems.

Given Φ'_C from (4.16), we can construct various iterative schemes for minimizing error functional (4.4) (see [19]).

The differential approach considered has a number of important advantages, primarily, in terms of the amount of computations. When $u(r, t)$ and $w(r, t)$ are determined using, for example, explicit schemes, the number of addition and multiplication operations required for computing the gradient is $O(n^3\tau)$, where τ is the number of grid points in time. This estimate is much lower than that in the integral approach.

5. CONCLUSIONS

The inverse problem considered is a nonlinear large-scale one that is difficult to implement even on modern supercomputers. Both integral and differential approaches seem promising and make it possible to analyze various diagnostic schemes for model problems.

An advantage of the integral approach is that it provides a simple inverse problem formulation suitable for any locations of sensors and sources, including semi_infinite or infinite media with inhomogeneity. The integral approach can be successfully used in the case of a simple inhomogeneity structure (e.g., for several small inhomogeneities).

The solution methods based on the differential approach require a smaller amount of computations. In this case, computational difficulties are caused by the necessity of setting additional boundary condi_ tions that are not related to the physical characteristics of the objects under study, but are a consequence of using difference schemes in a bounded spatial domain.

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