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Abstract. Multilayer composite materials are often used in building structures. The direct calculation of layered structures requires large expenditures of computer time. Therefore, the homogenization method is used. This method reduces the problem of a layered material with isotropic layers to the problem of a homogeneous transversely isotropic medium. The material considered in the article is also elastic-creeping. In the equations of state of such a material, terms of the convolution type with difference creep (relaxation) kernels are added to the terms of the usual theory of elasticity. The creep (relaxation) kernels are represented by decreasing exponential functions depending on two parameters. This problem becomes a problem of the theory of elasticity with a parameter after applying the Laplace transform in time to it. The inverse Laplace transform can be done in a computer algebra package, for example, Wolfram Mathematica, Wolfram-alpha. The obtained characteristics of the material are used to solve the problem of a layered elastic-creeping beam with hinge support. Formulas are given for determining displacements in the case of layers parallel to the beam axis.

1. Introduction

In the construction of industrial and residential buildings, a structure such as a beam plays an important role. This element bears the main loads of the building in the vertical direction. Therefore, a very important point in the construction process is the calculation and selection of building materials for this building part. Such complex service life will depend on its reliability.

By the method of fastening, the construction beam acts as a horizontal or inclined beam, more often working in bending. An additional important function of the horizontal beam is the distribution of the distributed load under the action of a vertically directed moment of force. It is this orientation that allows the product to significantly reduce the pressure on vertical structures (columns, posts) and to firmly strengthen them. Composite materials play an important role in the design of such building elements. The well-known homogenization theory can be used to calculate them [1,2]. In addition, such materials have the property of creep, that is, changes in time of the stress-strain state under constant loads. For example, to these questions are devoted the following works [4-6], the theory of the hereditary Boltzmann-Volterra mechanics are presented. In [7] the problem of bending deformation of layered metal-composite beams-walls, operating under conditions of steady creep of materials of all layers, was solved. Equations are obtained that allow describing the stress-strain state in a beam with varying degrees of accuracy. It is shown that in the case of a copper-steel metal composition, neither the classical theory nor the first version of Timoshenko's theory guarantee reliable results on the compliance of the structure even within the 20% accuracy, which is considered acceptable when studying the mechanical behavior of structural elements under creep conditions.

In the modern world, the industrial and construction sectors are constantly developing and improving. There is a continuous tightening of standards, reconstruction of buildings and structures, where it is often necessary to strengthen the supporting structures and their elements.

Composite reinforcement systems perform excellently in both normal and harsh environments. The most commonly used external reinforcement using carbon fiber composites. At the same time, the materials are selected in such a way that, in the optimal combination, they give a qualitatively new type of design. Or, in other words, in composite structures, composites are arranged so that under operational conditions they better correspond to their functional purpose.

The homogenization problems of elastic-creeping media were considered in [8-11].

Effective (averaged) modules of a layered elastic-creeping medium were obtained in paper [12].

The paper [13] considers the problem of a heterogeneous pipe consisting of a layered material. The elastic properties of this material depend only on the distance from the section center. The two approaches presented in this article. They allow to obtain a solution in an analytical form.

This article examines an elastic-creeping layered beam with longitudinal, pairwise alternating isotropic layers on two hinged supports. The creep kernels of the layers are exponential functions depending on two parameters.

2. Problem specification and decision

We consider an inhomogeneous beam of length *l*, consisting of pairwise alternating layers parallel to its axis. The layers of elastic-creeping materials are assumed to be isotropic therefore the stiffness tensor c_{ijkl} and relaxation tensor R_{ijkl} $(i,j,k,l=1\div 4)$, respectively, have the form [5]:

$$c_{ijkh}=\lambda\delta_{ij}\delta_{kh}+\mu(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk}), \tag{1}$$

$$R_{ijkh}=-(D_v(t)-\frac{1}{3}D_{sh}(t))\delta_{ij}\delta_{kh}-\frac{1}{2}D_{sh}(t)(\delta_{ik}\delta_{jh}+\delta_{ih}\delta_{jk}), \tag{2}$$

In (1, 2) λ,μ are Lame parameters, $D_{sh}(t)$ and $D_v(t)$ are the regular parts of the shear and the bulk relaxation respectively, δ_{ij} is Kronecker symbol. We admit $(D_v(t))_s=k_s(D_{sh}(t))_s$, k_s is a constant, $k_s>0,(s=1,2)$. Further, $D_{sh}(t)$ is denoted by $D(t)$. Further, the relaxation kernels are decreasing exponential functions.

Equilibrium equations in the theory of elasticity have the form [3]:

$$\frac{\partial\sigma_{ij}(x,t)}{\partial x_j}=f_i(x,t). \tag{3}$$

In (3) we designated: $x=(x_1,x_2,x_3)$, $\sigma_{ij}(x,t)$ - are deformation components, $f_i(x,t)$ are components of a vector of external forces, $i,j=1,2,3$ and variable *t* specifies time. The beam axis coincides with the x_3 - axis. For our problem the stress tensor components for each layer have the following form:

$$\sigma_{ij}^{(s)}=m_{ijkh}^{(s)}*e_{kh}^{(s)}. \tag{4}$$

$$m_{ijkh}=c_{ijkh}\delta(t)+R_{ijkh}. \tag{5}$$

($s=1,2$ is the layer number, $e_{kh}^{(s)}$ are deformation components for each layer, $\delta(t)$ is Dirac-delta, Einstein convention for repeated indices is used). The * means the next convolution operation

$$R_{ijkh}*e_{kh}=\int_0^t R_{ijkh}(t-\tau)e_{kh}(\tau)d\tau. \tag{6}$$

We assume the values of the stress components σ_{ij} are invariants with respect to the origin of the time reference, therefore the relaxation kernels R_{ijkh} depend on the difference $t-\tau$. Ideal contact conditions are assumed to be satisfied on the horizontal surfaces of the layers.
Since this beam is a layered medium all elastic modulus and relaxation kernels are periodic functions of the coordinate $\xi = \dfrac{y}{\varepsilon}$ (ε is the relative cell period) and are piecewise constant functions of this variable, i.e., elastic modulus and relaxation kernels have the form [9]:

$$\lambda(\xi)=\begin{cases}\lambda_1, \xi\in[0;h]\\ \lambda_2, \xi\in[1-h;1]\end{cases} \quad \mu(\xi)=\begin{cases}\mu_1, \xi\in[0;h]\\ \mu_2, \xi\in[1-h;1]\end{cases} \quad D(\xi,t)=\begin{cases}d_1 e^{-\alpha_1 t}, \xi\in[0;h]\\ d_2 e^{-\alpha_2 t}, \xi\in[1-h;1]\end{cases}. \tag{7}$$

In (7) $\lambda_1, \mu_1, \lambda_2, \mu_2$ are Lame parameters for each layer, $\alpha_1, d_1, \alpha_2, d_2$ are positive constants, t is the variable that specifies time, h is the constant that determines the layers thickness ratio of different materials.
The Laplace transform in the time domain is applied to (3) taking into account (4)

$$\tilde{f}(p)=\int_0^\infty f(t)e^{-pt}dt. \tag{8}$$

After this we have a system of equations of elasticity theory with a complex parameter *p*. The homogenization method [1, 2] is applied to this system. As a result, we obtain a homogeneous beam consisting of a transversely isotropic material.

Let us show how the elastic compliance of the obtained material is calculated in Laplace images, that is, they will be functions depending on the parameter *p*.

Further, the designations of elastic compliance components with two indices, which are more common in the technical literature, are used.

The equations of state written with elastic compliance $a_{ij}\,(i,j=1\div3)$ are as follows

$$\begin{cases}\varepsilon_{11}=a_{11}\sigma_{11}+a_{12}\sigma_{22}+a_{13}\sigma_{33}\\ \varepsilon_{22}=a_{12}\sigma_{11}+a_{11}\sigma_{22}+a_{13}\sigma_{33}\\ \varepsilon_{33}=a_{13}\sigma_{11}+a_{13}\sigma_{22}+a_{33}\sigma_{33}\end{cases}. \tag{9}$$

In [11] expressions for the elastic compliance components are obtained using the inverse of the elastic moduli matrix:

$$a_{11}=\frac{\langle\lambda+2\mu\rangle-\left\langle\dfrac{\lambda^2}{\lambda+2\mu}\right\rangle}{\left(\langle\lambda+2\mu\rangle-\langle\lambda\rangle\right)\left(\langle\lambda+2\mu\rangle+\langle\lambda\rangle-2\left\langle\dfrac{\lambda^2}{\lambda+2\mu}\right\rangle\right)}, \tag{10}$$

$$a_{12}=\frac{\left\langle\dfrac{\lambda^2}{\lambda+2\mu}\right\rangle-\langle\lambda\rangle}{\left(\langle\lambda+2\mu\rangle-\langle\lambda\rangle\right)\left(\langle\lambda+2\mu\rangle+\langle\lambda\rangle-2\left\langle\dfrac{\lambda^2}{\lambda+2\mu}\right\rangle\right)}, \tag{11}$$

$$a_{13}=-\frac{\left\langle\dfrac{\lambda}{\lambda+2\mu}\right\rangle}{\langle\lambda+2\mu\rangle+\langle\lambda\rangle-2\left\langle\dfrac{\lambda^2}{\lambda+2\mu}\right\rangle}, \tag{12}$$

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$$a_{33} = \left\langle \frac{1}{\lambda + 2\mu} \right\rangle + \frac{2\left\langle \frac{\lambda}{\lambda + 2\mu} \right\rangle^2}{\langle \lambda + 2\mu \rangle + \langle \lambda \rangle - 2\left\langle \frac{\lambda^2}{\lambda + 2\mu} \right\rangle}. \quad (13)$$

B(10-(13) brackets $\langle \cdot \rangle$ denote averaging operation:

$$\langle f \rangle = \int_0^1 f(\xi) d\xi. \quad (14)$$

As a result, we get

$$\tilde{a}_{11} = \frac{A_{11}}{B_{11}}, \tilde{a}_{12} = \frac{A_{12}}{B_{12}}, \tilde{a}_{13} = \frac{A_{13}}{B_{13}}, \quad (15)$$

$$A_{11} = h(p+\alpha_2)[L_2(p+\alpha_2) - G_2][2\mu_1(p+\alpha_1) - g_1]\left[2(\lambda_1+\mu_1)(p+\alpha_1) - g_1\left(2k_1 + \frac{1}{3}\right)\right] + \quad (16)$$

$$+(1-h)(p+\alpha_1)[L_1(p+\alpha_1) - G_1][2\mu_2(p+\alpha_2) - g_2]\left[2(\lambda_2+\mu_2)(p+\alpha_2) - g_2\left(2k_2 + \frac{1}{3}\right)\right],$$

$$A_{12} = h(p+\alpha_2)[\lambda_1(p+\alpha_1) - j_1][L_2(p+\alpha_2) - G_2][g_1 - 2\mu_1(p+\alpha_1)] + \quad (17)$$

$$+(1-h)(p+\alpha_1)[\lambda_2(p+\alpha_2) - j_2][L_1(p+\alpha_1) - G_1][g_2 - 2\mu_2(p+\alpha_2)],$$

$$A_{13} = \{h[\lambda_1(p+\alpha_1) - j_1][L_2(p+\alpha_2) - G_2] + \quad (18)$$

$$+(1-h)[\lambda_2(p+\alpha_2) - j_2][L_1(p+\alpha_1) - G_1]\}(p+\alpha_1)(p+\alpha_2),$$

$$B_{11} = B_{12} = h(p+\alpha_2)[L_2(p+\alpha_2) - G_2][2\mu_1(p+\alpha_1) - g_1][2(\lambda_1+\mu_1)(p+\alpha_1) - 3k_1 g_1] + \quad (19)$$

$$+(1-h)(p+\alpha_1)[L_1(p+\alpha_1) - G_1][2\mu_2(p+\alpha_2) - g_2][2(\lambda_2+\mu_2)(p+\alpha_2) - 3k_2 g_2],$$

$$B_{13} = h(p+\alpha_2)[L_2(p+\alpha_2) - G_2][2\mu_1(p+\alpha_1) - g_1][(3\lambda_1 + 2\mu_1)(p+\alpha_1) - 3k_1 g_1] + \quad (20)$$

$$+(1-h)(p+\alpha_1)[L_1(p+\alpha_1) - G_1][2\mu_2(p+\alpha_2) - g_2][(3\lambda_2 + 2\mu_2)(p+\alpha_2) - 3k_2 g_2].$$

In (16)-(20) the following notation was introduced

$$L_s = \lambda_s + 2\mu_s, G_s = g_s\left(k_s + \frac{2}{3}\right), j_s = g_s\left(k_s - \frac{1}{3}\right), (s = 1, 2).$$

It follows from the physical meaning of the problem that roots of the polynomials B_{11}, B_{12}, B_{13} in the denominators of (15) must be different real negative numbers. Otherwise, in the relation $\boldsymbol{\varepsilon} = a\boldsymbol{\sigma}$, where a is an elastic compliance matrix, for fixed $\boldsymbol{\sigma}$ and for $t \to +\infty$ exponentially growing or oscillating components of the strain tensor would be obtained.

We decompose the polynomials B_{11}, B_{12}, B_{13} (19), (20), into linear factors and use for example the symbolic mathematics software package Wolfram Mathematica to perform the inverse Laplace transform. We obtain the coefficients a_{11}, a_{12}, a_{13} as sums of some constants multiplied by the delta-function and decreasing exponential functions which depend on time. Their exponents have coefficients equal to the given values $-\alpha_1, -\alpha_2$ and coefficients equal to the roots of denominators polynomials of the corresponding fraction.

The obtained expressions for the components of elastic compliance can be used to calculate the displacements of an elastic-creeping layered beam for example, for the following fixing conditions. At the ends of the beam, forces are applied, resulting in bending moments acting in a plane passing through the beam axis z and one of the main axes of inertia y of the cross section. Let M is the bending moment, and I is the moment of inertia of the beam cross-section relative to the main x axis. As is known, in this case for a transversely isotropic body, the distribution of nonzero stresses and strains has the form [15]:

$$\sigma_{33} = \frac{M}{I} x_2, \tag{21}$$

$$\varepsilon_{11} = \frac{M}{I} a_{13} x_2, \varepsilon_{22} = \frac{M}{I} a_{12} x_2, \varepsilon_{33} = \frac{M}{I} a_{11} x_2. \tag{22}$$

The displacements u, v, w are determined from equations (22) by integration. Arbitrary constants that will be included in the obtained expressions are determined from the conditions for fixing the ends. Let there be hinged supports at the ends, then the displacements must satisfy the conditions for $x_1 = x_2 = x_3 = 0$ and $x_1 = x_2 = 0, x_3 = l$ conditions $u = v = w = 0$, and in addition, the condition that the axis element near one of the supports cannot rotate, that is, when $x_1 = x_2 = x_3 = 0$

$$\frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_2} = 0. \tag{23}$$

Then we will have

$$\begin{cases} u = \frac{M}{I} a_{13} x_1 x_2 \\ v = \frac{M}{2I}[-a_{13} x_1^2 + a_{21} x_2^2 + a_{11}(l x_3 - x_3^2)] \\ w = \frac{M}{2I} a_{11} x_2 (2x_3 - l) \end{cases}. \tag{24}$$

We can also write the equation of the curved beam axis

$$x_2 = \frac{M a_{11}}{2I}(l x_3 - x_3^2). \tag{25}$$

From equation (25), the largest deflection of the beam (in the middle of the span) can be determined.

3. Conclusion

In the article formulas are obtained, according to which one can learn the values of the elastic compliance of a layered composite beam with the arrangement of layers along its axis. In this case, in the calculations, you can use a computer algebra package Wolfram Mathematica. The material obtained using the averaging method is transversely isotropic, as a result of which the formulas of the mechanics of an anisotropic body are used to find the displacement components.

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