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Abstract

The problem of rolling without sliding of a homogeneous ball on a fixed surface under the action of gravity is a classical problem of nonholonomic system dynamics. Usually, when considering this problem, following the E. J. Routh approach [1] it is convenient to define explicitly the equation of the surface, on which the ball's centre is moving. This surface is equidistant to the surface, over which the contact point is moving. From the classical works of E. J. Routh [1] and F. Noether [2] it was known that if the ball rolls on a surface such that its centre moves along a surface of revolution, then the problem is reduced to solving the second order linear differential equation. However it is impossible to find the general solution of this differential equation for an arbitrary surface of revolution. Therefore it is interesting to study for which surface of revolution the corresponding second order linear differential equation admits the explicit solution, for example, Liouvillian solution. To solve this problem it is possible to apply the Kovacic algorithm [3] to the corresponding second order linear differential equation. In this paper we present our own method to derive the corresponding second order linear differential equation. In the case when the centre of the ball moves along the paraboloid of revolution we prove that the corresponding second order linear differential equation admits a Liouvillian solution.

Keywords (separated by '-'): Rolling without sliding - Homogeneous ball - Surface of revolution - Kovacic algorithm

Application of the Kovacic Algorithm to the Problem of Rolling of a Heavy Homogeneous Ball on a Surface of Revolution

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Abstract. The problem of rolling without sliding of a homogeneous ball on a fixed surface under the action of gravity is a classical problem of nonholonomic system dynamics. Usually, when considering this problem, following the E. J. Routh approach [1] it is convenient to define explicitly the equation of the surface, on which the ball's centre is moving. This surface is equidistant to the surface, over which the contact point is moving. From the classical works of E. J. Routh [1] and F. Noether [2] it was known that if the ball rolls on a surface such that its centre moves along a surface of revolution, then the problem is reduced to solving the second order linear differential equation. However it is impossible to find the general solution of this differential equation for an arbitrary surface of revolution. Therefore it is interesting to study for which surface of revolution the corresponding second order linear differential equation admits the explicit solution, for example, Liouvillian solution. To solve this problem it is possible to apply the Kovacic algorithm [3] to the corresponding second order linear differential equation. In this paper we present our own method to derive the corresponding second order linear differential equation. In the case when the centre of the ball moves along the paraboloid of revolution we prove that the corresponding second order linear differential equation admits a Liouvillian solution. AQ1

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1 Introduction

Investigation of various problems of classical mechanics and mathematical physics is reduced to solving the second–order linear differential equation with

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variable coefficients. In 1986, the American mathematician J. Kovacic proposed an algorithm for solving the second–order linear differential equation with rational coefficients in the case where the solution can be expressed in terms of so-called Liouvillian functions [3]. Liouvillian functions are constructed from rational functions by algebraic operations, taking exponentials and integration [3,4]. If a second–order linear differential equation has no Liouvillian solutions, the Kovacic algorithm also allows one to ascertain this fact. Therefore, the Kovacic algorithm is a very effective method for investigation the problems which solution are reduced to the integration the second–order linear differential equation. However, this algorithm is not very known for the specialists in mechanics and mechanical engineering. The goal of this paper is to avoid this problem and to made the Kovacic algorithm more popular for the investigation of various problems, where we need to solve the second–order linear differential equation. In this paper we discuss the application of the Kovacic algorithm to the problem of motion of a heavy homogeneous ball on a fixed perfectly rough surface of revolution.

The paper is organized as follows. In the Sect. 2 we present the detailed formulation of the problem and we give our own way to reduce the problem to the integration the second–order linear differential equation. In the Sect. 3 we prove that the problem of motion of a heavy homogeneous ball on a surface of revolution such that the centre of a ball moves along a paraboloid of revolution in integrated in terms of Liouvillian functions.

2 Problem Formulation. General Equations of Motion

Let us consider the problem of motion of a heavy homogeneous ball on any fixed perfectly rough surface under the action of any forces whose resultant passes through the centre of the ball [1]. Let G be the centre of gravity of the ball and let the moving axes GC, GA, GB be respectively a normal to the surface and some two lines at right angles to be afterwards chosen at our convenience. Let $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ be the unit vectors of these axes GA, GB and GC respectively. Let $\boldsymbol{\Omega} = \theta_1\boldsymbol{e}_1+\theta_2\boldsymbol{e}_2+\theta_3\boldsymbol{e}_3$ be the angular velocity of these axes; $\boldsymbol{v}_G = u\boldsymbol{e}_1+v\boldsymbol{e}_2+w\boldsymbol{e}_3$ be the velocity of G (then $w = 0$ since the ball is always in contact with the supporting surface); $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_1+\omega_2\boldsymbol{e}_2+\omega_3\boldsymbol{e}_3$ be the angular velocity of the body about these axes. Let $\boldsymbol{R} = F\boldsymbol{e}_1 + F'\boldsymbol{e}_2 + R\boldsymbol{e}_3$ be the reaction, acting on the ball from the surface and let $\boldsymbol{P} = X\boldsymbol{e}_1 + Y\boldsymbol{e}_2 + P\boldsymbol{e}_3$ be the impressed force on the centre of gravity of the ball. Let m be the mass of the ball, a – its radius, J – the moment of inertia of the ball about a diameter. We shall suppose that the ball rolls on the convex side of the fixed surface and that the positive direction of the axis GC is drawn outwards from the surface. Then the equations of motion of the ball can be written in vector form:

$$m\dot{\boldsymbol{v}}_G + [\boldsymbol{\Omega} \times \boldsymbol{v}_G] = \boldsymbol{P} + \boldsymbol{R}, \tag{1}$$

$$J\dot{\boldsymbol{\omega}} + [\boldsymbol{\Omega} \times J\boldsymbol{\omega}] = \left[\overrightarrow{GK} \times \boldsymbol{R}\right]. \tag{2}$$

Equations (1) and (2) represents the behavior of momentum and angular momentum of the ball respectively. Here $\overrightarrow{GK} = -a\boldsymbol{e}_3$ is the radius – vector of the ball's point of contact with the surface relative to its centre of gravity. Since the point of contact of the sphere and surface is at rest we have

$$\boldsymbol{v}_G + \left[\boldsymbol{\omega} \times \overrightarrow{GK}\right] = 0. \tag{3}$$

In scalar form Eqs. (3)–(4) can be written as follows:

$$m\dot{u} - m\theta_3 v = X + F, \quad m\dot{v} + m\theta_3 u = Y + F', \quad m\theta_1 v - m\theta_2 u = P + R; \tag{4}$$

$$J\dot{\omega}_1 + J\theta_2\omega_3 - J\theta_3\omega_2 = F'a, \quad J\dot{\omega}_2 + J\theta_3\omega_1 - J\theta_1\omega_3 = -Fa, \quad \dot{\omega}_3 + \theta_1\omega_2 - \theta_2\omega_1 = 0; \tag{5}$$

$$u - a\omega_2 = 0, \quad v + a\omega_1 = 0. \tag{6}$$

Eliminating F, F', ω_1, ω_2 from the Eqs. (4)–(6) we get:

$$\dot{u} - \theta_3 v = \frac{a^2X}{J + ma^2} + \frac{Ja\theta_1\omega_3}{J + ma^2}, \quad \dot{v} + \theta_3 u = \frac{a^2Y}{J + ma^2} + \frac{Ja\theta_2\omega_3}{J + ma^2}. \tag{7}$$

The meaning of the Eq. (7) may be found as follows. They are the two equations of motion of the centre of the ball, which we should have obtained if the given surface had been smooth and the centre G had been acted on by acceleration forces

$$\frac{Ja\theta_1\omega_3}{J + ma^2} \quad \text{and} \quad \frac{Ja\theta_2\omega_3}{J + ma^2}$$

along the axes GA, GB and by the same impressed forces as before reduced in the ratio

$$\frac{a^2}{J + ma^2}.$$

The centre G of the ball moves along a surface formed by producing all the normals to the given surface a constant length equal to the radius of the ball. Let us take the axes GA, GB to be tangents to the lines of curvature of this surface. Let us find expression for the angular velocity $\boldsymbol{\Omega}$ of the chosen moving coordinate system GA, GB, GC in terms of the components u and v of the velocity $\boldsymbol{v}_G$ of the ball. We will assume that the surface along which the centre of the ball moves is given with respect to some fixed coordinate system by the equation

$$\boldsymbol{r} = \boldsymbol{r}\left(q_1, q_2\right), \tag{8}$$

where q_1 and q_2 are gaussian coordinates on this surface. We shall assume that a coordinate grid on the surface (8) consists of curvature lines whose directions at every point are given by unit vectors:

$$\boldsymbol{e}_1 = \frac{1}{h_1}\frac{\partial \boldsymbol{r}}{\partial q_1}, \quad \boldsymbol{e}_2 = \frac{1}{h_2}\frac{\partial \boldsymbol{r}}{\partial q_2}, \quad (\boldsymbol{e}_i \cdot \boldsymbol{e}_j) = \delta_{i\,j}. \tag{9}$$

Here we have denoted by h_1, h_2 the Lame's parameters

$$h_i\left(q_1,\, q_2\right) = \left|\frac{\partial \boldsymbol{r}}{\partial q_i}\right|, \quad i = 1, 2.$$

The vector $\boldsymbol{e}_3 = [\boldsymbol{e}_1 \times \boldsymbol{e}_2]$ is a normal vector to the surface (8) at point $(q_1,\, q_2)$. The velocity of centre of the ball $\boldsymbol{v}_G$ may be written as follows:

$$\boldsymbol{v}_G = \frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{r}}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{r}}{\partial q_2}\dot{q}_2 = u\boldsymbol{e}_1 + v\boldsymbol{e}_2.$$

Therefore we have the following equation connecting the velocities u and v with the coordinates q_1, q_2 and their derivatives

$$u = h_1\dot{q}_1, \quad v = h_2\dot{q}_2. \tag{10}$$

Let $k_i\left(q_1,\, q_2\right)$, $i = 1, 2$ be principal curvatures of the surface (8). Then we have the following equations:

$$\frac{\partial \boldsymbol{e}_3}{\partial q_1} = -h_1 k_1 \boldsymbol{e}_1, \quad \frac{\partial \boldsymbol{e}_3}{\partial q_2} = -h_2 k_2 \boldsymbol{e}_2. \tag{11}$$

Equations (11) follow from the Rodrigues's theorem well known in differential geometry (see e.g. [5]), where we have to additionally account for the fact that our coordinate grid on the surface (8) is orthogonal and consists of curvature lines. Taking into account (9) and (11) it is easy to derive the following equations:

$$\begin{aligned} &\tfrac{\partial \boldsymbol{e}_1}{\partial q_1} = -\tfrac{1}{h_2}\tfrac{\partial h_1}{\partial q_2}\boldsymbol{e}_2 + h_1 k_1 \boldsymbol{e}_3, \quad \tfrac{\partial \boldsymbol{e}_1}{\partial q_2} = \tfrac{1}{h_1}\tfrac{\partial h_2}{\partial q_1}\boldsymbol{e}_2, \\ &\tfrac{\partial \boldsymbol{e}_2}{\partial q_1} = \tfrac{1}{h_2}\tfrac{\partial h_1}{\partial q_2}\boldsymbol{e}_1, \quad \tfrac{\partial \boldsymbol{e}_2}{\partial q_2} = -\tfrac{1}{h_1}\tfrac{\partial h_2}{\partial q_1}\boldsymbol{e}_1 + h_2 k_2 \boldsymbol{e}_3. \end{aligned} \tag{12}$$

The angular velocity of the coordinate system GA, GB, GC can be found by the well known formula:

$$\boldsymbol{\Omega} = \left(\dot{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3\right)\boldsymbol{e}_1 + \left(\dot{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1\right)\boldsymbol{e}_2 + \left(\dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2\right)\boldsymbol{e}_3,$$

where

$$\dot{\boldsymbol{e}}_i = \frac{d\boldsymbol{e}_i}{dt} = \frac{\partial \boldsymbol{e}_i}{\partial q_1}\dot{q}_1 + \frac{\partial \boldsymbol{e}_i}{\partial q_2}\dot{q}_2, \quad i = 1, 2, 3.$$

Taking into account (11)–(12) we obtain for the angular velocity $\boldsymbol{\Omega}$ the following expression

$$\boldsymbol{\Omega} = h_2 k_2 \dot{q}_2 \boldsymbol{e}_1 - h_1 k_1 \dot{q}_1 \boldsymbol{e}_2 + \left(\frac{\dot{q}_2}{h_1}\frac{\partial h_2}{\partial q_1} - \frac{\dot{q}_1}{h_2}\frac{\partial h_1}{\partial q_2}\right)\boldsymbol{e}_3.$$

This expression can be rewritten in the form

$$\boldsymbol{\Omega} = k_2 v \boldsymbol{e}_1 - k_1 u \boldsymbol{e}_2 + \frac{1}{h_1 h_2}\left(\frac{\partial h_2}{\partial q_1}v - \frac{\partial h_1}{\partial q_2}u\right)\boldsymbol{e}_3.$$

if we take into account Eq. (10). Therefore we have the following expressions for the components θ_1, θ_2, θ_3 of the angular velocity $\boldsymbol{\Omega}$:

$$\theta_1 = k_2 v, \quad \theta_2 = -k_1 u, \quad \theta_3 = \frac{1}{h_1 h_2}\left(\frac{\partial h_2}{\partial q_1} v - \frac{\partial h_1}{\partial q_2} u\right). \tag{13}$$

We suppose now that the surface along which the centre of the ball moves is a surface of revolution, given with respect to some fixed coordinate system by the equation

$$\boldsymbol{r} = \begin{pmatrix} \rho\left(q_1\right)\cos q_2 \\ \rho\left(q_1\right)\sin q_2 \\ \zeta\left(q_1\right) \end{pmatrix}. \tag{14}$$

In this case the Lame's parameters h_1 and h_2 take the form:

$$h_1 = h_1\left(q_1\right) = \sqrt{\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2}, \quad h_2 = h_2\left(q_1\right) = \rho\left(q_1\right), \tag{15}$$

and the principal curvatures k_1 and k_2 may be written as follows:

$$k_1 = k_1\left(q_1\right) = \frac{\left(\frac{d^2\zeta}{dq_1^2}\frac{d\rho}{dq_1} - \frac{d\zeta}{dq_1}\frac{d^2\rho}{dq_1^2}\right)}{\left(\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2\right)^{\frac{3}{2}}}, \quad k_2 = k_2\left(q_1\right) = \frac{\frac{d\zeta}{dq_1}}{\rho\sqrt{\left(\frac{d\rho}{dq_1}\right)^2 + \left(\frac{d\zeta}{dq_1}\right)^2}}. \tag{16}$$

In this case the meridians and parallels are the lines of curvature. Let the axis Z of upward vertical be symmetry axis of the considered surface of revolution. Except the coordinates q_1 and q_2 we introduce the Euler angles θ, ψ and φ, where the angle the axis GC makes with the axis of Z equals θ and ψ is the angle the plane containing Z and GC makes with any fixed vertical plane. We suppose, that the components θ_1, θ_2, θ_3 of the angular velocity $\boldsymbol{\Omega}$ are defined by the Euler kinematic equations:

$$\theta_1 = \dot\psi \sin\theta \sin\varphi + \dot\theta \cos\varphi, \quad \theta_2 = \dot\psi \sin\theta \cos\varphi - \dot\theta \sin\varphi, \quad \theta_3 = \dot\psi \cos\theta + \dot\varphi,$$

in which we put $\varphi = -\pi/2$. Therefore we have

$$\theta_1 = -\dot\psi \sin\theta, \quad \theta_2 = \dot\theta, \quad \theta_3 = \dot\psi \cos\theta. \tag{17}$$

Comparing these equations with Eq. (13) we find:

$$-\dot\psi \sin\theta = k_2 h_2 \dot q_2, \quad \dot\theta = -k_1 h_1 \dot q_1, \quad \dot\psi \cos\theta = \frac{1}{h_1}\frac{dh_2}{dq_1}\dot q_2. \tag{18}$$

From the second equation of the system (18) the connection between θ and q_1 is determined. Therefore we can assume that the surface (14) is defined by the variables θ and q_2, i.e.

$$\rho|_{q_1 = q_1(\theta)} = \sigma\left(\theta\right), \quad \zeta|_{q_1 = q_1(\theta)} = \tau\left(\theta\right). \tag{19}$$

The Lame parameters, calculated according to (15), are determined as follows:

$$h_1 = h_1(\theta) = \sqrt{\left(\frac{d\sigma}{d\theta}\right)^2 + \left(\frac{d\tau}{d\theta}\right)^2}, \quad h_2 = h_2(\theta) = \sigma(\theta), \tag{20}$$

and the principal curvatures $k_1 = k_1(\theta)$ and $k_2 = k_2(\theta)$ are calculated by the formulas:

$$k_1 = k_1(\theta) = \frac{\left(\frac{d^2\tau}{d\theta^2}\frac{d\sigma}{d\theta} - \frac{d\tau}{d\theta}\frac{d^2\sigma}{d\theta^2}\right)}{\left(\left(\frac{d\sigma}{d\theta}\right)^2 + \left(\frac{d\tau}{d\theta}\right)^2\right)^{\frac{3}{2}}}, \quad k_2 = k_2(\theta) = \frac{\frac{d\tau}{d\theta}}{\sigma\sqrt{\left(\frac{d\sigma}{d\theta}\right)^2 + \left(\frac{d\tau}{d\theta}\right)^2}}. \tag{21}$$

From the second equation of the system (18) we obtain, taking into account (10), that

$$u = -\frac{\dot{\theta}}{k_1}. \tag{22}$$

From Eq. (6) we get

$$\omega_1 = -\frac{v}{a}, \quad \omega_2 = \frac{u}{a} = -\frac{\dot{\theta}}{ak_1}.$$

Therefore, from the third equation of the system (5) we obtain:

$$\dot{\omega}_3 = \theta_2\omega_1 - \theta_1\omega_2 = \frac{v\dot{\theta}}{ak_1}(k_2 - k_1). \tag{23}$$

Equation (23) can be rewritten as follows:

$$\frac{d\omega_3}{d\theta} = \frac{v}{ak_1}(k_2 - k_1). \tag{24}$$

Now we suppose the ball rolls on a surface under the action of gravity. Then we have

$$Y = 0, \quad \theta_3 = \frac{1}{h_1h_2}\frac{dh_2}{d\theta}v$$

and the second equation of the system (7) takes the form:

$$\frac{dv}{d\theta} - \frac{v}{h_1h_2k_1}\frac{dh_2}{d\theta} = \frac{Ja}{J + ma^2}\omega_3. \tag{25}$$

Differentiating repeatedly (25) and taking into account (24) we have

$$\frac{d}{d\theta}\left(\frac{dv}{d\theta} - \frac{v}{h_1h_2k_1}\frac{dh_2}{d\theta}\right) = \frac{J}{J + ma^2}\frac{v}{k_1}(k_2 - k_1). \tag{26}$$

Thus, the problem of rolling of a ball on a fixed perfectly rough surface under the action of gravity, under the assumption that the ball's centre moves along a given surface of revolution, is reduced to integration the second order linear differential equation (26) with respect to the ball's velocity component v. Therefore, it is interesting to study for which surfaces of revolution equation (26) is integrable in Liouvillian functions. For study the problem of existence of Liouvillian solution of a given second order linear differential equation the Kovacic algorithm is usually used [3]. Below we prove that the problem of rolling of a heavy homogeneous ball on a fixed perfectly rough surface is integrable in Liouvillian functions when the centre of the ball moves along a paraboloid of revolution.

3 Rolling of a Ball on a Paraboloid of Revolution

Let the perfectly rough surface on which the ball rolls be such that the centre of the ball moves along a paraboloid of revolution. We write equation of the paraboloid in the form (14):

$$\boldsymbol{r} = \begin{pmatrix} Rq_1 \cos q_2 \\ Rq_1 \sin q_2 \\ -\frac{Rq_1^2}{2} \end{pmatrix}.$$

In the considered case

$$\rho(q_1) = Rq_1, \quad \zeta(q_1) = -\frac{Rq_1^2}{2}.$$

Here R is a parameter having the dimension of length. The Lame's parameters h_1 and h_2 calculated by (15) have a form:

$$h_1 = R\sqrt{1+q_1^2}, \quad h_2 = Rq_1,$$

and the principal curvatures k_1 and k_2, calculated by (16), have a form:

$$k_1 = -\frac{1}{R\left(1+q_1^2\right)^{\frac{3}{2}}}, \quad k_2 = -\frac{1}{R\sqrt{1+q_1^2}}.$$

From the second equation of the system (18) we find the connection between variables q_1 and θ:

$$\dot{\theta} = \frac{\dot{q}_1}{1+q_1^2}.$$

Therefore we have

$$q_1 = \tan\theta. \tag{27}$$

Taking into account (27), we can suppose, that the expressions for $\rho(q_1)$ and $\zeta(q_1)$ are rewritten as follows:

$$\sigma(\theta) = R\tan\theta, \quad \tau(\theta) = -\frac{R}{2}\tan^2\theta.$$

As a result we obtain the following expressions for the Lame's parameters h_1 and h_2 and the principal curvatures k_1 and k_2:

$$h_1 = \frac{R}{\cos^3\theta}, \quad h_2 = R\tan\theta, \quad k_1 = -\frac{\cos^3\theta}{R}, \quad k_2 = -\frac{\cos\theta}{R}.$$

The second order linear differential equation (26) can be represented in the form:

$$\frac{d}{d\theta}\left(\frac{dv}{d\theta} + \frac{v}{\sin\theta\cos\theta}\right) = \frac{J}{J+ma^2}\frac{\sin^2\theta}{\cos^2\theta}v. \tag{28}$$

Thus, the problem of motion of a heavy homogeneous ball on a perfectly rough surface such that the centre of the ball moves along a paraboloid of revolution is reduced to integration the second order linear differential equation (28). Let us change the independent variable in Eq. (28) by formula $x = \cos^2\theta$ and denote:

$$\frac{J}{J+ma^2} = n^2 < 1.$$

Then Eq. (28) is reduced to the equation with rational coefficients:

$$\frac{d^2v}{dx^2} + \frac{1}{x-1}\frac{dv}{dx} - \frac{\left(n^2x^2 + 2\left(1-n^2\right)x + n^2 - 1\right)}{4x^2\left(x-1\right)^2}v = 0. \tag{29}$$

Since Eq. (29) is the second–order linear differential equation with rational coefficients, we can apply the Kovacic algorithm to this differential equation. The goal of this algorithm is to find a solution of the differential equation

$$\frac{d^2v}{dx^2} + a\left(x\right)\frac{dv}{dx} + b\left(x\right)v = 0, \tag{30}$$

where $a\left(x\right)$ and $b\left(x\right)$ are rational functions of one (in general case complex) variable x. The first step of the algorithm is to reduce the differential equation (30) to a simpler form, using the following formula

$$y\left(x\right) = v\left(x\right)\exp\left(\frac{1}{2}\int a\left(x\right)dx\right). \tag{31}$$

Then Eq. (30) takes the form

$$\frac{d^2y}{dx^2} = R\left(x\right)y, \quad R\left(x\right) = \frac{1}{2}\frac{da}{dx} + \frac{a^2}{4} - b, \tag{32}$$

where $R\left(x\right)$ is also rational function of one variable x. The Kovacic algorithm allows one to find explicitly the solution of Eq. (32), expressed in terms of Liouvillian functions.

Applying to the differential equation (29) the transformation of the form (31), we reduce it to the differential equation

$$\frac{d^2y}{dx^2} = \frac{n^2-1}{4x^2}y. \tag{33}$$

Application of the Kovacic algorithm to the second order linear differential equation (33) shows that the general solution of this equation can be represented in the form:

$$y = C_1 x^{\frac{1+n}{2}} + C_2 x^{\frac{1-n}{2}},$$

where C_1 and C_2 are arbitrary constants. Therefore, the general solution of Eq. (33) is expressed in terms of Liouvillian functions. For the differential equation (28), we corresponding general solution has the form:

$$v(\theta) = \frac{\cos\theta}{\sin\theta}\left(K_1(\cos\theta)^n + K_2(\cos\theta)^{-n}\right),$$

where K_1 and K_2 are arbitrary constants. Thus the problem of motion of a heavy homogeneous ball of a surface of revolution such that the centre of the ball moves along the paraboloid of revolution is integrable in Liouvillian functions.

4 Conclusions

In this paper we apply the Kovacic algorithm to the problem of rolling of a heavy homogeneous ball on a fixed surface such that the centre of the ball moves along a given surface of revolution. This problem is reduced to solving the second–order linear differential equation with respect to the projection of velocity of the ball's centre onto the tangent to the parallel of the corresponding of revolution and we present here our own method to derive the corresponding equation. In the case when the centre of the ball moves along the paraboloid of revolution we are presenting the corresponding linear differential equation in explicit form and reduce its coefficients to a form of rational functions. Using the Kovacic algorithm we prove that the general solution of the corresponding second–order linear differential equation is expressed in terms of Liouvillian functions for all values of parameters of the problem.

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Chapter 15

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