Advanced Course in Classical Logic (spring 2018): Tasks

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1. Using the Deduction Theorem, derive the following infinitary formulas.
   Here Φ is an at most countable set of infinitary formulas.

   a) $(A \to B) \to \big[\bigwedge(\Phi \cup \{A\} \to \bigwedge(\Phi \cup \{B\})\big]$.
   b) $(A \leftrightarrow B) \to \big[\bigwedge(\Phi \cup \{A\} \leftrightarrow \bigwedge(\Phi \cup \{B\})\big]$.
   c) $(\Phi \to A) \leftrightarrow (\bigwedge\Phi \to A)$. Recall that $(\Phi \to A) := \neg\bigwedge(\Phi \cup \{\neg A\})$.

2. Are the following structures isomorphic? Are they elementary equivalent? Use games for the latter question. For your formula that distinguishes the two structures, prove (using games) that its quantifier depth is minimal.
   a) $(\mathbb{N}, <)$ and $(\mathbb{N} + \mathbb{Z}, <)$; **b)** $(\mathbb{N}, <)$ and $(\mathbb{N} + \mathbb{N}, <)$; **c)** $(\mathbb{N} + \mathbb{N}, <)$ and $(\mathbb{N} + \mathbb{N} + \mathbb{N}, <)$.

3. Tasks about filters and ultrafilters (solve them independently):

   a) Assume that Φ is a non-principal ultrafilter, $a \in X \in \Phi$, and $|X| > 1$. Then $(X \setminus \{a\}) \in \Phi$.
   b) Consider a principal filter $\Phi = (X) = \{Y \subseteq I \mid X \subseteq Y\}$. Prove that Φ is an ultrafilter $\Leftrightarrow$ X is a singleton.
   c) If an ultrafilter Φ contains some finite set $X \subseteq I$, then Φ is principal, moreover, $\Phi = \pi_a$, for some $a \in X$.
   d) Assume that I is an infinite set, Φ is an ultrafilter over I.
   Prove that Φ is non-principal $\Leftrightarrow$ Φ contains every cofinite subset of I.
   f) Let Φ be an ultrafilter over a set I and consider any subsets $A_1, \ldots, A_n \subseteq I$, where $n \geqslant 2$.
   Prove: $A_1 \cup \ldots \cup A_n \in \Phi \iff A_1 \in \Phi$ or $\ldots$ or $A_n \in \Phi$.
   Does this hold for infinitely many subsets $A_1, A_2, \ldots \subseteq I$?

4. Prove that the ultraproduct of normal models is a normal model.
   (A model $M = (D, *)$ is called *normal* if the symbol $=$ is interpreted as the equality relation $\{\langle a, a\rangle \mid a \in D\}$.)

5. Let D be a 2-element set, U an arbitrary ultrafilter (over any set I). How many elements does the ultrapower D^U have? Similarly for an n-element set D.

6. Prove that the following class of models is axiomatizable, but not finitely axiomatizable:
   a) all infinite linear orders; **b)** all fields of characteristic 0; **c)** all acyclic graphs.

7. Assume that A is a sentence (i.e., a closed formula) in the signature $\{=\}$.
   a) If A is true in some countable model, then A is true in every countable model.
   b) If A is true in some infinite model, then A is true in every infinite model.
   c) Consequently, all infinite models are elementarily equivalent in the signature $\{=\}$.
   d) Prove the same statement (c) using games.

8. A (non-oriented) *graph* is a structure $M = (V, E)$, where V is a set and E is an *irreflexive* and *symmetric* binary relation on V. A graph is called *connected* if any two distinct vertices are linked via a chain of edges.

   Task: Prove that the class of all connected graphs is not axiomatizable, i.e., there is no set of formulas Γ in the signature $\{R, =\}$ such that, for any graph M, we have: M is connected iff $M \models \Gamma$.

   Hint: consider $\Delta = \{\mathsf{Dist}_{>n}(a, b) \mid n \geqslant 1\}$, where a, b are constant symbols and the first-order sentence $\mathsf{Dist}_{>n}(a, b)$ says that the edge distance between a and b is greater than n (write this formula explicitly). Now use compactness.

9. The following are equivalent definitions of the *transitive closure* R^+ of a binary relation $R \subseteq D \times D$; prove it:
   a) R^+ is the minimal transitive relation that contains R: $R^+ = \min\{S \supseteq R \mid S \text{ is transitve}\}$.
   b) the relation R^+ links two points $a, b \in D$ iff there is a chain (containing at least one edge) that links a and b:

   $$a\, R^+\, b \iff \exists n \geqslant 1\ \exists x_0 \ldots \exists x_n (x_0 = a \wedge x_n = b \wedge x_0\, R\, x_1 \wedge \ldots \wedge x_{n-1}\, R\, x_n).$$

   c) $R^+ = \bigcup_{n \geqslant 1} R^n$. Here $R^1 = R$ and $R^{n+1} = R \circ R^n$. Recall that $R \circ S := \{(a, c) \mid \exists b\, (a\, R\, b \wedge b\, S\, c)\}$.
   d) R^+ is obtained in an iterative process of adding 2-step edges. This means that we start with the relation $S_1 = R$, and on $(n+1)$-th step we find all possible points a, b, c such that $a\, S_n\, b\, S_n\, c$ and add the edge (a, c):

   $$S_1 = R, \quad S_{n+1} = S_n \cup (S_n \circ S_n); \quad R^+ = \textstyle\bigcup_{n \geqslant 1} S_n.$$

10. Prove that the *transitive closure* R^+ is not expressible in the first-order language.
    This means that there is no first-order formula $A(x, y)$ with two free variables x, y in the signature $\{R, =\}$ such that, for any model $M = (D, R)$ and any points $e, d \in D$, we have $M \models A[e, d]$ iff $e\, R^+\, d$.

    Hint: consider the set of formulas $\Delta = \{\neg R(a, b),\ \neg R^2(a, b),\ \neg R^3(a, b), \ldots\}$. Use compactness.