ACCL Lecture 12:

The hierarchy of “species” of elementary classes of models

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Advanced Course in Classical Logic
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First-order formulas and models

We consider the following logical system $S = (\mathcal{L}, \mathcal{M}, \models)$:

$\mathcal{L}$ — closed formulas over a signature $\Sigma = (\mathrm{Pred}, \mathrm{Func}, \mathrm{Const})$.

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   Examples: groups, semigroups, monoids, etc.

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Task

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$$\mathsf{Models}(\mathsf{Theory}(\mathsf{Models}(\Gamma))) = \mathsf{Models}(\Gamma)$$

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Definition

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Example

The class of groups, the class of fields, the class of linear orders, the class of Boolean algebras, etc.

Lemma

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2. *FIN-AX is closed under intersection (of two classes):*

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Theorem (Keisler, 1961)

Let $\mathbb{K}$ be any class of first-order models (of a fixed signature).

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$\mathbb{K}$ is axiomatizable	$\equiv$	uProd	
$\mathbb{K}$ is finitely axiomatizable	$\equiv$	uProd	uProd

Why non-symmetric?

Because this is not the whole story!

The hierarchy starts growing

Definition

A class of models $\mathbb{K}$ is called finitely axiomatizable (FIN-AX), if $\mathbb{K}$ is definable by a single formula $A \in \mathcal{L}$: $\mathbb{K} = \text{Models}(A)$.

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$$\boxed{\Cup\Cap\mathbb{E} = \Cap\Cup\mathbb{E}}$$

The hierarchy of 4 "species" of classes

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\begin{tikzcd}
 & \boxed{\text{quasi-axiomatizable classes}} & \\
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Criterion for quasi-axiomatizability

Theorem (Criterion for $\mathbb{U}\Cap\mathbb{E}$)

For any class of models $\mathbb{K}$, the following statements are equivalent:

(1a) $\mathbb{K} \in \mathbb{U}\Cap\mathbb{E}$ *($\mathbb{K}$ is the union of intersections of AX classes)*

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(1b)⇒(2): Similarly. □

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(2)$\Rightarrow$(1a):

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$$\mathbb{K} \;=\; \bigcup_{M \in \mathbb{K}} [M], \qquad \text{where } [M] = \{N \mid M \equiv N\}.$$

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$(\subseteq)$ If $M \in \mathbb{K}$, then (since $M \in [M]$) the model M belongs to the right hand side of the equality.

$(\supseteq)$ For every $M \in \mathbb{K}$ we have $[M] \subseteq \mathbb{K}$, and so $\bigcup_{M\in\mathbb{K}} [M] \subseteq \mathbb{K}$. □

More properties

$\boxed{\mathbb{E} \text{ means FIN-AX, } \Cap\mathbb{E} \text{ means AX, } \Cup\mathbb{E} \text{ means co-AX, } \Cup\Cap\mathbb{E} \text{ means Q-AX}}$

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Lemma

1. $\mathbb{E}$ *is closed under finite* $\cap$ *and* $\cup$*:*

   $\mathbb{K}_1$ *and* $\mathbb{K}_2$ *are FIN-AX* $\quad\Longrightarrow\quad$ $\mathbb{K}_1 \cap \mathbb{K}_2$ *and* $\mathbb{K}_1 \cap \mathbb{K}_2$ *are FIN-AX.*

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2. *$\Cap\mathbb{E}$ is closed under arbitrary $\cap$.*

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Criteria for all 4 "species"

Theorem (Keisler, 1961)

Let $\mathbb{K}$ be any class of first-order models (of a fixed signature).

- $\mathbb{K}$ *is axiomatizable* $\iff$
  $\mathbb{K}$ *is closed under* $\equiv$ *and ultraproducts*
- $\mathbb{K}$ *is finitely axiomatizable* $\iff$
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	Both	$\mathbb{K}$	$\overline{\mathbb{K}}$
$\mathbb{K} \in \Cup\Cap\mathbb{E}$	$\equiv$		
$\mathbb{K} \in \Cap\mathbb{E}$	$\equiv$	uProd	
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The table is symmetric now!

Criteria for all 4 "species"

Theorem (Keisler, 1961)

Let $\mathbb{K}$ be any class of first-order models (of a fixed signature).

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Tasks for students:

- Take the minimal signature $\Sigma = (\{P, =\}, \varnothing, \varnothing)$, where $\mathrm{ar}(P) = 1$. So we have one unary predicate symbol.

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End of lecture 12. Thank you!