

ACCL Lecture 12:

The hierarchy of “species” of elementary classes of models

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Advanced Course in Classical Logic
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First-order formulas and models

We consider the following **logical system** $S = (\mathcal{L}, \mathcal{M}, \models)$:

\mathcal{L} — **closed formulas** over a signature $\Sigma = (\text{Pred}, \text{Func}, \text{Const})$.

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Examples: groups, semigroups, monoids, etc.

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$$\text{Models}(A) := \{M \in \mathcal{M} \mid M \models A\}$$

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Task

$$\text{Theory}(\text{Models}(\text{Theory}(\mathbb{K}))) = \text{Theory}(\mathbb{K})$$

$$\text{Models}(\text{Theory}(\text{Models}(\Gamma))) = \text{Models}(\Gamma)$$

Finitely axiomatizable class

Definition

A class of models \mathbb{K} is called **finitely axiomatizable** (FIN-AX), if \mathbb{K} is definable by a single formula $A \in \mathcal{L}$ (or by finitely many formulas):

$$\mathbb{K} = \text{Models}(A).$$

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Example

The class of groups, the class of fields, the class of linear orders, the class of Boolean algebras, etc.

Lemma

$$\textcircled{1} \quad \mathbb{K} \text{ is FIN-AX} \quad \Longleftrightarrow \quad \overline{\mathbb{K}} \text{ is FIN-AX.}$$

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- ① \mathbb{K} is FIN-AX $\iff \overline{\mathbb{K}}$ is FIN-AX.
- ② FIN-AX is closed under intersection (of two classes):
classes \mathbb{K}_1 and \mathbb{K}_2 are FIN-AX $\implies \mathbb{K}_1 \cap \mathbb{K}_2$ is FIN-AX.

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Proof.

(1) $\overline{\mathbb{K}} = \text{Models}(\neg A).$

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(Finite) axiomatizability criterion

Theorem (Keisler, 1961)

Let \mathbb{K} be any class of first-order models (of a fixed signature).

- \mathbb{K} is *axiomatizable* \iff
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Why non-symmetric?

Because this is not the whole story!

The hierarchy starts growing

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A class of models \mathbb{K} is called **finitely axiomatizable** (FIN-AX), if \mathbb{K} is definable by a single formula $A \in \mathcal{L}$: $\mathbb{K} = \text{Models}(A)$.

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Should we continue? \mathbb{NUE} ? \mathbb{UNUE} ? \mathbb{NUNE} ? ...

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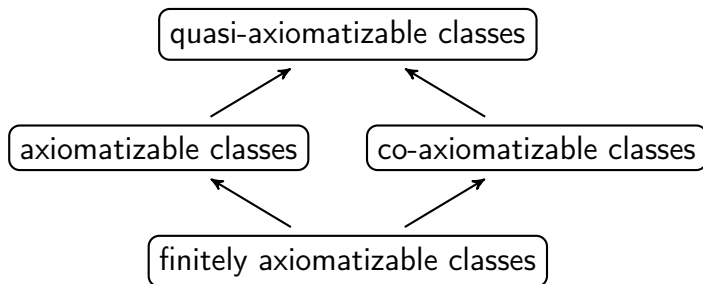
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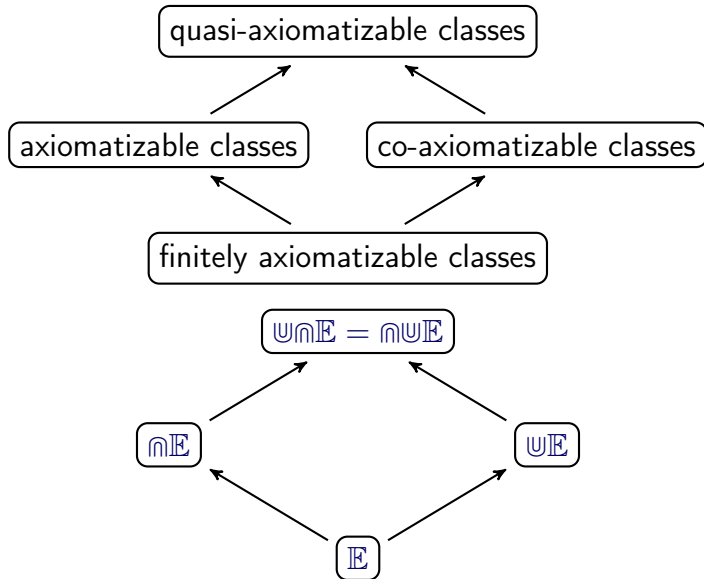
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$$\mathbb{WNE} = \mathbb{NWE}$$

The hierarchy of 4 “species” of classes



The hierarchy of 4 “species” of classes



Criterion for quasi-axiomatizability

Theorem (Criterion for $\cup\cap E$)

For any class of models \mathbb{K} , the following statements are equivalent:

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(1b) \Rightarrow (2): Similarly.



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(\supseteq) For every $M \in \mathbb{K}$ we have $[M] \subseteq \mathbb{K}$, and so $\bigcup_{M \in \mathbb{K}} [M] \subseteq \mathbb{K}$. □

More properties

\mathbb{E} means FIN-AX, \mathbb{NE} means AX, \mathbb{UE} means co-AX, \mathbb{UNE} means Q-AX

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① \mathbb{E} is closed under finite \cap and \cup :

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Criteria for all 4 “species”

Theorem (Keisler, 1961)

Let \mathbb{K} be any class of first-order models (of a fixed signature).

- \mathbb{K} is *axiomatizable* \iff
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$\mathbb{K} \in \cap \mathbb{E}$	\equiv	uProd	
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The table is symmetric now!

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$\mathbb{K} \in \cap E$	\equiv	compact	
$\mathbb{K} \in \cup E$	\equiv		compact
$\mathbb{K} \in E$	\equiv	compact	compact

Examples

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- ④ The class of infinite fields of **characteristic p** , where $p > 0$ is fixed — is **axiomatizable**.

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- ④ The class of infinite fields of **characteristic** p , where $p > 0$ is fixed — is **axiomatizable**.
- ⑤ The class of infinite fields of all positive **characteristics** $\text{ch}(F) > 0$ — it is Q-AX ($\bigcup \mathbb{N} \mathbb{E} = \mathbb{N} \bigcup \mathbb{E}$).

Examples

- 1 The class of fields — is **finitely axiomatizable**.
- 2 The class of finite fields — is **co-axiomatizable**,
but not FIN-AX, because it is not compact: take $\Gamma = \{\exists^n x \mid n \geq 1\}$.
- 3 The class of infinite fields — is **axiomatizable**.
But not FIN-AX: because its complement is not compact.
- 4 The class of infinite fields of **characteristic p** , where $p > 0$ is fixed — is **axiomatizable**.
- 5 The class of infinite fields of all positive **characteristics $\text{ch}(F) > 0$** — it is Q-AX ($\bigcup \mathbb{E} = \bigcap \mathbb{E}$).
But it is not AX. And it is not co-AX.

Further thinking

Tasks for students:

- Take the **minimal** signature $\Sigma = (\{P, =\}, \emptyset, \emptyset)$, where $\text{ar}(P) = 1$.
So we have one unary predicate symbol.

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Build classes of models that are FIN-AX, AX, co-AX, Q-AX.

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Show that we cannot build a Q-AX class of models!

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- What about the signature without equality?
 $\Sigma = (\{=\}, \emptyset, \emptyset)$, where $\text{ar}(P) = 1$.
Can we build classes of models that are FIN-AX, AX, co-AX.

Further thinking

Tasks for students:

- Take the **minimal** signature $\Sigma = (\{P, =\}, \emptyset, \emptyset)$, where $\text{ar}(P) = 1$.
So we have one unary predicate symbol.
Build classes of models that are FIN-AX, AX, co-AX, Q-AX.
- Now try a smaller signature: $\Sigma = (\{=\}, \emptyset, \emptyset)$.
Build classes of models that are FIN-AX, AX, co-AX.
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Can we build classes of models that are FIN-AX, AX, co-AX.

End of lecture 12. Thank you!