# ACCL Lecture 10: Representation of finite Boolean algebras. Ultrafilters in Boolean algebras. Representation of arbitrary Boolean algebras (Stone's representation theorem)

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Advanced Course in Classical Logic
April 28th, 2021

# Abstract Boolean algebras

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Boolean algebra: $\mathcal{B} = (D, \wedge, \vee, -, 0, 1)$, where $D \neq \varnothing$ is a set, $0, 1 \in D$, the operations $\wedge, \vee \colon D \times D \to D$ and $- \colon D \to D$ satisfy the laws:

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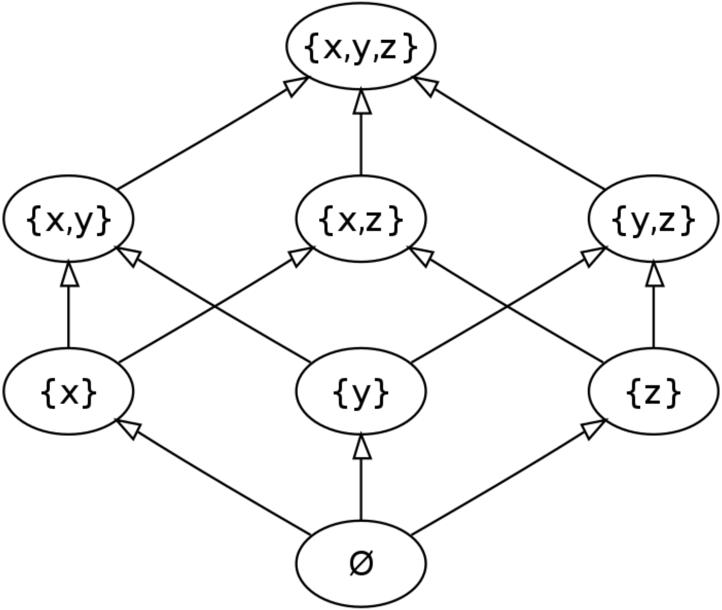
Terminology: $\wedge$ meet, $\vee$ join, $-$ complement.

# Concrete Boolean algebra: $\wp(\{x, y, z\})$



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- any Boolean algebra is isomorphic to an algebra of the form (b),

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**Definition**

Boolean algebras $\mathcal{B}$ and $\mathcal{B}'$ are isomorphic ($\mathcal{B} \cong \mathcal{B}'$) if there is an isomorphism $h\colon \mathcal{B} \to \mathcal{B}'$, i.e., a bijection that preserves $\vee, ^{-}$ (and $\wedge, 0, 1$).

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For any finite Boolean algebra $\mathcal{B}$ there exists a set $W$ such that $\mathcal{B} \cong (2^W, \cap, \cup, -, \varnothing, W)$.

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Hence our algebra is not isomorphic to any algebra of the form $2^W$.

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**Proposition**

*The relation $\leqslant$ is a partial order on $D$, i.e., it is:*

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| *reflexive* | $\forall a \, (a \leqslant a)$ |
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For this reason, we read "$a \leqslant b$" as "$a$ is contained in $b$" or "$b$ contains $a$".

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This process is finite. When it stops, we get an atom $s \leqslant a$. $\square$

# Properties of atoms

**Lemma (Disjunction property for atoms)**

*Let $s$ be an atom and $a, b$ be elements of a Boolean algebra $\mathcal{B}$.*

$$s \leqslant (a \vee b) \quad \Rightarrow \quad s \leqslant a \ \text{ or } \ s \leqslant b.$$

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Intuitively: there is no "strongest" formula.

# Stone's representation theorem

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How can we prove this theorem without having atoms?

# Filters and ultrafilters over a set

**Definition**

Let $W \neq \varnothing$. A filter over $\mathcal{I}$ — is a family of subsets $\Phi \subseteq 2^{\mathcal{I}}$ such that

**(1)** $\varnothing \notin \Phi$

**(2)** $W \in \Phi$

**(3)** $X \in \Phi$ and $X \subseteq Y \;\Rightarrow\; Y \in \Phi$ for all $X, Y \subseteq \mathcal{I}$

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So we replace $X \subseteq \mathcal{I}$ with $a \in D$, and write $a \leqslant b$ instead of $X \subseteq Y$.

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A subset $E \subseteq D$ has the finite intersection property if

$$\text{for all } a_1, \ldots, a_n \in E \quad \text{we have} \quad (a_1 \wedge \ldots \wedge a_n) \neq 0.$$

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An ultrafilter in a Boolean algebra $\mathcal{B}$ additionally satisfies:

**(5)** $a \in \Phi$ or $\overline{a} \in \Phi$ for all $a \in D$.

A subset $E \subseteq D$ has the finite intersection property if

$$\text{for all } a_1, \ldots, a_n \in E \quad \text{we have} \quad (a_1 \wedge \ldots \wedge a_n) \neq 0.$$

## Theorem

**(1)** *$E \subseteq D$ has the finite intersection property $\Longrightarrow$ $E \subseteq \Phi$ for some filter $\Phi$.*

# Filters and ultrafilters in Boolean algebras

Let $\mathcal{B} = (D, \wedge, \vee, -, 0, 1)$ be a Boolean algebra. We also have $\leqslant$.

## Definition

A filter in a Boolean algebra $\mathcal{B}$ is a subset $\Phi \subseteq D$ such that

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**(1)** *$E \subseteq D$ has the finite intersection property $\Longrightarrow$ $E \subseteq \Phi$ for some filter $\Phi$.*

**(2)** *Any filter is contained in some ultrafilter.*

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**Theorem (Stone's representation theorem, 1936)**

*Any Boolean algebra is isomorphic to some field of sets, i.e., to some subalgebra $S \subseteq 2^W$, for some set $W$.*

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In general, $f$ is not surjective! □

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End of lecture 10. Thank you!