ACCL Lecture 9: Applications of ultraproducts: Compactness. Criterion of axiomatizability

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Advanced Course in Classical Logic
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Łoś's theorem (reminder)

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Suppose $M = \prod_{i\in I}^{\Phi} M_i$ is the ultraproduct of models.
Then, for any closed formula (sentence) A, we have:

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For every sentence A: $M \models A \iff M^{\Phi} \models A.$ *Thus:* $M \equiv M^{\Phi}$.

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Γ is closed under $\wedge$ means: $\quad A, B \in \Gamma \implies (A \wedge B) \in \Gamma.$

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Theorem (Compactness theorem)

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For every $i \in \mathrm{I}$, consider $X_i := \{\ell \in \mathrm{I} \mid M_\ell \models A_i\}$. So $X_i \subseteq \mathrm{I}$.

Proposition. *The family of sets $\Psi = \{X_i \mid i \in \mathrm{I}\}$ has the F.I.P.*

$\ell \in X_{i_1} \cap \ldots \cap X_{i_n}$, where $A_\ell := (A_{i_1} \wedge \ldots \wedge A_{i_n}) \in \Gamma$, since $M_\ell \models A_\ell$.

Hence $\Psi \subseteq \Phi$ for some ultrafilter Φ. Build $M := \prod_{i \in \mathrm{I}}^{\Phi} M_i$. Then $M \in \mathbb{K}$.
Now $M \models A_i$ for every $i \in \mathrm{I}$, since $\{\ell \in \mathrm{I} \mid M_\ell \models A_i\} = X_i \in \Phi$. $\square$

Ultraproducts $\Longrightarrow$ compactness

Theorem (Compactness theorem)

If a class of models $\mathbb{K}$ is closed under ultraproducts, then $\mathbb{K}$ is compact.

We will use Definition 2 of compactness.

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Corollary (Мальцев). *The class of all models (over Σ) is compact.*

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Definition

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Eureka! These two conditions are sufficient!

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$$\Longrightarrow$$
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The last implication was proved in Keisler's theorem. □

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($\Longleftarrow$) By Theorem 1, $\mathbb{K}$ and $\overline{\mathbb{K}}$ are both elementary:

$$\mathbb{K} = \text{Models}(\Gamma) \quad \text{and} \quad \overline{\mathbb{K}} = \text{Models}(\Delta).$$

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Exercise: $\mathbb{K} = \mathrm{Models}(A_1 \wedge \ldots \wedge A_m)$,
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Moreover, the converse also holds (exercise):

any non-principal ultrafilter Φ
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