ACCL Lecture 6: Ehrenfeucht games: a criterion of elementary equivalence of two models

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Advanced Course in Classical Logic
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First-order predicate logic: Syntax

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A signature $\Sigma = (\mathrm{Pred}, \mathrm{Func}, \mathrm{Const})$

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$\exists y\,(\forall x\, P(x, y, x) \ \wedge \neg Q(y)) \to \forall z R(x, z)$ — free variable!

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This is a closed formula or sentence.

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Sentence A: $\forall x \exists y \, (x < y)$. Then...

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So, a sentence means some statement (true or false) about a model.

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$M \cong N$, if there is a bijection $f\colon D \to G$, such that, for all $a, b \in D$:

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$M \cong N \quad \Longrightarrow \quad M \equiv N.$ *The converse does not hold in general.*

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$(\mathbb{Z}, <) \equiv (\mathbb{Z} + \mathbb{Z}, <)$? Yes, but why?

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The "aim" of the first player $\mathsf{P}_{\neq}$ is to show that $M \not\equiv N$.
The "aim" of the second player $\mathsf{P}_{\equiv}$ is to show that $M \equiv N$.

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YES $\Longrightarrow$ $\mathrm{P}_{\not\equiv}$ wins (Player 1)
NO $\Longrightarrow$ $\mathrm{P}_{\equiv}$ wins (Player 2).

Important notion: a winning strategy for some player.

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$$\begin{gathered} q(\, P(x_1, \ldots, x_s)\,) = 0, \\ q(\neg A) = q(A), \\ q(A \wedge B) = \max(\, q(A), q(B)\,). \\ q(\forall x\, B) = 1 + q(B), \end{gathered}$$

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Theorem (Main$_n$)

$M \equiv_n N \iff \mathrm{P}_{\equiv}$ *has a winning strategy in* $Game_n(M, N)$.

Theorem 1 $\Longleftarrow$ Theorem 2.

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. P_1 wins in 3 rounds.
  But P_2 wins in 2 rounds.

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. P_1 wins in 3 rounds.
  But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.

Interactive

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- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$		
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$		
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$		
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$		

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. P_1 wins in 3 rounds.
  But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$		
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$		
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$		

The end of Lecture 6. Questions?

- $(\mathbb{N},<)$ and $(\mathbb{Z},<)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
- $(\mathbb{Z},<)$ and $(\mathbb{Q},<)$. P_1 wins in 3 rounds.
  But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N}+\mathbb{N}$	$-$	
$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$-$	
$\mathbb{Q}$	$\mathbb{Q}+\mathbb{Q}$		
$\mathbb{R}$	$\mathbb{R}+\mathbb{R}$		

The end of Lecture 6. Questions?

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
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  But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$	$-$	
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$		

The end of Lecture 6. Questions?

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  But P_2 wins in 1 round.
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- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N}+\mathbb{N}$	$-$	
$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$-$	
$\mathbb{Q}$	$\mathbb{Q}+\mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R}+\mathbb{R}$	$-$	

The end of Lecture 6. Questions?

Interactive

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- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	$-$
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$	$-$	
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$	$-$	

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- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. P_1 wins in 3 rounds. But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N}+\mathbb{N}$	$-$	$-$
$\mathbb{Z}$	$\mathbb{Z}+\mathbb{Z}$	$-$	$+$
$\mathbb{Q}$	$\mathbb{Q}+\mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R}+\mathbb{R}$	$-$	

The end of Lecture 6. Questions?

Interactive

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- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	$-$
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$	$-$	$+$
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$	$-$	$+$

The end of Lecture 6. Questions?

Interactive

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M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	$-$
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$	$-$	$+$
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$	$-$	$+$

- $\mathbb{N} \equiv \mathbb{N} + \mathbb{Z}$. Show this using games.

The end of Lecture 6. Questions?

Interactive

- $(\mathbb{N}, <)$ and $(\mathbb{Z}, <)$. P_1 wins in 2 rounds.
  But P_2 wins in 1 round.
- $(\mathbb{Z}, <)$ and $(\mathbb{Q}, <)$. P_1 wins in 3 rounds.
  But P_2 wins in 2 rounds.
- $\mathbb{Q} \equiv \mathbb{R}$. Show this using games.
- Using games, prove:

M	N	$M \cong N$	$M \equiv N$
$\mathbb{N}$	$\mathbb{N} + \mathbb{N}$	$-$	$-$
$\mathbb{Z}$	$\mathbb{Z} + \mathbb{Z}$	$-$	$+$
$\mathbb{Q}$	$\mathbb{Q} + \mathbb{Q}$	$+$	$+$
$\mathbb{R}$	$\mathbb{R} + \mathbb{R}$	$-$	$+$

- $\mathbb{N} \equiv \mathbb{N} + \mathbb{Z}$. Show this using games.
- $\mathbb{N} + \mathbb{N} \not\equiv \mathbb{N} + \mathbb{N} + \mathbb{N}$. Find the formula with minimal $q(A)$.

The end of Lecture 6. Questions?